

Lecture 21

Anomalies

Up to this point we have talked considered symmetries, both global and gauged, as either been exact or spontaneously broken. In both these cases the lagrangian (or hamiltonian) remain invariant under symmetry transformations. In the case of spontaneous symmetry breaking, it is the ground state that is not invariant under them. However, it is possible to consider symmetries that are *explicitly* broken. In this case, there are terms in the lagrangian that are not invariant under the symmetry. However, the symmetry may still be a useful tool in many cases. For instance, if the explicit symmetry breaking is small, i.e. if it is controlled by a small parameter, the predictions obtained in the limit where the symmetry is exactly valid will suffer corrections which are typically some power of this small parameter. This is the case, for example, for the chiral symmetry of QCD at low energies. In the limit of massless quarks, the global (two flavor) symmetry of QCD is $SU(2)_L \times SU(2)_R$. It is spontaneously broken down to $SU(2)_V$, or isospin. The pions, massless in the limit of massless quarks, acquire a small mass in the presence of the quark masses (coming from the *spontaneous* breaking of electroweak symmetry. In general, observables computed using the exact symmetry limit will receive corrections which are small as long as the typical momenta in the process of interest is small compared with the hadronic scale responsible for the spontaneous breaking. Thus, we would have a controlled approximation (e.g. in powers of momentum or the quark mass over the hadronic scale, so that the *explicit* symmetry breaking is a small effect on the predictions. This is, for instance, the case in chiral perturbation theory, an effective theory valid at low energies describing interactions of pions and other hadrons.

There is, however, another case of explicit symmetry breaking in which the symmetry predictions are rendered potentially useless. This is the case of anomalous symmetries. These are symmetries that are classically preserved but that are explicitly broken by quantum corrections. An example we will not go into here is scale invariance. In the absence of fermion masses, the QED and QCD actions are invariant under scale transformations

$$x^\mu \rightarrow \kappa x^\mu , \tag{21.1}$$

where κ is a arbitrary real constant. This symmetry relies on the fact that there are no dimensionful quantities in

$$\bar{\psi} i \not{D} \psi , \quad (21.2)$$

where the covariant derivative is the appropriate for the QED or QCD case. However, quantum corrections introduce scale dependence of the parameters of the lagrangian. We can see in this case that the so called scale anomaly is induced by the presence of a non zero $\beta(g)$ function, with g the coupling. In fact it is possible to see that the non conservation of the current associated to wcale invariance (the energy momentum tensor) is proportional to $\beta(g)$. But here we will concentrate on the anomaly associated to chiral transformation symmetry.

21.1 The Chiral Anomaly

Let us cosider theory with a massless fermion described by

$$\mathcal{L} = \bar{\psi} i \not{D} \psi = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R , \quad (21.3)$$

where the covariant derivative is assumed to correspond to some generic gauge theory. Independently of the gauge transformation properties of fermions, the lagrangian above is invariant under the *global* symmetry defined by

$$\psi_L \longrightarrow e^{i\alpha_L} \psi_L, \quad \psi_R \longrightarrow e^{i\alpha_R} \psi_R , \quad (21.4)$$

where α_L abd α_R are real constants. The currents associated with the symmetry transformations above are

$$j_L^\mu = \bar{\psi}_L \gamma^\mu \psi_L, \quad j_R^\mu = \bar{\psi}_R \gamma^\mu \psi_R . \quad (21.5)$$

It will be more convenient to define the vector and axial currents

$$j_V^\mu = \bar{\psi} \gamma^\mu \psi, \quad j_A^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi , \quad (21.6)$$

where $j_V^\mu = j_R^\mu + j_L^\mu$ and $j_A^\mu = j_R^\mu - j_L^\mu$. Both the vector and axial currents are *classically* conserved, i.e. the satisfy

$$\partial_\mu j_V^\mu = 0, \quad \partial_\mu j_A^\mu = 0 , \quad (21.7)$$

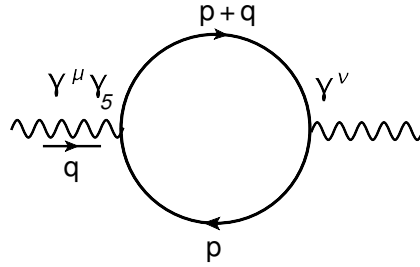


Figure 21.1: Fermion loop contribution to the product of a chiral and a vector currents.

which just results from the application of the Dirac equation. However, as we will see below, quantum corrections will spoil current conservation for one of these currents.

