Notes on Spherical Collapse

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I. SPHERICAL COLLAPSE

The background scale factor a and a top-hat overdensity of radius r are described by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\rho_{\rm m} + \rho_{\rm DE}\right] \quad (1)$$

$$\frac{\ddot{r}}{r} = -\frac{4\pi G}{3}\left[(1+3w_{\rm cluster})\rho_{\rm cluster} + (1+3w_{\rm eff})\rho_{\rm eff}\right] \quad (2)$$

The critical density is defined as

$$\rho_{\rm crit}(a) = \frac{3H^2(a)}{8\pi G} \quad (3)$$

with $H(a) = \dot{a}/a$ and densities relative to critical

$$\Omega_\alpha(a) = \frac{\rho_\alpha(a)}{\rho_{\rm crit}(a)} \quad (4)$$

It is useful to normalize a and r to their values at turn-around time $t_{\rm ta}$:

$$x = \frac{a}{a_{\rm ta}}, \quad (5)$$

$$y = \frac{r}{r_{\rm ta}} \quad (6)$$

Assuming that $w_{\rm m} = 0$ and $w_{\rm DE} = -1$ in the background, we have

$$\rho_{\rm m} = \rho_{\rm m,ta}x^{-3}, \quad (7)$$

$$\rho_{\rm DE} = \rho_{\rm DE,ta} \quad (8)$$

and Eq.(1) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_{\rm ta}^2}{\rho_{\rm crit,ta}}\left[\rho_{\rm m,ta}x^{-3} + \rho_{\rm DE}\right] \quad (9)$$

or

$$\left(\frac{\dot{x}}{x}\right)^2 = \left[\Omega_{\rm m,ta}x^{-3} + \frac{\rho_{\rm DE}}{\rho_{\rm crit,ta}}\right] \quad (10)$$

where the dot in Eq.(12) denotes derivative with respect to the scaled time $\tau = H_{\rm ta}t$. Inside the perturbation we assume $w_{\rm cluster} = 0$ so that

$$\rho_{\rm cluster} = \rho_{\rm cluster,ta}y^{-3} \quad (11)$$

This also results simply from mass conservation inside the cluster after turn-around. We have

$$\frac{\ddot{r}}{r} = -\frac{H_{\rm ta}^2}{2\rho_{\rm crit,ta}}\left[\rho_{\rm cluster,ta}y^{-3} + (1+3w_{\rm eff})\rho_{\rm eff}\right]$$

$$\frac{\ddot{r}}{H_{\rm ta}^2 r} = -\frac{1}{2}\left[\frac{\rho_{\rm m,ta}}{\rho_{\rm crit,ta}}\frac{\rho_{\rm cluster,ta}}{\rho_{\rm m,ta}}y^{-3} + (1+3w_{\rm eff})\frac{\rho_{\rm eff}}{\rho_{\rm crit,ta}}\right] \quad (12)$$

or

$$\frac{\ddot{y}}{y} = -\frac{1}{2}\left[\Omega_{\rm m,ta}\zeta y^{-3} + (1+3w_{\rm eff})\frac{\rho_{\rm eff}}{\rho_{\rm crit,ta}}\right] \quad (13)$$

where

$$\zeta = \frac{\rho_{\rm cluster,ta}}{\rho_{\rm m,ta}} \quad (14)$$

To solve the equations we must provide a prescription for how $\rho_{\rm eff}$ scales (with a or r). For DE models we simply have $\rho_{\rm eff} = \rho_{\rm DE}$ and for modifications of gravity it will in general have some dependency on a and r. In the latter case, Birkhoff's Theorem does not apply since the density of the "stuff" causing the background expansion scales differently within and outside the perturbation.

A. Initial Conditions

We can obtain the appropriate initial conditions to evolve these equations by assuming that the effects of $\rho_{\rm DE}$ on the background and and $\rho_{\rm eff}$ on the perturbation are negligible very early on. We have

$$\left(\frac{\dot{x}}{x}\right)^2 = \Omega_{\rm m,ta}x^{-3} \quad (15)$$

$$\frac{\ddot{y}}{y} = -\frac{1}{2}\Omega_{\rm m,ta}\zeta y^{-3} \quad (16)$$

The first equation is just the solution for a matter dominated universe:

$$\tau = \int d\tau = \int \frac{x^{1/2}dx}{\sqrt{\Omega_{\rm m,ta}}} = \frac{2}{3}\frac{x^{3/2}}{\sqrt{\Omega_{\rm m,ta}}} \tag{17}$$

or

$$x = \left(\frac{3}{2}\right)^{2/3} \Omega_{\rm m,ta}^{1/3}\tau^{2/3} \tag{18}$$

