

# Chapter 6

## Review

Let us review what we have seen so far before we proceed.

### 6.1 Electromagnetism

Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (\text{Gauss Law}) \quad (6.1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Nonexistence of Magnetic Monopoles}) \quad (6.2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{Faraday induction Law}) \quad (6.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (\text{Ampere Law}) \quad (6.4)$$

naturally imply charge conservation (divergence of Ampere's Law):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (6.5)$$

We may define electromagnetic potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad (6.6)$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (6.7)$$

which under gauge transformations

$$\phi' = \phi - \frac{\partial f}{\partial t} \quad (6.8)$$

$$\mathbf{A}' = \mathbf{A} + \nabla f \quad (6.9)$$

produce the same electromagnetic fields

$$\mathbf{E}' = \mathbf{E} \quad (6.10)$$

$$\mathbf{B}' = \mathbf{B} \quad (6.11)$$

The Lorenz gauge

$$\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0 \quad (\text{Lorenz Gauge}) \quad (6.12)$$

is particularly useful for electromagnetic waves. In fact, inserting the potentials in the Maxwell Eqs. and imposing the Lorenz gauge, we obtain

$$\square^2 \phi = -\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = -\frac{\rho}{\epsilon_0} \quad (6.13)$$

$$\square^2 \mathbf{A} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} = -\mu_0 \mathbf{j} \quad (6.14)$$

i.e., the potentials propagate according to the classical non-homogenous wave equation with constant speed equal to the speed of light  $c^2 = 1/\mu_0 \epsilon_0$ . **Unification: E&M  $\leftrightarrow$  Optics.**

Finally, given the E&M fields, corresponding E&M forces  $\mathbf{F}$  act on particles as:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (6.15)$$

## 6.2 Special Relativity

**Postulate 1:** The laws of physics are the same in all inertial frames.

**Postulate 2:** The speed of light is the same in all inertial frames.

Postulate 2 follows from postulate 1, since E&M is a set of physical laws.

### 6.2.1 Coordinates and Metric

Contravariant coordinates

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \quad (6.16)$$

Line element  $ds$

$$ds^2 = \eta_{\mu\nu} dx^\nu dx^\mu \quad (6.17)$$

Metric  $\eta_{\mu\nu}$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.18)$$

Covariant coordinates  $x_\mu$

$$x_\mu = \eta_{\mu\nu} x^\nu = (-ct, x, y, z) \quad (6.19)$$

Similarly,

$$x^\mu = \eta^{\mu\nu} x_\nu, \quad (6.20)$$

where  $\eta^{\mu\nu}$  inverse metric. Flat space:  $\eta^{\mu\nu} = \eta_{\mu\nu}$ .

Einstein sum convention: crossed repeated indices are summed over, e.g.  $\eta^{\mu\nu} x_\nu \equiv \sum_{\nu=0}^3 \eta^{\mu\nu} x_\nu$

### 6.2.2 Invariance of the Line Element:

Under 3d spatial rotations, coordinates transform as

$$x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} x^{\nu} = \Lambda^{\mu'}_{\nu} x^{\nu} \quad (6.21)$$

with

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.22)$$

such that the 3d spatial line element

$$l^2 = x^2 + y^2 + z^2 = (x')^2 + (y')^2 + (z')^2 = l'^2 \quad (6.23)$$

is invariant.

Similarly, under a boost with velocity  $v$  in the x-direction, the Lorentz transformations with

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.24)$$

where

$$\beta = \frac{v}{c} < 1 \quad (6.25)$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} > 1 \quad (6.26)$$

leave the 4-d line element  $s^2 = -c^2 t^2 + x^2 + y^2 + z^2$  invariant.

### 6.2.3 Time Dilation and Space Contraction

As a result, we have time dilation:

$$\Delta t' = \Delta t / \gamma \quad (6.27)$$

and space contraction

$$\Delta x' = \gamma \Delta x \quad (6.28)$$

### 6.2.4 Tensors

Tensors defined according to their Lorentz transformations:

$$T'^{\mu\nu} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} T^{\alpha\beta} \quad (6.29)$$

scalar: tensor of rank 0 (invariant), vector: rank 1, matrix: rank 2, etc...

Example: 4-velocity  $U^\mu$ :

$$U^\mu = \frac{dx^\mu}{d\tau} = \left( \frac{dx^0}{d\tau}, \frac{dx^i}{d\tau} \right) = \left( \frac{cdt}{d\tau}, \gamma \frac{dx^i}{dt} \right) = (\gamma c, \gamma \mathbf{v}) = \gamma(c, \mathbf{v}) \quad (6.30)$$

4-momentum (massive particles):

$$P^\mu \equiv mU^\mu = (\gamma mc, \gamma m\mathbf{v}) \equiv \left( \frac{E}{c}, \mathbf{p} \right) \quad \text{Momentum (massive particles)} \quad (6.31)$$

Classical limit ( $v \ll c$  we have  $\gamma = (1 - \beta^2)^{-1/2} \approx 1 + \beta^2/2 + O(\beta^4)$ ):

$$E = \gamma mc^2 \approx mc^2 + \frac{1}{2}mv^2 + O(\beta^4) \quad (6.32)$$

$$\mathbf{p} = \gamma m\mathbf{v} \approx m\mathbf{v} + O(\beta^3) \quad (6.33)$$

More generally, for massive and massless particles:

$$P^\mu = \frac{dx^\mu}{d\lambda} \equiv \left( \frac{E}{c}, \mathbf{p} \right) \quad \text{Momentum (massive and massless particles)} \quad (6.34)$$

where  $\lambda$  parametrizes the trajectory. Massive particles:  $\lambda = \tau/m$ . Massless particles:  $\tau = m = 0$ , so choose something else or replace  $\lambda \rightarrow t$ . Finally

$$P^\mu P_\mu = - \left( \frac{E}{c} \right)^2 + p^2 = -m^2 c^2 \quad \rightarrow \quad E^2 = (pc)^2 + (mc^2)^2 \quad (6.35)$$