The loop corrections we need to consider must have an odd number of γ_5 matrices in them. Otherwise, the result will not test the chiral aspect of the couplings. The first quantum correction we can think of is a fermion loop with one insertion of $\gamma^\mu \gamma_5$ (an axial current coupling) and another of γ^μ . For instance, this could be a one loop correction to the two point function of chiral gauge bosons. This is depicted in Figure 21.1 These could appear in theories with both vector and chiral gauge bosons, as for instance in the SM. However, in 4D this diagram will not contribute to an anomaly. To see this, we notice that the diagram is proportional to the factor

$$\text{Tr} \left[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma_5 \right] p_\alpha q_\beta = -4i \epsilon^{\mu\alpha\nu\beta} p_\alpha q_\beta . \quad (21.8)$$

But to check current conservation we need to multiply this diagram by q_ν to check for the vector current conservation, or by q_μ to check the axial current conservation. Since both of these vanish, then this diagram does not generate an anomaly in 4D¹. However, as we will see below, the triangle diagrams of Figure 21.2, corresponding to quantum corrections to three point functions, will result in anomalous contributions and current non conservation. They correspond to the insertion of one axial current in one of the vertices and two vector currents in the other two. The fact that we have two diagrams is a reflection of Bose symmetry, i.e. the diagram should be invariant under the exchange of the two vector vertices. If we call the sum of the two diagrams $\Delta^{\lambda\mu\nu}(k_1, k_2)$ then Bose symmetry requires that

$$\Delta^{\lambda\mu\nu}(k_1, k_2) = \Delta^{\lambda\nu\mu}(k_2, k_1) , \quad (21.9)$$

which is trivially satisfied by the sum of diagrams (a) and (b). The amplitude is given by

¹In 2D, however, this loop correction is in fact a source of the anomaly

$$\begin{aligned}
i \Delta^{\lambda\mu\nu}(k_1, k_2) &= (-1) \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\gamma^\lambda \gamma_5 \frac{i}{\not{p} - \not{q}} \gamma^\nu \frac{i}{\not{p} - \not{k}_1} \gamma^\mu \frac{i}{\not{p}} \right] \\
&+ (-1) \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\gamma^\lambda \gamma_5 \frac{i}{\not{p} - \not{q}} \gamma^\mu \frac{i}{\not{p} - \not{k}_2} \gamma^\nu \frac{i}{\not{p}} \right]
\end{aligned} \tag{21.10}$$

Conservation of the vector current corresponds to contracting the amplitude above with the momenta associated with the vector like vertices (k_1 and k_2) obtaining

$$k_{1\mu} \Delta^{\lambda\mu\nu}(k_1, k_2) = 0, \quad k_{2\nu} \Delta^{\lambda\mu\nu}(k_1, k_2) = 0, \tag{21.11}$$

whereas conservation of the axial current would imply

$$q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) = 0. \tag{21.12}$$

For instance, let us compute²

$$\begin{aligned}
k_{1\mu} \Delta^{\lambda\mu\nu}(k_1, k_2) &= i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\gamma^\lambda \gamma_5 \frac{i}{\not{p} - \not{q}} \gamma^\nu \frac{i}{\not{p} - \not{k}_1} \not{k}_1 \frac{i}{\not{p}} \right] \\
&+ i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\gamma^\lambda \gamma_5 \frac{i}{\not{p} - \not{q}} \not{k}_1 \frac{i}{\not{p} - \not{k}_2} \gamma^\nu \frac{i}{\not{p}} \right].
\end{aligned} \tag{21.13}$$