The equation for y can be similarly integrated after multiplying it by $2\dot{y}y$

$$\begin{aligned} 2\ddot{y}\dot{y} &= -\Omega_{\rm m,ta}\zeta\dot{y}y^{-2} \\ \frac{d}{d\tau}\left(\dot{y}^2\right) &= \Omega_{\rm m,ta}\zeta\frac{d}{d\tau}\left(\frac{1}{y}\right) \end{aligned} \tag{19}$$

so that

$$\dot{y}^2 = \Omega_{\rm m,ta}\zeta\left(\frac{1}{y}-1\right) \tag{20}$$

where we integrated and used the boundary condition at turn-around ($\dot{y}=0$ when $y=1$). Very early on, the first term on the RHS dominates and we get an equation similar to Eq.(16)

$$\left(\frac{\dot{y}}{y}\right)^2 = \Omega_{\rm m,ta}\zeta y^{-3} \tag{21}$$

from which the solution can be obtained imediately

$$y = \left(\frac{3}{2}\right)\left(\Omega_{\rm m,ta}\zeta\right)^{1/3}\tau^{2/3} \tag{22}$$

or

$$y = \zeta^{1/3}x \tag{23}$$

B. Solution in ΛCDM

Here we have $w_{\rm eff} = w_{\rm DE} = -1$, and $\rho_{\rm eff} = \rho_{\rm DE} = \rho_\Lambda = \rho_{\Lambda,\rm ta}$.

$$\begin{aligned} \left(\frac{\dot{x}}{x}\right)^2 &= \left[\Omega_{\rm m,ta}x^{-3} + \Omega_{\Lambda,\rm ta}\right] \\ \frac{\ddot{y}}{y} &= -\frac{1}{2}\left[\Omega_{\rm m,ta}\zeta y^{-3} - 2\Omega_{\Lambda,\rm ta}\right] \end{aligned} \tag{24, 25}$$

In this case, there is an analytical solutions for the background and the perturbation evolution is that of a closed universe with scaled energy densities. For the background we have (Appendix)

$$x = \left(\frac{\Omega_{\rm m,ta}}{\Omega_{\Lambda,\rm ta}}\right)^{1/3}\sinh^{2/3}\left(\frac{3\sqrt{\Omega_{\Lambda,\rm ta}}}{2}\tau\right) \tag{26}$$

which reduces to $x \sim \tau^{2/3}$ at low τ and $x \sim \exp\sqrt{\Omega_{\Lambda,\rm ta}}\tau$ at high τ). For the perturbation, after multiplying Eq.(26) by $2\dot{y}y$, we have

$$2\ddot{y}\dot{y} = -\left[\Omega_{\rm m,ta}\zeta\dot{y}y^{-2} - 2\Omega_{\Lambda,\rm ta}\dot{y}y\right] \tag{27}$$

so that

$$\frac{d}{d\tau}\left(\dot{y}^2\right) = \frac{d}{d\tau}\left(\frac{\Omega_{\rm m,ta}\zeta}{y} + \Omega_\Lambda y^2\right) \tag{28}$$

which, after integrating and using the boundary condition at turn-around produces

$$\dot{y}^2 = \left(\Omega_{\rm m,ta}\zeta(\frac{1}{y}-1) + \Omega_\Lambda(y^2-1)\right) \tag{29}$$

This can be rewritten as

$$\begin{aligned} \left(\frac{\dot{y}}{y}\right)^2 &= \left(\Omega_{\rm m,ta}\zeta y^{-3} + \Omega_\Lambda + \frac{-(\Omega_{\rm m,ta}\zeta + \Omega_\Lambda)}{y^2}\right) \\ &= \left(\Omega_{\rm cluster,ta}y^{-3} + \Omega_\Lambda + \frac{-(\Omega_{\rm cluster,ta} + \Omega_\Lambda)}{y^2}\right) \end{aligned} \tag{30}$$

This is the same equation of a closed universe with a matter density scaled from the background by ζ and curvature density $-\Omega_{\rm k} = \Omega_{\rm cluster,ta} + \Omega_\Lambda$. In a true closed universe however, we would have $\Omega_{\rm k} = 1 - \Omega_{\rm cluster,ta} - \Omega_\Lambda$. Here since $-\Omega_{\rm k} > 0$, it is in fact a closed universe. For a universe with matter only, an analytic solution exists with a parametrized cycloid (see Appendix) and we can obtain many parameters, including $\zeta \sim 5.5517$ and $\delta_{\rm c}^{\rm lin} \sim 1.68647$.