### 6.2.5 Doppler Effect

Applying the Lorentz transformations to  $P^\mu = (E/c, \mathbf{p})$  for a photon, we have

$$E'_\gamma = \sqrt{\frac{1-\beta}{1+\beta}} E_\gamma \quad (6.36)$$

and since  $E_\gamma = h\nu$ :

$$\nu' = \sqrt{\frac{1-\beta}{1+\beta}} \nu \quad (6.37)$$

or

$$\lambda' = \sqrt{\frac{1+\beta}{1-\beta}} \lambda \quad \text{Doppler Redshift} \quad (6.38)$$

The redshift  $z$  is defined as

$$z = \frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \approx \sqrt{(1+\beta)^2} - 1 = \frac{v}{c} \quad (6.39)$$

### 6.2.6 Covariant Formulation

Finally, one can show that the electromagnetic equations can be written in terms of tensors in a covariant form. Defining:

$$j^\mu = (c\rho, \mathbf{j}) \quad (6.40)$$

$$A^\alpha = (\phi/c, \mathbf{A}) \quad (6.41)$$

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \quad (6.42)$$

$$f^\mu = qF^{\mu\nu}U_\nu \quad (6.43)$$

we have charge conservation:

$$\frac{\partial j^\mu}{\partial x^\mu} = 0, \quad (6.44)$$

Wave equation:

$$\square A^\alpha = -\mu_0 j^\alpha, \quad (6.45)$$

Gauge transformation:

$$A'^\alpha = A^\alpha + \frac{\partial f}{\partial x_\alpha} \quad (6.46)$$

Lorenz gauge:

$$\frac{\partial A^\alpha}{\partial x^\alpha} = 0 \quad (6.47)$$

Maxwell's equations:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 j^\mu \quad (6.48)$$

$$\frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} = 0 \quad (6.49)$$

and Lorentz force:

$$f^\mu = qF^{\mu\nu}U_\nu \quad (6.50)$$

### 6.2.7 Energy-Momentum Tensor

The energy-momentum tensor  $T^{\mu\nu}$  is generally defined as

$T^{\mu\nu}$  = "flux of  $P^\mu$  across surface of constant  $x^\nu$ " =  $P^\mu$  per surface  $\perp$  to  $x^\nu$ .

e.g.

$T^{00}$ : density of  $P^0 = E$  : energy density

$T^{ii}$ : flux of  $P^i$  in the  $x^i$  direction : force  $f^i$  per area  $\perp$  to  $x^i$  = pressure

For a perfect fluid:

$$T^{\alpha\beta} = (\rho + P)U^\alpha U^\beta + P\eta^{\alpha\beta} \quad (6.51)$$

## 6.3 General Relativity

### 6.3.1 Equivalence Principle

**Locally inertial frames:** freely-falling frames in small enough regions for which special relativity holds locally.

**Weak Equivalence Principle (WEP):** "In small enough regions of space-time, the *motion* of freely-falling particles is the same in a uniform gravitational field and in a uniformly accelerated frame, i.e. the *laws of Mechanics* take the same form as in an unaccelerated frame in the absence of gravitation. As a result, at every point of space-time in an arbitrary gravitational field, it is possible to choose a "locally inertial frame" such that in small enough regions the *laws of Mechanics* reduce to those of special relativity."

**Strong Equivalence Principle (SEP):** Replace *laws of Mechanics* by *laws of Physics* above.

### 6.3.2 Geodesics

K' frame: freely-falling coordinates  $\xi^\alpha$ ,

K frame: coordinates  $x^\beta$ .

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \quad (6.52)$$

Change  $\xi^\alpha \rightarrow x^\beta$ :

$$\frac{d^2x^\gamma}{d\tau^2} + \Gamma_{\mu\nu}^\gamma \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \quad (6.53)$$

where the *affine connection*  $\Gamma_{\mu\nu}^\gamma$

$$\Gamma_{\mu\nu}^\gamma = \frac{\partial x^\gamma}{\partial \xi^\beta} \frac{\partial^2 \xi^\beta}{\partial x^\mu \partial x^\nu} \quad (6.54)$$

Similarly, the *metric tensor*  $g_{\mu\nu}$  in coordinates  $x^\mu$ :

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} \quad (6.55)$$

### 6.3.3 Metric and Connection

Differentiating Eq. 1.163, changing indices and adding:

$$\Gamma_{\mu\lambda}^\sigma = \frac{g^{\sigma\nu}}{2} \left( \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} \right) \quad (6.56)$$

One can show that in the Newtonian limit with

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}(x), \quad \text{with } h_{\alpha\beta}(x) \ll \eta_{\alpha\beta} \quad (6.57)$$

the geodesics equation gives

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{c^2}{2} \nabla h_{00} = -\nabla\phi \quad (6.58)$$

and with appropriate boundary conditions

$$g_{00} = -(1 + 2\phi) \quad (6.59)$$

### 6.3.4 Time Dilation and Gravitational Redshift

Therefore, the ratio of times between 1 and 2 is

$$\frac{dt_2}{dt_1} = \left( \frac{g_{00}(x_2)}{g_{00}(x_1)} \right)^{-1/2} \quad (6.60)$$

i.e. the ratio of frequencies  $\nu \propto 1/dt$  will be

$$\frac{\nu_2}{\nu_1} = \left( \frac{g_{00}(x_2)}{g_{00}(x_1)} \right)^{1/2} \quad (6.61)$$

Weak field regime:  $g_{00} = -(1 + 2\phi)$  and

$$\frac{\delta\nu}{\nu_1} = \frac{\nu_2 - \nu_1}{\nu_1} \approx \phi(x_2) - \phi(x_1) \quad (6.62)$$

### 6.3.5 General Covariance

**Equivalence Principle:** Gravitational effects can be obtained by writing equations for general gravitational fields in a locally inertial frame where gravitational effects disappear (e.g.  $d\xi^2/d\tau^2 = 0$ ) and transforming to the Laboratory coordinates to find the equation in the Lab. frame.

