PGF5299: Physical Cosmology II

Problem Set 6

(Due October 28, 2014)

1) Spherically Symmetric Lens: Recall that the reduced deflection angle vector $\hat{\vec{\alpha}} = (\alpha_1, \alpha_2)$ is given by

$$\hat{\vec{\alpha}}(\vec{r}_\perp) = 2\int_{-\infty}^{\infty} \vec{\nabla}_\perp \Phi(r_\parallel, \vec{r}_\perp)\, dr_\parallel \tag{1}$$

For a point lens with mass M, the potential is $\Phi(r_\parallel, \vec{r}_\perp) = GM/(r_\parallel^2 + r_\perp^2)^{1/2}$. Therefore, for a vector $\vec{\xi} = \vec{r}_\perp$ on the lens plane at a distance ξ away from M, the deflection becomes

$$\vec{\alpha}(\vec{\xi}) = 4GM\frac{\vec{\xi}}{\xi^2} \tag{2}$$

For a more general lens, we can consider the lens made up of point masses on the lens plane, each one at a distance ξ_i from an origin so that at ξ the reduced deflection is

$$\begin{aligned}
\hat{\vec{\alpha}}(\vec{\xi}) &= \sum_i \alpha_i(\vec{\xi} - \vec{\xi}_i) = 4G\sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2} = 4G\overbrace{\int d^2\xi' \underbrace{\int dz\ \rho(r)}_{\Sigma(\xi')}}^{dM} \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \\
&= 4G\int d^2\xi' \frac{\Sigma(\xi')(\vec{\xi} - \vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2}
\end{aligned} \tag{3}$$

or in terms of $\vec{\theta} = \vec{\xi}/D_L$, the deflection $\vec{\alpha}(\vec{\xi}) = \hat{\vec{\alpha}}(\vec{\xi})D_{LS}/D_S$ (in wich $\vec{\theta} = \vec{\beta} + \vec{\alpha}$)

$$\begin{aligned}
\vec{\alpha}(\vec{\theta}) &= \frac{D_{LS}}{D_S} 4G \int d^2\theta' \frac{D_L \Sigma(D_L\theta')(\vec{\theta} - \vec{\theta}')}{|\vec{\theta} - \vec{\theta}'|^2} = \frac{1}{\pi}\int d^2\theta' \Sigma(D_L\theta') \frac{4\pi G D_{LS} D_L}{D_S} \frac{(\vec{\theta} - \vec{\theta}')}{|\vec{\theta} - \vec{\theta}'|^2} \\
&= \frac{1}{\pi}\int d^2\theta' \kappa(\theta') \frac{(\vec{\theta} - \vec{\theta}')}{|\vec{\theta} - \vec{\theta}'|^2}
\end{aligned} \tag{4}$$

where the convergence $\kappa(\theta) = \Sigma(D_L\vec{\theta})/\Sigma_{crit}$ and $\Sigma_{crit}^{-1} = 4\pi G D_L D_{LS}/D_S$

a) For a spherically symmetric lens, argue that the deflection must point towards the lens center (i.e. $\vec{\alpha} = \alpha(\theta)\vec{\theta}/\theta$) and show that its magnitude is

$$\alpha(\theta) \quad = \quad \frac{M(<\theta)}{\pi D_L^2 \Sigma_{crit}} \frac{1}{\theta} = \frac{m(\theta)}{\theta} \tag{5}$$

where $M(<\theta)$ is the mass contained within θ and $m(\theta)$ is an average surface density normalized by the critical surface density.

b) Since $\vec{\alpha} = \alpha(\theta)\vec{\theta}/\theta$ and $\theta = (\theta_1^2 + \theta_2^2)^{1/2}$, compute $\partial\alpha_i/\partial\theta_j$, for $i, j = 1, 2$. Express the results in terms of $\alpha(\theta)$ and $d\alpha/d\theta$.

c) Use the previous relations to compute $\kappa(\theta)$, $\gamma_1(\vec{\theta})$, $\gamma_2(\vec{\theta})$ in terms of $\alpha(\theta)$ and $d\alpha/d\theta$.

d) Compute the shear amplitude $\gamma(\theta) = (\gamma_1^2 + \gamma_2^2)^{1/2}$ and the average convergence $\bar{\kappa}(\theta)$

$$\bar{\kappa}(\theta) = \frac{1}{\pi\theta^2} \int 2\pi\theta' \, d\theta' \kappa(\theta') \tag{6}$$

and show that

$$\gamma(\theta) = \bar{\kappa}(\theta) - \kappa(\theta) \tag{7}$$

2) NFW Profile: Recall the spherical NFW halo profile is given by:

$$\rho(r|M_{vir}, z) = \frac{\rho_s}{cr/r_{vir}(1 + cr/r_{vir})^2} \tag{8}$$

a) Show analytically that the mass contained within the virial radius of this halo is

$$M_{vir} \quad = \quad \int_0^{r_{vir}} dr \; 4\pi r^2 \rho(r) \quad = \quad 4\pi\rho_s \frac{r_{vir}^3}{c^3} f^{-1} \tag{9}$$

$$\text{where} \quad f(c) \quad = \quad \left[\ln(1+c) - \frac{c}{1+c}\right]^{-1} \tag{10}$$

b) Using $r = |\vec{r}|$ with $\vec{r} = (r_\parallel, \vec{r}_\perp)$ and $\vec{r}_\perp = D_L\vec{\theta}$, compute either analytically or numerically the projected surface density profile

$$\Sigma(\theta) = \int_{-r_{vir}}^{r_{vir}} dr_\parallel \; \rho(r) \tag{11}$$

and plot the convergence $\kappa(\theta) = \Sigma(\theta)/\Sigma_{\rm crit}$ versus θ induced by a halo of mass $M_{vir} = 14.0$ at $z_l = 0.5$ on a source galaxy at $z_s = 2.0$. *Note:* Look at Takada and Jain 2003.