

Apêndice H

Integral $\int \frac{d\phi}{a - \rho \cos(\phi)}$

Vamos calcular a integral I abaixo (onde $a > \rho$)

$$I = \int_0^{2\pi} \frac{d\phi}{a - \rho \cos(\phi)} \quad (\text{H.1})$$

Primeiro vamos dividir a integral em duas partes

$$I = \int_0^{\pi} \frac{d\phi}{a - \rho \cos(\phi)} + \int_{\pi}^{2\pi} \frac{d\phi}{a - \rho \cos(\phi)} \quad (\text{H.2})$$

Na segunda integral vamos mudar a variável $\phi \rightarrow \phi' = 2\pi - \phi$, ou similarmente $\phi = 2\pi - \phi'$. Temos $d\phi = -d\phi'$ e os limites de integração $\phi = \pi \rightarrow \phi' = \pi$ e $\phi = 2\pi \rightarrow \phi' = 0$. Além disso $\cos(\phi) = \cos(2\pi - \phi') = \cos(-\phi') = \cos(\phi')$. Assim

$$I = \int_0^{\pi} \frac{d\phi}{a - \rho \cos(\phi)} + \int_{\pi}^0 \frac{-d\phi'}{a - \rho \cos(\phi')} = 2 \int_0^{\pi} \frac{d\phi}{a - \rho \cos(\phi)}. \quad (\text{H.3})$$

Agora vamos mudar $\phi \rightarrow u = \tan(\phi/2)$. Temos

$$\frac{du}{d\phi} = \frac{d}{du} \left[\frac{\sin(\phi/2)}{\cos(\phi/2)} \right] = \frac{1}{2} \left[1 + \frac{\sin^2(\phi/2)}{\cos^2(\phi/2)} \right] = \frac{1}{2} [1 + \tan^2(\phi/2)] = \frac{1}{2 \cos^2(\phi/2)}, \quad (\text{H.4})$$

ou seja

$$d\phi = \frac{2du}{1 + u^2} \quad (\text{H.5})$$

Além disso, temos

$$\cos \phi = 2 \cos^2(\phi/2) - 1 = \frac{2}{1 + \tan^2(\phi/2)} - 1 = \frac{2}{1 + u^2} - 1 = \frac{1 - u^2}{1 + u^2} \quad (\text{H.6})$$

Assim, a integral fica

$$\begin{aligned} I &= 2 \int \left(\frac{2du}{1 + u^2} \right) \frac{1}{a - \rho \left(\frac{1 - u^2}{1 + u^2} \right)} = 2 \int \frac{2du}{a(1 + u^2) - \rho(1 - u^2)} = 4 \int \frac{du}{(a - \rho) + (a + \rho)u^2} \\ &= \frac{4}{a - \rho} \int_{\phi=0}^{\phi=\pi} \frac{du}{1 + \left(\frac{a + \rho}{a - \rho} \right) u^2} \end{aligned} \quad (\text{H.7})$$

Agora mudamos de variável novamente $u \rightarrow y = \sqrt{\frac{a+\rho}{a-\rho}}u$, com $du = \sqrt{\frac{a-\rho}{a+\rho}}dy$. A integral fica:

$$\begin{aligned} I &= \frac{4}{a-\rho} \int \sqrt{\frac{a-\rho}{a+\rho}} \frac{dy}{1+y^2} = \frac{4}{\sqrt{a-\rho}\sqrt{a+\rho}} \int_{\phi=0}^{\phi=\pi} \frac{dy}{1+y^2} \\ &= \frac{4}{\sqrt{a^2-\rho^2}} \arctan(y) \Big|_{\phi=0}^{\phi=\pi} \end{aligned} \quad (\text{H.8})$$

Voltando às variáveis originais

$$I = \frac{4}{\sqrt{a^2-\rho^2}} \arctan \left(\sqrt{\frac{a+\rho}{a-\rho}} u \right) \Big|_{\phi=0}^{\phi=\pi} = \frac{4}{\sqrt{a^2-\rho^2}} \arctan \left(\sqrt{\frac{a+\rho}{a-\rho}} \tan(\phi/2) \right) \Big|_{\phi=0}^{\phi=\pi} \quad (\text{H.9})$$

Com $\tan(\pi/2) = \infty$ e $\tan(0) = 0$, e portanto $\arctan(\infty) = \pi/2$, temos

$$I = \frac{4}{\sqrt{a^2-\rho^2}} [\arctan(\infty) - \arctan(0)] = \frac{4}{\sqrt{a^2-\rho^2}} \frac{\pi}{2} \quad (\text{H.10})$$

ou finalmente

$$I = \int_0^{2\pi} \frac{d\phi}{a - \rho \cos(\phi)} = \frac{2\pi}{\sqrt{a^2-\rho^2}} \quad (\text{H.11})$$