But we can write

$$\not{k}_1 = \not{p} - (\not{p} - \not{k}_1) \quad \not{k}_1 = \not{p} - \not{k}_2 - (\not{p} - \not{q}), \tag{21.14}$$

where in the second equality we used $q = k_1 + k_2$. Thus, we can rewrite (21.13) as

$$k_{1\mu} \Delta^{\lambda\mu\nu}(k_1, k_2) = i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\gamma^\lambda \gamma_5 \frac{i}{\not{p} - \not{q}} \gamma^\nu \frac{i}{\not{p} - \not{k}_1} - \gamma^\lambda \gamma_5 \frac{1}{\not{p} - \not{k}_2} \gamma^\nu \frac{i}{\not{p}} \right]. \tag{21.15}$$

²Here we notice that what is relevant for computing the divergence of the vector current is $(-ik_{1\mu}) i \Delta^{\lambda\mu\nu}$, and similarly for the axial current with $(-iq_\lambda) i \Delta^{\lambda\mu\nu}$. So we omit the factors of i from here on.

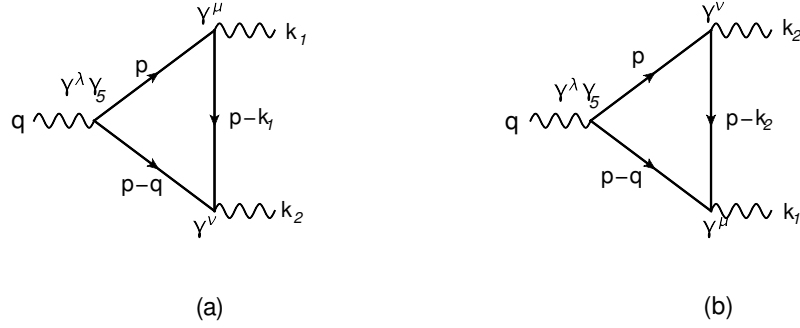


Figure 21.2: Feynman diagrams relevant for the calculation of the anomaly in chiral theories in four dimensions.

Just staring at the expression above it looks like it should vanish. The reason is that a mere shift in the integration momentum p seems to make these two integrals identical and therefore the two terms in (21.15) cancel. For instance, if we make the shift

$$p \rightarrow p - k_1 , \quad (21.16)$$

in the second term we obtain the first and they therefore cancel. However, one has to be very careful with shifts of integration variables when integrals are divergent. In the case above, the integrals in (21.13) are linearly divergent. To understand this point, let us consider a simple example. Let us compute the integral

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dx \left\{ f(x+y) - f(x) \right\} = \int_{-\infty}^{\infty} dx \left\{ y f'(x) + \frac{1}{2} y^2 f''(x) + \dots \right\} , \\ &= y \left(f(\infty) - f(-\infty) \right) + \frac{y^2}{2} \left(f'(\infty) - f'(-\infty) \right) + \dots . \end{aligned} \quad (21.17)$$

If the integral is linearly divergent, just as it is in our case, then we have

$$f'(\pm\infty) = f''(\pm\infty) = \dots = 0 . \quad (21.18)$$

However, if $f(\infty) \neq f(-\infty)$, then the shift $x \rightarrow x + y$ cannot be correctly assumed so that the integral vanishes. To get closer to our case in (21.15), we consider the integral in the euclidean

$$I_4 = i \int d^4 p \left\{ F(p+k) - F(p) \right\} = i \int d^4 p \left\{ k^\mu \partial_\mu F(p) + \dots \right\} . \quad (21.19)$$

Once again, if we consider the case when the integral is linearly divergent, then the dots in (21.19) will not contribute and we have

$$I_4 = i k^\mu \int d^3 S_\mu F(p) \Big|_{p \rightarrow \infty} = i k^\mu \left(\frac{p_\mu}{p} \right) S_3(p) F(p) \Big|_{p \rightarrow \infty} . \quad (21.20)$$

In the first equality above, we used Gauss law in euclidean 4D. So $S_3(p) = 2\pi^2 p^3$ in the next equality is the three dimensional sphere. Thus, upon return to Minkowski, we obtain

$$I_4 = \lim_{p \rightarrow \infty} i k^\mu p_\mu 2\pi^2 p^2 F(p) . \quad (21.21)$$