In Fig. 1 , we show the collapse density $\delta_{\rm c}$ as a function of collapse redshift $z_{\rm c}$ for a flat universe and different values of $\Omega_{\rm m,0}$. When $\Omega_{\rm m,0} = 1.0$, one can show that $\delta_{\rm c} = 3/5(3\pi/2)^{2/3} \sim 1.68647$ (see Appendix). For lower values of $\Omega_{\rm m,0}$, the collapse density is smaller at lower collapse redshifts.

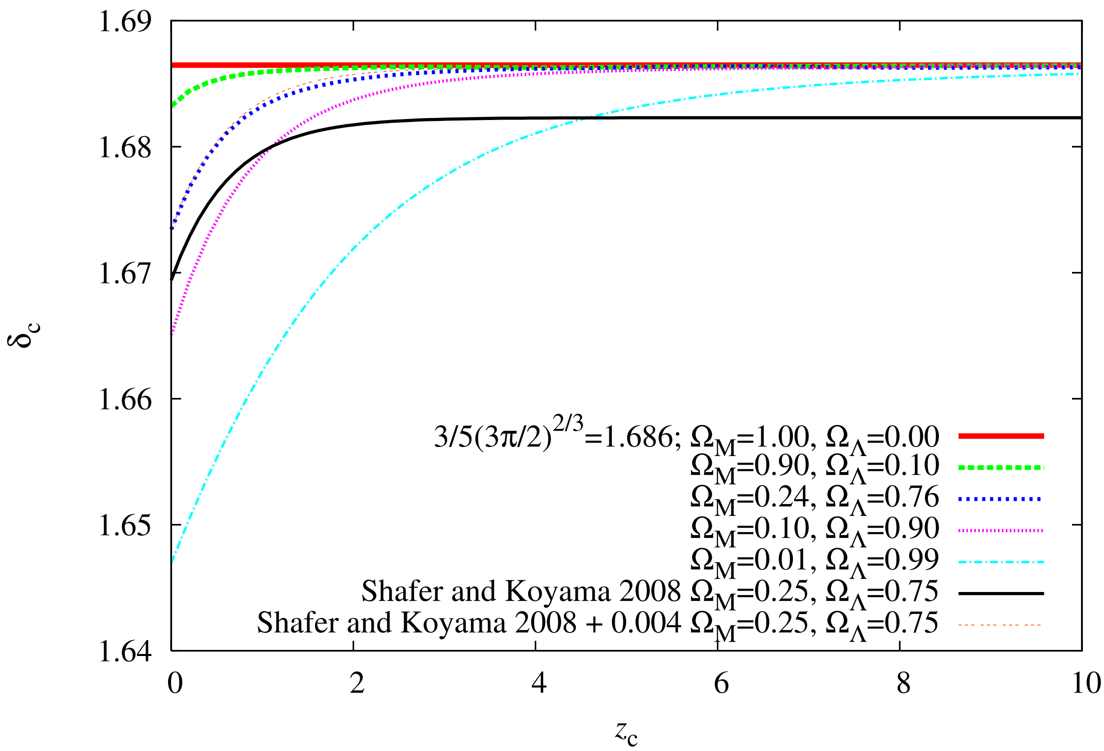

FIG. 1: Linear collapse density δ_c as a function of collapse redshift z_c for different values of Ω_m and Ω_Λ in a flat ΛCDM cosmology. These results were generated evolving Eqs.(25) and (31). Evolving Eq. (26) seems to be much less stable than (31) and I often get my code to crash. Note that Eq.(31) automatically imposes the boundary conditions ($y = 1$ and $\dot{y} = 0$ at turn-around). However, it is necessary to evolve Eq.(31) for anything beyond ΛCDM.