Principle of General Covariance: alternative to the Equivalence Principle (same physical content).

**Principle of General Covariance:** A physical equation holds in general gravitational fields (i.e. in general relativity) if:

a) the equation holds in the absence of gravitation; i.e. it agrees with special relativity when  $g_{\mu\nu} = \eta_{\mu\nu}$  and  $\Gamma^\alpha_{\mu\nu} = 0$ .

b) the equation is *generally* covariant, i.e. it preserves its form under a *general* coordinate transformation.

### Volume Element

Define the determinant of the metric:

$$g = \text{Det } g_{\mu\nu} \quad (6.63)$$

from which we can show that

$$\sqrt{-g'} d^4x' = \sqrt{-g} d^4x \quad (6.64)$$

i.e.  $\sqrt{-g} d^4x$  is an *invariant* (scalar) volume element.

### 6.3.6 Transformation of the Affine Connection

The affine connection was defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \quad (6.65)$$

and is not a tensor as it transforms as

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} - \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma} \quad (6.66)$$

### 6.3.7 Covariant Differentiation

For a contravariant vector:

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}, \quad (6.67)$$

and its derivative is

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}. \quad (6.68)$$

Combining the transformations for  $\Gamma_{\mu\nu}^{\lambda}$  and  $V^{\nu}$  we have

$$\Gamma_{\lambda\kappa}^{\mu} V'^{\kappa} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \Gamma_{\rho\sigma}^{\nu} V^{\sigma} - \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\sigma}}_{\frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}} \quad (6.69)$$

Adding the two equations above, the inhomogeneous terms cancel out and we get

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} + \Gamma_{\lambda\kappa}^{\mu} V'^{\kappa} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left( \frac{\partial V^{\nu}}{\partial x^{\rho}} + \Gamma_{\rho\sigma}^{\nu} V^{\sigma} \right) \quad (6.70)$$

The combination in brackets is the *covariant derivative*, which transforms as a *tensor*:

$$\nabla_{\lambda} V^{\mu} = V^{\mu}{}_{;\lambda} = \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma_{\lambda\kappa}^{\mu} V^{\kappa} \quad (6.71)$$

Extended to a general tensor:

$$T^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial T^{\mu\sigma}}{\partial x^{\rho}} \lambda + \Gamma_{\rho\nu}^{\mu} T^{\nu\sigma}{}_{\lambda} + \Gamma_{\rho\nu}^{\sigma} T^{\mu\nu}{}_{\lambda} - \Gamma_{\lambda\rho}^{\kappa} T^{\mu\sigma}{}_{\kappa} \quad (6.72)$$

The covariant derivative of the metric is zero, as can be checked, using Eq. 1.172:

$$g_{\mu\nu}{}_{;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \Gamma_{\lambda\mu}^{\rho} g_{\rho\nu} - \Gamma_{\lambda\nu}^{\rho} g_{\mu\rho} = 0 \quad (6.73)$$

Importance of covariant derivatives for forming covariant equations:

- 1) They transform tensors into tensors, i.e. if  $A^{\mu\nu}$  is a tensor, so is  $\nabla_{\lambda} A^{\mu\nu}$ .
- 2) They reduce to ordinary derivatives in the absence of gravity (when  $g_{\mu\nu} = \eta_{\mu\nu}$  and  $\Gamma_{\mu\nu}^{\lambda} = 0$ ).

Therefore, the principle of general covariance allows us to apply the following algorithm to obtain equations that are generally covariant and true in the presence of gravity:

- a) Write the equation in special relativity (which holds in the absence of gravitation)
- b) Replace  $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$
- c) Replace  $\partial/\partial x^{\mu} \rightarrow \nabla_{\mu}$ .



## 6.4 Curvature

The connection is not a tensor, but the combination defined as the *Riemann curvature tensor*

$$R^\lambda{}_{\mu\nu\kappa} = \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta} \quad (\text{Riemann Tensor}) \quad (6.74)$$

is indeed a tensor:

$$R'^\tau{}_{\rho\sigma\eta} = \frac{\partial x'^\tau}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} \frac{\partial x^\kappa}{\partial x'^\eta} R^\lambda{}_{\mu\nu\kappa} \quad (6.75)$$

Tensors of lower rank by contracting the Riemann Tensor. Ricci tensor:

$$R_{\mu\nu} = g^{\lambda\kappa} R_{\lambda\mu\kappa\nu} = R^\kappa{}_{\mu\kappa\nu} \quad (\text{Ricci Tensor}) \quad (6.76)$$

Ricci scalar:

$$R = g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu \quad (\text{Ricci Scalar}) \quad (6.77)$$

It can also be shown that these are the only tensor and scalar that can be formed from the Riemann tensor and the metric.

### 6.4.1 Commutation of Covariant Derivatives

Covariant derivative to a covariant vector  $V_\mu$  twice in reverse order leads to

$$V_{\mu;\nu;\kappa} - V_{\mu;\kappa;\nu} = -R^\sigma{}_{\mu\nu\kappa} V_\sigma \quad (6.78)$$

Therefore, if the Riemann tensor vanishes, covariant derivatives commute (as they should in flat space). For a space-time with curvature, covariant derivatives do not commute.

One can show a number of properties of the Riemann Tensor, these lead to the Bianchi Identities, which imply:

$$(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\mu} = 0 \quad (6.79)$$

## 6.5 Einstein Equations

Finally, imposing that the gravitational field equations must satisfy certain conditions, such as being tensorial, containing at most 2 derivatives of the metric, being consistent with the Bianchi identities, and reducing to Newtonian gravity in the appropriate limit, one finds that

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (\text{Einstein Equations}) \quad (6.80)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad (6.81)$$

This result can also be obtained by the Einstein-Hilbert action:

$$S_{\text{EH,vac}} = \int d^4x \sqrt{-g} R. \quad (6.82)$$

if we require this action to be stationary under variations with respect to the metric  $g^{\mu\nu}$ .

