Derivation of Spherical Collapse Equations

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Notes on Spherical Collapse

I. COMBINING FLUID EQUATIONS

Consider the nonlinear continuity and Euler equations for a pressureless fluid in a gravitational potential:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot (1+\delta)\mathbf{v} \;=\; 0\,, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}(\mathbf{v}\cdot\nabla)\mathbf{v} + H\mathbf{v} \;=\; -\frac{1}{a}\nabla\Psi\,, \tag{2}$$

Taking the derivative of the first equation, and using the original first and second equations, we have

$$\frac{\partial^2 \delta}{\partial t^2} - \frac{\dot{a}}{a^2}\nabla \cdot (1+\delta)\mathbf{v} + \frac{1}{a}\nabla \cdot \left[\frac{\partial \delta}{\partial t}\mathbf{v} + (1+\delta)\frac{\partial \mathbf{v}}{\partial t}\right] = 0\,, \tag{3}$$

$$\frac{\partial^2 \delta}{\partial t^2} + H\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot \left[-\left(\frac{1}{a}\nabla \cdot (1+\delta)\mathbf{v}\right)\mathbf{v} + (1+\delta)\left(-\frac{1}{a}(\mathbf{v}\cdot\nabla)\mathbf{v} - H\mathbf{v} - \frac{1}{a}\nabla\Psi\right)\right] = 0\,, \tag{4}$$

$$\frac{\partial^2 \delta}{\partial t^2} + H\frac{\partial \delta}{\partial t} + \frac{1}{a^2}\nabla \cdot \left[-\left(\nabla \cdot (1+\delta)\mathbf{v}\right)\mathbf{v} - \left((1+\delta)(\mathbf{v}\cdot\nabla)\right)\mathbf{v}\right] - H\frac{1}{a}\nabla \cdot (1+\delta)\mathbf{v} = \frac{1}{a^2}\nabla \cdot (1+\delta)\nabla\Psi\,, \tag{5}$$

$$\frac{\partial^2 \delta}{\partial t^2} + 2H\frac{\partial \delta}{\partial t} + \frac{1}{a^2}\nabla \cdot \left[-\left(\nabla \cdot (1+\delta)\mathbf{v}\right)\mathbf{v} - \left((1+\delta)(\mathbf{v}\cdot\nabla)\right)\mathbf{v}\right] = \frac{1}{a^2}\nabla \cdot (1+\delta)\nabla\Psi\,, \tag{6}$$

We can write the third term as

$$\begin{aligned}
\nabla \cdot \left[-\left(\nabla \cdot (1+\delta)\mathbf{v}\right)\mathbf{v} - \left((1+\delta)(\mathbf{v}\cdot\nabla)\right)\mathbf{v}\right] \;&=\; -\nabla_j\left(\nabla_i(1+\delta)v_i\right)v_j - \nabla_j\left((1+\delta)(v_i\nabla_i)\right)v_j && (7)\\
&=\; -\nabla_j\nabla_i\left[(1+\delta)v_i\right]v_j - \nabla_j\left[(1+\delta)v_i\right]\nabla_i v_j && (8)\\
&=\; -\nabla_j\nabla_i\left[(1+\delta)v_i v_j\right] && (9)\\
&=\; -\frac{\partial^2 (1+\delta)v_i v_j}{\partial x_i \partial x_j} && (10)
\end{aligned}$$

Therefore

$$\frac{\partial^2 \delta}{\partial t^2} + 2H\frac{\partial \delta}{\partial t} - \frac{1}{a^2}\frac{\partial^2 (1+\delta)v_i v_j}{\partial x_i \partial x_j} = \frac{1}{a^2}\nabla \cdot (1+\delta)\nabla\Psi\,, \tag{11}$$

For a top-hat profile, δ is spatially constant, i.e. it is only a function of time. Therefore from the continuity equation:

$$\nabla \cdot \mathbf{v} = \frac{a}{1+\delta}\frac{\partial \delta}{\partial t} \tag{12}$$

Since the RHS is only a function of time, the LHS must also be spatially constant. If that's the case, and we preserve spherical symmetry, we must take $\mathbf{v} = A\mathbf{r}$ for a constant A, such that

$$\nabla \cdot \mathbf{v} \;=\; \frac{\partial v_i}{\partial x_i} = A\frac{\partial x_i}{\partial x_i} = 3A \tag{13}$$

$$\frac{\partial v_i v_j}{\partial x_i \partial x_j} \;=\; A^2\frac{\partial}{\partial x_i}\left(\frac{\partial (x_i x_j)}{\partial x_j}\right) = A^2\frac{\partial}{\partial x_i}\left(\delta_{ij}x_j + 3x_i\right) = A^2\frac{\partial}{\partial x_i}\left(4x_i\right) = 12A^2 \tag{14}$$

Therefore

$$\frac{\partial v_i v_j}{\partial x_i \partial x_j} = 12A^2 = 12\left(\frac{\nabla\cdot\mathbf{v}}{3}\right)^2 = \frac{4}{3}\frac{a^2\dot{\delta}^2}{(1+\delta)^2} \tag{15}$$

And we have

$$\frac{\partial^2\delta}{\partial t^2} + 2H\frac{\partial\delta}{\partial t} - \frac{4}{3}\frac{\dot{\delta}^2}{(1+\delta)} = \frac{(1+\delta)}{a^2}\nabla^2\Psi\,, \tag{16}$$