Appendix A: Background evolution in flat ΛCDM

The Friedman equation in a closed universe with CDM only is given by

$$\left(\frac{\dot{x}}{x}\right)^2 = \left(\Omega_{\rm m,ta}x^{-3} + \Omega_{\Lambda,\rm ta}\right) \tag{A1}$$

which can be rewritten as

$$\frac{dx}{d\tau} = \sqrt{\Omega_{\rm m,ta}x^{-1} + \Omega_{\Lambda,\rm ta}x^2} \tag{A2}$$

or

$$\begin{aligned}\tau &= \int dt = \int \frac{dx}{\sqrt{\Omega_{\rm m,ta}x^{-1} + \Omega_{\Lambda,\rm ta}x^2}} \\ &= \int \frac{x^{1/2}dx}{\sqrt{\Omega_{\rm m} + \Omega_{\Lambda,\rm ta}x^3}} \\ &= \frac{1}{\sqrt{\Omega_{\rm m,ta}}}\int \frac{x^{1/2}dx}{\sqrt{1 + (\Omega_{\Lambda,\rm ta}/\Omega_{\rm m,ta})x^3}}\end{aligned}$$

Change $u^2 = \Omega_{\Lambda,\rm ta}/\Omega_{\rm m,ta}x^3$, so that $u = \sqrt{\Omega_{\Lambda,\rm ta}/\Omega_{\rm m,ta}}x^{3/2}$ and $du = 3/2\sqrt{\Omega_{\Lambda,\rm ta}/\Omega_{\rm m,ta}}x^{1/2}dx$ we have

$$\begin{aligned}\tau &= \frac{1}{\sqrt{\Omega_{\rm m,ta}}}\frac{2}{3}\sqrt{\frac{\Omega_{\rm m,ta}}{\Omega_{\Lambda,\rm ta}}}\int \frac{du}{\sqrt{1+u^2}} \\ &= \frac{2}{3}\frac{1}{\sqrt{\Omega_{\Lambda,\rm ta}}}\sinh^{-1}u \\ &= \frac{2}{3}\frac{1}{\sqrt{\Omega_{\Lambda,\rm ta}}}\sinh^{-1}\sqrt{\Omega_{\Lambda,\rm ta}/\Omega_{\rm m,ta}x^{3/2}}\end{aligned}$$

or inverting

$$x = \left(\frac{\Omega_{\rm m,ta}}{\Omega_{\Lambda,\rm ta}}\right)^{1/3}\sinh^{2/3}\left(\frac{3\sqrt{\Omega_{\Lambda,\rm ta}}}{2}\tau\right) \tag{A3}$$

Notice that for small τ

$$\begin{aligned}x &\sim \left(\frac{\Omega_{\rm m,ta}}{\Omega_{\Lambda,\rm ta}}\right)^{1/3}\left(\frac{3\sqrt{\Omega_{\Lambda,\rm ta}}}{2}\tau\right)^{2/3} \\ &\sim \left(\frac{3}{2}\right)^{2/3}\Omega_{\rm m,ta}^{1/3}\tau^{2/3}\end{aligned} \tag{A4}$$

and for large τ

$$x \sim \left(\frac{\Omega_{\rm m,ta}}{\Omega_{\Lambda,\rm ta}}\right)^{1/3} \exp\left(\frac{3\sqrt{\Omega_{\Lambda,\rm ta}}}{2}\tau\right)^{2/3}$$
$$\sim \left(\frac{\Omega_{\rm m,ta}}{\Omega_{\Lambda,\rm ta}}\right)^{1/3} \exp\left(\sqrt{\Omega_{\Lambda,\rm ta}}\tau\right) \quad (A5)$$

Appendix B: CDM Closed Universe Solution

The Friedman equation in a closed universe with CDM only is given by

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\Omega_{\rm m}a^{-3} + \frac{\Omega_{\rm k}}{a^2}\right) \quad (B1)$$

which can be written as

$$\frac{da}{dt} = H_0\sqrt{\Omega_{\rm m}a^{-1} + \Omega_{\rm k}} \quad (B2)$$

or

$$t = \int dt = \frac{1}{H_0}\int \frac{da}{\sqrt{\Omega_{\rm m}a^{-1}+\Omega_{\rm k}}}$$
$$= \frac{1}{H_0}\int \frac{a^{1/2}da}{\sqrt{\Omega_{\rm m}+\Omega_{\rm k}a}}$$
$$= \frac{1}{H_0\sqrt{\Omega_{\rm m}}}\int \frac{a^{1/2}da}{\sqrt{1+(\Omega_{\rm k}/\Omega_{\rm m})a}}$$

It is easier to first solve for the conformal time η, defined by $d\eta = dt/a$. We have

$$\eta = \int d\eta = \int \frac{dt}{a} = \frac{1}{H_0\sqrt{\Omega_{\rm m}}}\int \frac{a^{-1/2}da}{\sqrt{1+(\Omega_{\rm k}/\Omega_{\rm m})a}}$$