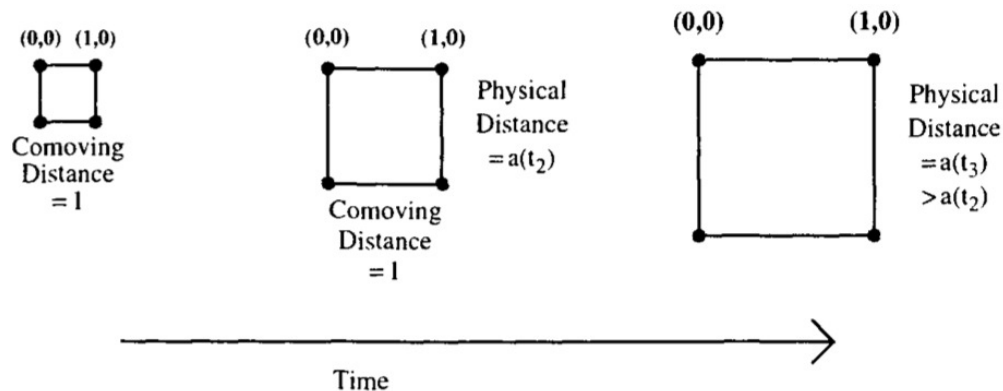


Figure 6.1: Scale factor and expansion. Comoving coordinates do not change, but physical coordinates expand with the scale factor  $a(t)$ . (Dodelson).

## 6.6 Expansion of the Universe

**Cosmological Principle:** Assumption that the Universe is homogeneous (same at every point, therefore symmetric under translations) and isotropic (same in all directions, therefore symmetric under rotations).

Expanding universe: useful to define *comoving coordinates*  $\mathbf{x}$ : do not change with the expansion, parametrized in terms of the scale factor  $a(t)$  (see Fig. 6.1).

Then physical distances  $r$  change with change such that

$$\text{physical distance} = a(t) \times \text{comoving distance.} \quad (6.83)$$

or

$$\mathbf{r}(t) = a(t)\mathbf{x} \quad (6.84)$$

## 6.7 The Friedmann-Robertson-Walker metric

Generalizes Minkowski metric to include expansion on the spatial hypersurfaces, maintaining spatial isotropy and homogeneity. *Flat* Universe it is given by

$$ds^2 = -dt^2 + a^2(t)dl^2 \quad (6.85)$$

where

$$dl^2 = dx^2 + dy^2 + dz^2 = dD^2 + D^2d\alpha^2 \quad (6.86)$$

and

$$d\alpha^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad (6.87)$$

For universes with curvature  $k$ , generalize

$$dl^2 = R^2 [dD^2 + f_k^2(D)d\alpha^2] \quad (6.88)$$

$$= R^2 \left[ \frac{dD_A^2}{1 - kD_A^2} + D_A^2 d\alpha^2 \right] \quad (3d \text{ curved space}) \quad (6.89)$$

such that:

$$D_A = f_k(D) = \frac{\sin(\sqrt{k}D)}{\sqrt{k}} = \begin{cases} \sinh(D), & k = -1, & \text{Negative Curvature,} & \text{Open Universe} \\ D, & k = 0, & \text{Zero Curvature,} & \text{Flat Universe} \\ \sin(D), & k = +1, & \text{Positive Curvature,} & \text{Closed Universe} \end{cases} \quad (6.90)$$

## 6.8 The Friedmann Equations

(FRW metric + Einstein Equations)  $\rightarrow$  Friedmann Equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \quad (6.91)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \quad (6.92)$$

with curvature, generalizes to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (6.93)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \quad (6.94)$$

In a universe with no curvature, the density is called critical

$$\rho_{\text{crit}}(t) = \frac{3H^2(t)}{8\pi G} \quad (6.95)$$

Define the density parameter

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_{\text{crit}}(t)} \quad (6.96)$$

and the Friedmann equation becomes

$$E^2(t) = \frac{H^2(t)}{H_0^2} = [\Omega_k a^{-2} + \Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda] \quad (6.97)$$

where

$$\Omega_k = -k/H_0^2 = 1 - (\Omega_m + \Omega_r + \Omega_\Lambda) \quad (6.98)$$

For a Universe with both matter and cosmological constant, we have

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3} \left(\frac{3\sqrt{\Omega_\Lambda} H_0 t}{2}\right) \quad (\text{Matter + Cosmological Constant}) \quad (6.99)$$

In the context of an expanding universe, the gravitational (dynamical) redshift is due to the stretch of space-time itself and relates to the scale factor

$$1 + z = \frac{1}{a} \quad (6.100)$$

## 6.9 Cosmological Distances

### 6.9.1 Comoving Radial Distance

The comoving radial distance  $D$  can be obtained by considering the a radial path of photons, in which we have  $d\alpha^2 = 0$  (radial) and  $ds^2 = -dt^2 + a^2(t)dD^2 = 0$  (photons), so that  $D$  can be expressed as

$$\begin{aligned} D &= \int dD = \int_t^{\text{age}} \frac{dt}{a(t)} = \int_a^1 \frac{da}{\dot{a}a} = - \int_z^0 \frac{dz}{H(z)} \\ &= \int_0^z \frac{dz}{H(z)} \end{aligned} \quad (6.100)$$

where we used  $da = -a^2 dz$  and  $H(z) = \dot{a}/a$ . Notice that  $D$  depends on the curvature only via the Hubble parameter from the Friedmann's equations. We may also define a *physical* radial distance  $d_p = a(t)D$ .

### 6.9.2 Comoving Horizon

The comoving horizon  $D_H$  is similar to  $D$ , but instead of integrating from  $z = 0$  to a certain redshift  $z$ , we integrate from  $z$  to  $z = \infty$ , effectivelly finding the comoving size of the universe at  $z$ :

$$D_H = \int_0^t \frac{dt}{a(t)} = \int_0^a \frac{da}{\dot{a}a} = \int_z^\infty \frac{dz}{H(z)} \quad (6.101)$$

We may also define a *physical* horizon  $d_H = a(t)D_H$ .