Going back to our case, if we define

$$F(p) = \text{Tr} \left[\gamma^\lambda \gamma_5 \frac{1}{\not{p} - \not{k}_2} \gamma^\nu \frac{1}{\not{p}} \right] , \quad (21.22)$$

and using

$$\text{Tr} \left[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \gamma_5 \right] = -4i \epsilon^{\mu\nu\alpha\beta} , \quad (21.23)$$

we arrive at

$$F(p) = 4i \epsilon^{\alpha\nu\beta\lambda} \frac{k_{2\alpha} p_\beta}{(p - k_2)^2 p^2} . \quad (21.24)$$

Using this result in (21.21) we then obtain

$$k_{1\mu} \Delta^{\lambda\mu\nu} = \lim_{p \rightarrow \infty} \frac{1}{2\pi^2} k_1^\mu p_\mu p_\beta k_{2\alpha} \epsilon^{\alpha\nu\beta\lambda} \frac{1}{(p - k_2)^2} . \quad (21.25)$$

But making the replacement

$$\frac{p_\mu p_\beta}{p^2} \rightarrow \frac{g_{\mu\beta}}{4} , \quad (21.26)$$

we obtain

$$\boxed{k_{1\mu} \Delta^{\lambda\mu\nu} = -\frac{1}{8\pi^2} \epsilon^{\lambda\nu\beta\alpha} k_{1\beta} k_{2\alpha} \neq 0} . \quad (21.27)$$

Thus, we see that it looks like the vector current is not conserved. However, there is still much freedom in our choice of integration variable shift. In fact, different shifts in $\Delta^{\lambda\mu\nu}(k_1, k_2)$ would lead to different results due to the linear divergence. To understand this freedom, let us consider the most general shift

$$p_\mu \rightarrow p_\mu + a_\mu , \quad (21.28)$$

where a_μ does not depend on the integration variable p_μ , although it depends on the independent external momenta $k_{1\mu}$ and $k_{2\mu}$. Then, we write

$$\begin{aligned} i \Delta^{\lambda\mu\nu}(a, k_1, k_2) &= (-1) \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\gamma^\lambda \gamma_5 \frac{i}{\not{p} + \not{a} - \not{q}} \gamma^\nu \frac{i}{\not{p} + \not{a} - \not{k}_1} \gamma^\mu \frac{i}{\not{p} + \not{a}} \right] \\ &+ (-1) \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\gamma^\lambda \gamma_5 \frac{i}{\not{p} + \not{a} - \not{q}} \gamma^\mu \frac{i}{\not{p} + \not{a} - \not{k}_2} \gamma^\nu \frac{i}{\not{p} + \not{a}} \right] \end{aligned} \quad (21.29)$$

In particular, we are interested in the difference

$$\Delta^{\lambda\mu\nu}(a, k_1, k_2) - \Delta^{\lambda\mu\nu}(k_1, k_2) \equiv I_a, \quad (21.30)$$

and to compute it we start with the definition of

$$G(p) \equiv \text{Tr} \left[\gamma^\lambda \gamma_5 \frac{1}{\not{p} - \not{q}} \gamma^\nu \frac{1}{\not{p} - \not{k}_1} \gamma^\mu \frac{1}{\not{p}} \right], \quad (21.31)$$

so that, already in the euclidean, we can write

$$I_a = i \int \frac{d^4 p}{(2\pi)^4} \left(G(p + a) - G(p) \right) + (\mu, k_1) \rightarrow (\nu, k_2), f \quad (21.32)$$

where the last term accounts for the second diagrams shifted and not shifted. From (21.31) we obtain