During collapse, the total mass of the perturbation is conserved:

$$M = (4\pi/3)r^3\bar{\rho}_m(1+\delta) = C' = \text{const.} \tag{17}$$

or equivalently

$$r^3 = \frac{C}{\rho(1+\delta)} \tag{18}$$

From these, we have

$$3r^2\dot{r} = -C\frac{\dot{\bar{\rho}}}{\bar{\rho}^2(1+\delta)} - C\frac{\dot{\delta}}{\bar{\rho}(1+\delta)^2} \tag{19}$$

$$\rightarrow \quad \frac{\dot{r}}{r} = -\frac{1}{3}\left(\frac{\dot{\bar{\rho}}}{\bar{\rho}} + \frac{\dot{\delta}}{1+\delta}\right) \tag{20}$$

Differentiating once more, we have

$$\frac{\ddot{r}}{r} - \left(\frac{\dot{r}}{r}\right)^2 = -\frac{1}{3}\left[\frac{\ddot{\bar{\rho}}}{\bar{\rho}} - \left(\frac{\dot{\bar{\rho}}}{\bar{\rho}}\right)^2 + \frac{\ddot{\delta}}{(1+\delta)} - \left(\frac{\dot{\delta}}{1+\delta}\right)^2\right] \tag{21}$$

$$\rightarrow \quad \frac{\ddot{r}}{r} = -\frac{1}{3}\left[\frac{\ddot{\bar{\rho}}}{\bar{\rho}} - \left(\frac{\dot{\bar{\rho}}}{\bar{\rho}}\right)^2 + \frac{\ddot{\delta}}{(1+\delta)} - \left(\frac{\dot{\delta}}{1+\delta}\right)^2\right] + \frac{1}{9}\left[\left(\frac{\dot{\bar{\rho}}}{\bar{\rho}}\right)^2 + \left(\frac{\dot{\delta}}{1+\delta}\right)^2 + \frac{2\dot{\bar{\rho}}\dot{\delta}}{\bar{\rho}(1+\delta)}\right] \tag{22}$$

$$= -\frac{1}{3}\frac{\ddot{\bar{\rho}}}{\bar{\rho}} + \frac{4}{9}\frac{\dot{\bar{\rho}}}{\bar{\rho}} - \frac{1}{3}\frac{\ddot{\delta}}{1+\delta} + \frac{4}{9}\left(\frac{\dot{\delta}}{1+\delta}\right)^2 + \frac{2}{9}\frac{\dot{\bar{\rho}}\dot{\delta}}{\bar{\rho}(1+\delta)} \tag{23}$$

Now since $\bar{\rho}_m = \bar{\rho}_{m,0}a^{-3}$, we have

$$\dot{\bar{\rho}}_m = -3\bar{\rho}_{m,0}a^{-4}\dot{a} \tag{24}$$

$$\ddot{\bar{\rho}}_m = 12\bar{\rho}_{m,0}a^{-5}\dot{a}^2 - 3\bar{\rho}_{m,0}a^{-4}\ddot{a} \tag{25}$$

and since

$$H = \frac{\dot{a}}{a} \tag{26}$$

$$\dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \tag{27}$$

we have

$$\frac{\dot{\bar{\rho}}_m}{\bar{\rho}_m} = -3\frac{\dot{a}}{a} = -3H \tag{28}$$

$$\frac{\ddot{\bar{\rho}}_m}{\bar{\rho}_m} = 12\left(\frac{\dot{a}}{a}\right)^2 - 3\frac{\ddot{a}}{a} = 9H^2 - 3\dot{H} \tag{29}$$

Therefore:

$$\frac{\ddot{r}}{r} = -\frac{1}{3}\left(9H^2 - 3\dot{H}\right) + \frac{4}{9}(-3H)^2 - \frac{1}{3}\frac{\ddot{\delta}}{1+\delta} + \frac{4}{9}\left(\frac{\dot{\delta}}{1+\delta}\right)^2 + \frac{2}{9}\frac{(-3H)\dot{\delta}}{(1+\delta)} \tag{30}$$

$$= -3H^2 - \dot{H} + 4H^2 - \frac{1}{3}\frac{\ddot{\delta}}{1+\delta} + \frac{4}{9}\left(\frac{\dot{\delta}}{1+\delta}\right)^2 - \frac{2H\dot{\delta}}{3(1+\delta)} \tag{31}$$

$$= H^2 - \dot{H} - \frac{1}{3}\frac{1}{1+\delta}\left(\ddot{\delta} - \frac{4}{3}\frac{\dot{\delta}^2}{1+\delta} + 2H\dot{\delta}\right) \tag{32}$$

Finally, using the Friedmann equations

$$\dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \qquad \rightarrow \qquad H^2 - \dot{H} = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\bar{\rho}_m + (1+3w)\rho_{\rm DE}\right] \tag{33}$$

and using this along with the collapse equation, we have

$$\frac{\ddot{r}}{r} = -\frac{4\pi G}{3}\left[\bar{\rho}_m + (1+3w)\rho_{\rm DE}\right] - \frac{\nabla^2\Psi}{3a^2} \tag{34}$$

For ΛCDM $w = -1$, $\rho_{\rm DE} = \rho_\Lambda$ and $\nabla^2\Psi = 4\pi G a^2 \delta\rho_m$, so

$$\frac{\ddot{r}}{r} = -\frac{4\pi G}{3}\left[\bar{\rho}_m + (1+3w)\rho_{\rm DE}\right] - \frac{4\pi G}{3}(\rho_m - \bar{\rho}_m) \tag{35}$$

$$= -\frac{4\pi G}{3}\left[\rho_m + (1+3w)\rho_\Lambda\right] \tag{36}$$

So the collapse proceeds as if the perturbation were a separate closed universe with density ρ_m.