### 6.9.3 Angular Diameter Distance

The comoving angular diameter distance  $D_A$  is defined such that it gives an object's comoving size  $dl$  when it is multiplied by the object angular size  $d\alpha$

$$dl = D_A d\alpha \quad (6.102)$$

From the metric definition, with  $dD = 0$  we can see that it is given in terms of  $D$  by

$$D_A = f_k(D) = \frac{\sin(\sqrt{k}D)}{\sqrt{k}} = \begin{cases} \sinh(D), & k = -1, & \text{Negative Curvature, Open Universe} \\ D, & k = 0, & \text{Zero Curvature, Flat Universe} \\ \sin(D), & k = +1, & \text{Positive Curvature, Closed Universe} \end{cases} \quad (6.103)$$

or similarly, with  $k = -H_0^2 \Omega_k$ :

$$D_A = f_k(D) = \frac{\sin[\sqrt{-\Omega_k}H_0D]}{\sqrt{-\Omega_k}H_0} = \begin{cases} \frac{\sinh[\sqrt{\Omega_k}H_0D]}{\sqrt{\Omega_k}H_0D}, & \Omega_k > 0, & \text{Negative Curvature, Open Universe} \\ D, & \Omega_k = 0, & \text{Zero Curvature, Flat Universe} \\ \frac{\sin[\sqrt{-\Omega_k}H_0D]}{\sqrt{-\Omega_k}H_0D}, & \Omega_k < 0, & \text{Positive Curvature, Closed Universe} \end{cases} \quad (6.104)$$

### 6.9.4 Luminosity Distance

The *physical* luminosity distance  $d_L$  is defined such that the Euclidean relation remains valid for the comoving flux, i.e.

$$F = \frac{L}{4\pi d_L^2} \quad (6.105)$$

and comparing with the previous equation, we conclude that

$$d_L = \frac{D_A}{a} = \frac{d_A}{a^2} \quad (6.106)$$

In the case of a flat universe we have

$$d_L = \frac{D}{a} = \frac{d}{a^2} \quad (\text{Flat}) \quad (6.107)$$

In any case, the relation  $a^2 d_L = d_A$  is always true for FRW cosmologies, independent of curvature and/or cosmology. It provides a consistency check for the homogeneity and isotropy of the Universe.

Finally, the *comoving* luminosity distance is

$$D_L = \frac{d_L}{a} = \frac{D_A}{a^2} = \frac{f_k(D)}{a^2} \quad (6.108)$$

### 6.9.5 Comoving Volume

the comoving volume element in spherical coordinates is given by

$$dV(z) = (D_A d\theta)(D_A \sin \theta d\phi) dD = \frac{D_A^2(z)}{H(z)} dz d\Omega, \quad (6.108)$$

### 6.9.6 Comoving versus Physical

physical and comoving version. The physical distance  $d$  is always obtained by multiplying the comoving distance  $D$  by the scale factor  $a(t)$ . This holds also for the luminosity and angular-diameter distances such that:

$$d_p = a(t)D \quad (6.109)$$

$$d_H = a(t)D_H \quad (6.110)$$

$$d_A = a(t)D_A \quad (6.111)$$

$$d_L = a(t)D_L \quad (6.112)$$

and the physical volume is

$$dV_{\text{phys}} = (d_A d\theta)(d_A \sin \theta d\phi) d(d_p) = a^3(t) \frac{D_A^2(z)}{H(z)} dz d\Omega = a^3(t) dV \quad (6.113)$$

## 6.10 Energy Evolution

The Bianchi identity says that the covariant derivative of the Einstein Tensor is zero:

$$\nabla_\mu G^{\mu\nu} = 0 \quad (6.114)$$

which, through the Einstein equations, automatically imply that the Energy-Momentum tensor is covariantly conserved:

$$\nabla_\mu T^{\mu\nu} = 0 \quad (6.115)$$

the  $\nu = 0$  equation implies ( $T^{00} = g^{00}T^0_0 = \rho$  and  $T^{ij} = g^{ik}T^i_k = -\delta_{ik}/a^2(-\delta_{ik}P) = \delta_{ij}P/a^2$ ):

$$\begin{aligned} \nabla_\mu T^{\mu 0} &= \partial_\mu T^{\mu 0} + \Gamma_{\mu\lambda}^\mu T^{\lambda 0} + \Gamma_{\mu\lambda}^0 T^{\mu\lambda} \\ &= \partial_0 T^{00} + \Gamma_{0\lambda}^0 T^{\lambda 0} + \Gamma_{i\lambda}^i T^{\lambda 0} + \Gamma_{0\lambda}^0 T^{0\lambda} + \Gamma_{i\lambda}^0 T^{i\lambda} \\ &= \partial_0 T^{00} + \Gamma_{i\lambda}^i T^{\lambda 0} + \Gamma_{i\lambda}^0 T^{i\lambda} \\ &= \partial_0 T^{00} + \Gamma_{i0}^i T^{00} + \Gamma_{ij}^0 T^{ij} \\ &= \partial_0 \rho + \delta_{ii} \frac{\dot{a}}{a} \rho + (\delta_{ij} a \dot{a}) \left( \frac{\delta_{ij} P}{a^2} \right) \\ &= \frac{\partial \rho}{\partial t} + 3 \frac{\dot{a}}{a} \rho + 3 \frac{\dot{a}}{a} P = 0 \end{aligned} \quad (6.111)$$

or with  $P = w\rho$ :

$$\boxed{\frac{\partial \rho}{\partial t} + 3H(1+w)\rho = 0} \quad (6.112)$$

the general solution to this equation as

$$\begin{aligned} \frac{d\rho}{dt} &= -3 \frac{da/dt}{a} \rho [1 + w(t)] \\ \frac{d\rho}{\rho} &= -3 [1 + w(t)] \frac{da}{a} \\ d \ln \rho &= -3 [1 + w(t)] d \ln a \\ \ln \rho &= -3 \int [1 + w(a)] d \ln a + \text{const.} \\ \rho(a) &= \rho(1) \exp \left[ -3 \int_1^a \frac{(1 + w(a))}{a} da \right] \end{aligned} \quad (6.109)$$