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$$\begin{aligned} \lim_{p \rightarrow \infty} G(p) &= \frac{1}{p^6} \text{Tr} \left[\gamma^\lambda \gamma_5 \not{p} \gamma^\nu \not{p} \gamma^\mu \not{p} \right] \\ &= \frac{1}{p^6} \left\{ 2p^\mu \text{Tr} \left[\gamma^\lambda \gamma_5 \not{p} \gamma^\nu \not{p} \right] - \text{Tr} \left[\gamma^\lambda \gamma_5 \not{p} \gamma^\nu \gamma^\mu \right] p^2 \right\}, \end{aligned} \quad (21.33)$$

where to get from the first to the second line we used the Dirac algebra to write $\not{p} \gamma^\mu = 2p^\mu - \gamma^\mu \not{p}$. The first term in the second line in (21.33) vanishes and we obtain

$$\lim_{p \rightarrow \infty} G(p) = -\frac{4}{p^4} p_\alpha \epsilon^{\alpha\nu\mu\lambda} . \quad (21.34)$$

Following the same procedure that resulted in (21.21), we obtain

$$I_a = \lim_{p \rightarrow \infty} a^\sigma \frac{p_\sigma}{p} \frac{2\pi^2 p^3}{(2\pi)^4} G(p) . \quad (21.35)$$

Putting it all together and using once again (21.26) we obtain

$$\Delta^{\lambda\mu\nu}(a, k_1, k_2) - \Delta^{\lambda\mu\nu}(k_1, k_2) = -\frac{1}{8\pi^2} a_\alpha \epsilon^{\alpha\nu\mu\lambda} + (\mu, k_1) \rightarrow (\nu, k_2) , \quad (21.36)$$

where we should keep in mind that a^σ would change when switching $k_1 \leftrightarrow k_2$ in the second term. In fact, we can write the most general a as a linear combination of k_1 and k_2 as in

$$a = b_1 (k_1 + k_2) + b_2 (k_1 - k_2) , \quad (21.37)$$

where the coefficients b_1 and b_2 are to be chosen so as to preserve the symmetries we want to remain in the presence of quantum corrections. Substituting in (21.36) we obtain

$$\begin{aligned} \Delta^{\lambda\mu\nu}(a, k_1, k_2) - \Delta^{\lambda\mu\nu}(k_1, k_2) &= -\frac{1}{8\pi^2} \epsilon^{\alpha\nu\mu\lambda} \left[b_1 (k_1 + k_2)_\alpha + b_2 (k_1 - k_2)_\alpha \right] \\ &\quad - \frac{1}{8\pi^2} \epsilon^{\alpha\nu\mu\lambda} \left[b_1 (k_1 + k_2)_\alpha - b_2 (k_1 - k_2)_\alpha \right] \\ &= -\frac{1}{4\pi^2} b_2 \epsilon^{\alpha\nu\mu\lambda} (k_1 - k_2)_\alpha , \end{aligned} \quad (21.38)$$

We are now in a position to impose vector currents conservation, i.e. to impose that

$$k_{1\mu} \Delta^{\lambda\mu\nu}(a, k_1, k_2) = 0, \quad k_{2\nu} \Delta^{\lambda\mu\nu}(a, k_1, k_2) = 0 , \quad (21.39)$$

For instance, if we impose the first equality in (21.39) and use our result in (21.38) we obtain

$$\begin{aligned} k_{1\mu} \Delta^{\lambda\mu\nu}(a, k_1, k_2) &= k_{1\mu} \Delta^{\lambda\mu\nu}(k_1, k_2) + \frac{1}{4\pi^2} b_2 \epsilon^{\alpha\nu\mu\lambda} k_{1\mu} k_{2\alpha} \\ &= -\frac{1}{8\pi^2} \epsilon^{\lambda\nu\beta\alpha} k_{1\beta} k_{2\alpha} + \frac{1}{4\pi^2} b_2 \epsilon^{\alpha\nu\mu\lambda} k_{1\mu} k_{2\alpha} , \end{aligned} \quad (21.40)$$

where in the second line we made use of (21.27). Noticing that

$$\epsilon^{\alpha\nu\mu\lambda} = -\epsilon^{\lambda\nu\beta\alpha} , \quad (21.41)$$

we conclude that for (21.40) to vanish, therefore enforcing vector current conservation, we need