In terms of redshift  $z$ ,  $a = (1+z)^{-1}$ ,  $da = -(1+z)^{-2} dz$ , so that  $da/a = -dz/(1+z)$  and:

$$\rho(z) = \rho(0) \exp \left[ 3 \int_0^z \frac{[1 + w(z)]}{1+z} dz \right] \quad (6.110)$$

**Solutions for constant  $w$** 

We can find solutions for cases when the universe content is dominated by different species with constant  $w$ :

$$\rho(z) = \rho(0) \exp \left[ 3(1+w) \int_0^z \frac{dz}{1+z} \right] = \rho(0) \exp [3(1+w) \ln(1+z)] \quad (6.111)$$

or

$$\rho(z) = \rho(0)(1+z)^{3(1+w)} \quad (6.112)$$

**6.11 Equilibrium Thermodynamics**

distribution function  $f(\mathbf{x}, \mathbf{p}, t)$  of a species in phase space  $(\mathbf{x}, \mathbf{p})$  and time  $t$ , defined such that

$$N = f(\mathbf{x}, \mathbf{p}, t) d^3x d^3p \quad (6.113)$$

is the number of particles in phase space element  $d^3x d^3p$ .

In thermodynamical equilibrium, the distribution function is independent of position angular direction, and given by

$$f(\mathbf{x}, \mathbf{p}, t) = f(p, t) = \frac{1}{e^{(E-\mu)/T} \pm 1} \begin{cases} + & \text{Fermi-Dirac} \\ - & \text{Bose-Einstein} \end{cases} \quad (6.114)$$

where  $E = \sqrt{p^2 + m^2}$ , and both cases reduce to the Maxwell-Boltzmann distribution in the classical limit (high temperatures and low densities):

$$f(p, t) \propto e^{-(E-\mu)/T} \quad \text{Classical} \quad (6.115)$$

number density, energy density and pressure, respectively:

$$n(\mathbf{x}, t) = g \int \frac{d^3p}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t) \quad (6.116)$$

$$\rho(\mathbf{x}, t) = g \int \frac{d^3p}{(2\pi)^3} E f(\mathbf{x}, \mathbf{p}, t) \quad (6.117)$$

$$P(\mathbf{x}, t) = g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E} f(\mathbf{x}, \mathbf{p}, t) \quad (6.118)$$

The Boltzmann equation then implies

$$T \propto \frac{1}{a} \quad (6.119)$$

**6.12 Boltzmann Equations**

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx_i}{dt} \frac{\partial f}{\partial x_i} + \frac{dp}{dt} \frac{\partial f}{\partial p} + \frac{d\hat{p}_i}{dt} \frac{\partial f}{\partial \hat{p}_i} = \left( \frac{\partial f}{\partial t} \right)_C \quad (6.120)$$

In equilibrium, the distribution  $f(\mathbf{x}, \mathbf{p}, t) = f_0(p, t)$  is either the BE or FD distribution, and the collision term is zero (collisions/reactions in one direction cancelled by terms in opposite direction) such that the collisionless Boltzmann is satisfied and

$$\frac{df_0}{dt} = \frac{\partial f_0}{\partial t} + \frac{dx^i}{dt} \underbrace{\frac{\partial f_0}{\partial x^i}}_0 + \frac{dp}{dt} \frac{\partial f_0}{\partial p} + \frac{d\hat{p}^i}{dt} \underbrace{\frac{\partial f_0}{\partial \hat{p}^i}}_0 = 0 \quad (6.121)$$

$$\rightarrow \frac{\partial f_0}{\partial t} + \frac{dp}{dt} \frac{\partial f_0}{\partial p} = 0 \quad (6.122)$$

For photons

$$P^2 = g_{\mu\nu} P^\mu P^\nu = 0 \rightarrow P^0 = p \quad (6.123)$$

and the Geodesics equation gives

$$\frac{dp}{dt} = -Hp \quad (6.124)$$

For matter

$$P^2 = g_{\mu\nu} P^\mu P^\nu = -m^2 \rightarrow E^2 = p^2 + m^2 \quad (6.125)$$

and the Boltzmann equation leads to

$$\rho \propto \frac{1}{a^3} \quad (6.126)$$

## 6.13 Thermal History

The Boltzmann equation may be written as

$$\frac{\partial f}{\partial t} - Hp \frac{\partial f}{\partial p} = \frac{1}{E} \left( \frac{\partial f}{\partial t} \right)_C \quad (6.127)$$

or, similarly, integrating over momentum

$$a^{-3} \frac{d(na^3)}{dt} = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{E} \left( \frac{\partial f}{\partial t} \right)_C \quad (6.128)$$

For a general process

$$1 + 2 \leftrightarrow 3 + 4 \quad (6.129)$$

we may evaluate the collision term and obtain for particle 1

$$a^{-3} \frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left[ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right] \quad (6.130)$$

In chemical equilibrium the collision term is zero and we have the Saha equation

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \quad (6.131)$$



More generally, we must solve the differential equation while only kinetic equilibrium holds.

We used this equation to study a number of processes in the early universe, namely

- \* Neutrino decoupling
- \* Freeze out of neutrons,
- \* Big Bang nucleosynthesis: formation of light element nuclei.
- \* Recombination of electrons and protons allowing the decoupling of electrons and photons
- \* Production of relic dark matter particles.

## 6.14 Linear Perturbations in the Universe

Gravitational dynamics  $\rightarrow$  space-time *perturbations* in the metric and in the energy-momentum tensor components:

$$\delta g_{\mu\nu}(\mathbf{x}, t) : \Psi(\mathbf{x}, t), \Phi(\mathbf{x}, t) \quad (6.132)$$

$$\delta T_{\mu\nu}(\mathbf{x}, t) : \delta\rho(\mathbf{x}, t), v_i(\mathbf{x}, t), \delta P(\mathbf{x}, t), \Pi_{ij}(\mathbf{x}, t) \quad (6.133)$$

Fourier transform

$$\delta(\mathbf{k}, t) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \delta(\mathbf{x}, t) \quad (6.134)$$

and inverse

$$\delta(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \delta(\mathbf{k}, t) \quad (6.135)$$

lead to

$$\delta(\mathbf{x}) \rightarrow \delta(\mathbf{k}) \quad (6.136)$$

$$\frac{\partial}{\partial x_i} \delta(\mathbf{x}) \rightarrow ik_i \delta(\mathbf{k}) \quad (6.137)$$

$$\nabla^2 \delta(\mathbf{x}) \rightarrow -k^2 \delta(\mathbf{k}) \quad (6.138)$$

$$\int d^3x' \delta(x') W(x - x') \rightarrow \delta(\mathbf{k}) W(\mathbf{k}) \quad (6.139)$$

Metric perturbations (Conformal Newtonian Gauge):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(1 + 2\Psi) dt^2 + a^2(1 + 2\Phi) dx^2 \quad (6.140)$$

$\Psi(x, t)$ : Newtonian potential (time-time metric perturbation)

$\Phi(x, t)$  curvature potential (space-space metric perturbation).

This metric leads, in Fourier space to the connection symbols:

$$\Gamma_{00}^0 = \dot{\Psi}, \quad \Gamma_{0i}^0 = \Gamma_{i0}^0 = ik_i \Psi, \quad \Gamma_{ij}^0 = \delta_{ij} a^2 \left[ H + 2H(\Phi - \Psi) + \dot{\Phi} \right] \quad (6.141)$$

$$\Gamma_{00}^i = \frac{ik_i}{a^2}\Psi, \quad \Gamma_{0j}^i = \Gamma_{j0}^i = \delta_{ij}(H + \dot{\Phi}), \quad \Gamma_{jk}^i = i\Phi(\delta_{ki}k_j - \delta_{jk}k_i + \delta_{ij}k_k) \quad (6.142)$$

Ricci tensor:

$$R_{00} = -3\frac{\ddot{a}}{a} - \frac{k^2}{a^2}\Psi + 3H(\dot{\Psi} - 2\dot{\Phi}) - 3\ddot{\Phi} \quad (6.143)$$

$$R_{0i} = -2ik_i(\dot{\Phi} - H\Psi) \quad (6.144)$$

$$R_{ij} = \delta_{ij} \left[ (a\ddot{a} + 2a^2H^2)[1 + 2(\Phi - \Psi)] + a^2H(6\dot{\Phi} - \dot{\Psi}) + a^2\ddot{\Phi} + k^2\Phi \right] + k_ik_j(\Phi + \Psi) \quad (6.145)$$

Ricci scalar:

$$R = 6\left(\frac{\ddot{a}}{a} + H^2\right) + \frac{2k^2}{a^2}(\Psi + 2\Phi) - 6H(\dot{\Psi} - 4\dot{\Phi}) + 6\ddot{\Phi} - 12\Psi \left(\frac{\ddot{a}}{a} + H^2\right) \quad (6.146)$$

## 6.15 Perturbed Boltzmann Equations

FRW metric with perturbations in the Newtonian gauge  $\rightarrow$  Boltzmann equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \frac{p}{E} \frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial E} \left[ \frac{p^2}{E} \dot{\Phi} + \frac{\partial \Psi}{\partial x^i} \frac{p \hat{p}^i}{a} + \frac{p^2}{E} H \right] = \left( \frac{\partial f}{\partial t} \right)_C. \quad (6.146)$$

### 6.15.1 Photons

For photons,  $E = p$  and

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial f}{\partial x^i} - p \frac{\partial f}{\partial p} \left[ H + \dot{\Phi} + \frac{\partial \Psi}{\partial x^i} \frac{\hat{p}^i}{a} \right] \quad (6.147)$$

Perturbation in the distribution function around equilibrium Planck distribution  $f^0(p, t)$ :

$$f(\mathbf{x}, \mathbf{p}, t) = f^0(p, t) + \delta f(\mathbf{x}, \mathbf{p}, t) \quad (6.148)$$

or similarly in terms of perturbations in the temperature field

$$T(\mathbf{x}, \hat{p}, t) = T(t) + \delta T(\mathbf{x}, \hat{p}, t) = T(t) [1 + \Theta(\mathbf{x}, \hat{p}, t)] \quad (6.149)$$

where  $\Theta(\mathbf{x}, \hat{p}, t) = \delta T(\mathbf{x}, \hat{p}, t)/T(t)$ , so that

$$\begin{aligned} f(\mathbf{x}, \mathbf{p}, t) &= \left\{ \exp \left[ \frac{p}{T(\mathbf{x}, \hat{p}, t)} \right] - 1 \right\}^{-1} \\ &= f^0(p, t) - p \frac{\partial f^0}{\partial p} \Theta \end{aligned} \quad (6.149)$$

or

$$\delta f(\mathbf{x}, \mathbf{p}, t) = -p \frac{\partial f^0}{\partial p} \Theta \quad (6.150)$$

Keeping only first-order terms, we have

$$\left. \frac{df}{dt} \right|_{\text{1st order}} = -p \frac{\partial f^0}{\partial p} \left[ \dot{\Theta} + \frac{\hat{p}_i}{a} \frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \frac{\hat{p}_i}{a} \frac{\partial \Psi}{\partial x^i} \right] \quad (6.151)$$