$$\boxed{b_2 = -\frac{1}{2}} . \quad (21.42)$$

It is important to notice that b_1 can take any value: it does not enter into the enforcing of vector current conservation. Finally, we examine axial current conservation by computing

$$q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2) = q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) - \frac{1}{4\pi^2} b_2 q_\lambda \epsilon^{\alpha\nu\mu\lambda} (k_1 - k_2)_\alpha , \quad (21.43)$$

where we already took into account that since $q = k_1 + k_2$ once again b_1 does not enter here due to the antisymmetry of the epsilon tensor. To obtain a result from (21.43) we need to compute the first term. The calculation is completely analogous to the one performed to obtain (21.27) and we leave it as an exercise. The result is

$$q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) = -\frac{1}{4\pi^2} \epsilon^{\mu\nu\beta\alpha} k_{1\beta} k_{2\alpha} . \quad (21.44)$$

Substituting this result into (21.43) and using (21.42) for b_2 , we arrive at the final result for the axial current divergence in momentum space:

$$\boxed{q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2) = -\frac{1}{2\pi^2} \epsilon^{\mu\nu\lambda\alpha} k_{1\lambda} k_{2\alpha}} . \quad (21.45)$$

The result above means that the axial current is not conserved due to the non vanishing contraction of the triangle diagram with the momentum carried by the current. At first this appears surprising: we introduced the most general shift in the integration, a_μ , whose dependence on $k_{1\mu}$ and $k_{2\mu}$ introduces *two* arbitrary parameters, b_1 and b_2 . So in principle, one could think that we would have enough freedom to enforce the conservation of both the vector and the axial currents. However, as we saw above, the parameter b_1 does not enter neither in the vector nor in the axial current conservation conditions. Only b_2 does. This means that we can only *save one of the currents*. Here we chose to impose vector current conservation, since it is generally associated with gauge and even global symmetries we hold dear. This means that the axial current is inevitably anomalous. The reason for the disappearance of b_1 from the current conservation conditions is the presence

of the epsilon tensor which in turn appears as a consequence of the presence of the γ_5 in one of the vertices.

We can recast this result in terms of the axial current itself by going to position space. We can write

$$\partial_\lambda J_A^\lambda = (-iq_\lambda) \left(i\Delta^{\lambda\mu\nu}(a, k_1, k_2) \right) \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) (-ig)^2, \quad (21.46)$$

where we $\epsilon(k_1)$ and $\epsilon(k_2)$ are the polarizations of the two external vector gauge bosons, and we included a factor of the vector gauge coupling, $-ig$, for each of the two vector vertices. Then, using (21.45) we write

$$\partial_\lambda J_A^\lambda = \frac{g^2}{2\pi^2} \epsilon^{\mu\nu\lambda\alpha} k_{1\lambda} \epsilon_\mu^*(k_1) k_{2\alpha} \epsilon_\nu^*(k_2), \quad (21.47)$$

which can be rewritten as

$$\partial_\lambda J_A^\lambda = \frac{g^2}{16\pi^2} \epsilon^{\mu\nu\lambda\alpha} F_{\lambda\mu} F_{\alpha\nu}. \quad (21.48)$$

The factor of 8 between the last two expressions above can be obtained by noticing that $F_{\lambda\mu}$ actually generates two identical terms with $\lambda \leftrightarrow \nu$. Similarly for $F_{\alpha\nu}$. The final factor of 2 comes from the interchange of the momenta associated with each of the F 's. I.e. the expression (21.48) generates two diagrams with the momenta exchanged. Although we obtained the result above from computing a loop diagram it is possible to understand the origin of the axial anomaly in a more profound way. We will start doing this in the next lecture. But from this one, it is clear that there is a problem with axial transformations once quantum corrections are taken into account. Thus, it appears that the origin of the problem comes from the ultraviolet structure of the theory. We will explore this point in what follows.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Sections 19.1 and 19.2.
- *The Quantum Theory of Fields, Vol. II*, by S. Weinberg, Section 22.3.
- *Dynamics of the Standard Model*, J. F. Donoghue, E. Golowich and B. Holstein, Section III-3.