Compton scattering of photon off electrons is the main interaction:

$$e^-(\mathbf{q}) + \gamma(\mathbf{p}) \leftrightarrow e^-(\mathbf{q}') + \gamma(\mathbf{p}') \quad (6.152)$$

with amplitude

$$|\mathcal{M}|^2 \approx 8\pi\sigma_T m_e^2 \quad (6.153)$$

and the collision term is given by

$$\left( \frac{\partial f(\mathbf{p})}{\partial t} \right)_C = -p \frac{\partial f^0}{\partial p} n_e \sigma_T [\Theta_0 - \Theta + \hat{p} \cdot \mathbf{v}_b] \quad (6.154)$$

where

$$n_e = \int \frac{d^3q}{(2\pi)^3} f_e(\mathbf{q}) \quad (6.155)$$

$$n_e \mathbf{v}_b = \int \frac{d^3q}{(2\pi)^3} f_e(\mathbf{q}) \frac{\mathbf{q}}{m_e} \quad (6.156)$$

$$\Theta_l = \frac{1}{(-i)^l} \int_{-1}^1 \frac{d\mu}{2} P_l(\mu) \Theta \quad (6.157)$$

The full equation becomes

$$\dot{\Theta} + \frac{\hat{p}_i}{a} \frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \frac{\hat{p}_i}{a} \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T [\Theta_0 - \Theta + \hat{p} \cdot \mathbf{v}_b] \quad (6.158)$$

Then,

\* Change  $t \rightarrow \eta$ ,

\* Change to Fourier space,

\* use  $\mu = \cos(\theta) = \frac{\mathbf{k} \cdot \hat{p}}{k} = \frac{k_i \hat{p}_i}{k} \rightarrow \hat{p}_i k_i = \mu k$

\* Define optical depth:  $\tau' \equiv \frac{d\tau}{d\eta} = -n_e \sigma_T a$

and finally

$$\Theta' + ik\mu\Theta + \Phi' + ik\mu\Psi' = -\tau' [\Theta_0 - \Theta + \mu v_b] \quad (6.159)$$

### 6.15.2 Dark matter

For cold dark matter it is easier to simply use energy-momentum conservation. But following the Boltzmann equations we also obtain

$$\delta'_c + ik_i v_c^i + 3\Phi' = 0 \quad (6.160)$$

$$(v_c^i)' + \left(\frac{a'}{a}\right) v_c^i + ik_i \Psi = 0 \quad (6.161)$$

or in terms of  $\theta_c(\mathbf{x}, t) = \nabla \cdot \mathbf{v}_c(\mathbf{x}, t) = ikv_c$

$$\delta'_c + \theta_c + 3\Phi' = 0 \quad (6.162)$$

$$\theta'_c + \left(\frac{a'}{a}\right) \theta_c - k^2 \Psi = 0 \quad (6.163)$$

The 2 equations may be combined to give

$$\delta''_c + \left(\frac{a'}{a}\right) \delta'_c + 3 \left[ \Phi'' + \left(\frac{a'}{a}\right) \Phi' \right] = -k^2 \Psi \quad (6.164)$$

### 6.15.3 Baryons

For baryons, need to consider the interactions

$$e(q) + p(Q) \rightarrow e(q') + p(Q') \quad (6.165)$$

$$e(q) + \gamma(p) \rightarrow e(q') + \gamma(p') \quad (6.166)$$

to obtain

$$\delta'_b + ikv_b + 3\Phi' = 0 \quad (6.167)$$

$$v'_b + \left(\frac{a'}{a}\right) v_b + ik\Psi = \tau' \frac{4\rho_\gamma}{3\rho_b} [3i\Theta_1 + v_b] \quad (6.168)$$

### 6.15.4 Neutrinos

Massless neutrinos: similar to photons, but different temperature  $T^\nu$  and no collision term. Define  $\mathcal{N} = \delta T_\nu / T_\nu$ , such that

$$\mathcal{N}' + ik\mu\mathcal{N} + \Phi' + ik\mu\Psi' = 0 \quad (6.169)$$

Massive neutrinos: evolution starts as massless neutrinos while they are relativistic. Transition to transition to that of dark matter once they become non-relativistic.

See Ma & Bertschinger 1995 for a careful description of:

1) linear perturbations for all components above and Einstein Equations in both Conformal Newtonian Gauge and Synchronous Gauge.

2) a technique to solve the equations for photons and neutrinos in terms of a multipole expansion in Legendre polynomials.

## 6.16 Perturbed Einstein Equations

FRW metric with Newtonian perturbations + Einstein Equations:

$$-k^2\Phi - 3H(\dot{\Phi} - H\Psi) = -4\pi G\delta\rho \quad (6.170)$$

$$-k^2(\dot{\Phi} - H\Psi) = 4\pi G(\rho + P)(ik^i v_i) \quad (6.171)$$

$$\ddot{\Phi} - H(\dot{\Psi} - 3\dot{\Phi}) - \left(2\frac{\ddot{a}}{a} + H^2\right)\Psi + \frac{k^2}{3a^2}(\Psi + \Phi) = -4\pi G\delta P \quad (6.172)$$

$$-k^2(\Psi + \Phi) = 32\pi G\rho\Theta_2 \quad (6.173)$$

Consider first and third equation in a non-expanding universe and static fields:

$$-k^2\Phi = -4\pi G\delta\rho \quad (6.174)$$

$$\frac{k^2}{3a^2}(\Psi + \Phi) = -4\pi G\delta P \quad (6.175)$$

$$(6.176)$$

so adding the first and 3 times the second we have

$$\nabla^2\Psi = 4\pi G(\delta\rho + 3\delta P) \quad (6.177)$$

In General Relativity, pressure perturbation is also a source to the gravitational potential  $\Psi$ .

Finally, we saw initial conditions from the Boltzmann/Einstein equations themselves.

We ended the semester looking at the Inflation model as a solution to a number of problems in the Big Bang scenario (horizon problem, flatness problem, unwanted relics), as well as a means of producing and magnifying quantum perturbations in the early Universe, and its implementation with a slowly-rolling scalar field.