Changing $u^2 = -\Omega_{\rm k}/\Omega_{\rm m}a$, so that $u = \sqrt{-\Omega_{\rm k}/\Omega_{\rm m}}a^{1/2}$ and $du = 1/2\sqrt{-\Omega_{\rm k}/\Omega_{\rm m}}a^{-1/2}da$ we have

$$\eta = \frac{1}{H_0\sqrt{\Omega_{\rm m}}}2\sqrt{\frac{\Omega_{\rm m}}{-\Omega_{\rm k}}}\int \frac{du}{\sqrt{1-u^2}}$$
$$= \frac{2}{H_0\sqrt{-\Omega_{\rm k}}}\sin^{-1}u$$

or inverting

$$u = \sin(\theta/2) \quad (B3)$$
$$\theta = H_0\sqrt{-\Omega_{\rm k}}\eta \quad (B4)$$

Under the same change of variables ($a \to u$), since $u^2du = 1/2(-\Omega_{\rm k}/\Omega_{\rm m})^{3/2}a^{1/2}da$ the equation for t becomes

$$t = \frac{1}{H_0\sqrt{\Omega_{\rm m}}}\int \frac{a^{1/2}da}{\sqrt{1+(\Omega_{\rm k}/\Omega_{\rm m})a}}$$
$$= \frac{1}{H_0\sqrt{\Omega_{\rm m}}}2\left(\frac{\Omega_{\rm m}}{-\Omega_{\rm k}}\right)^{3/2}\int \frac{u^2du}{\sqrt{1-u^2}}$$
$$= \frac{2\Omega_{\rm m}}{H_0(-\Omega_{\rm k})^{3/2}}\int \frac{u^2du}{\sqrt{1-u^2}} \quad (B5)$$

or, changing $u = \sin(\theta/2)$, $du = \cos(\theta/2)d\theta/2$, and using $\cos(\theta) = \cos^2(\theta/2) - \sin^2(\theta/2) = 1 - 2\sin^2(\theta/2)$ we get

$$t = \frac{\Omega_{\rm m}}{H_0(-\Omega_{\rm k})^{3/2}}\int \sin^2(\theta/2)d\theta$$
$$= \frac{\Omega_{\rm m}}{2H_0(-\Omega_{\rm k})^{3/2}}\int 1-\cos(\theta)d\theta$$
$$= \frac{\Omega_{\rm m}}{2H_0(-\Omega_{\rm k})^{3/2}}(\theta - \sin(\theta))$$

Recall that $a = -(\Omega_{\rm m}/\Omega_{\rm k})u^2 = -(\Omega_{\rm m}/\Omega_{\rm k})\sin^2(\theta/2)$ so the parametric solution is

$$a = = \frac{\Omega_{\rm m}}{-2\Omega_{\rm k}}(1-\cos(\theta)) \quad (B6)$$
$$t = \frac{\Omega_{\rm m}}{2H_0(-\Omega_{\rm k})^{3/2}}(\theta - \sin(\theta)) \quad (B7)$$
$$\theta = H_0\sqrt{-\Omega_{\rm k}}\eta \quad (B8)$$

Appendix C: Top-hat perturbation predictions

A top-hat perturbation in a CDM universe evolves as a closed universe according to

$$\left(\frac{\dot y}{y}\right)^2 = \left(\Omega_{\rm cluster,ta}y^{-3} + \frac{-(\Omega_{\rm cluster,ta})}{y^2}\right) \quad (C1)$$

So, we can identify terms and immediately write the solution if we think this top-hat perturbation is actually merged in a ΛCDM background:

$$y = = \frac{1}{2}(1-\cos(\theta)) \quad (C2)$$
$$\tau = \frac{1}{2\sqrt{\Omega_{\rm cluster,ta}}}(\theta - \sin(\theta)) \quad (C3)$$
$$\tau = \frac{2}{3}\frac{x^{3/2}}{\sqrt{\Omega_{\rm m,ta}}} \quad \text{from background} \quad (C4)$$

Early on $\theta \sim \tau \sim y \sim x \sim 0$, to zeroth order.

At turn-around $\theta \sim \pi$, $y = 1$,$x = 1$. This allows us to get the relation

$$\tau_{\rm ta} = \frac{\pi}{2\sqrt{\Omega_{\rm cluster,ta}}} = \frac{2}{3\sqrt{\Omega_{\rm m,ta}}} \tag{C5}$$

or

$$\zeta = \frac{\Omega_{\rm cluster,ta}}{\Omega_{\rm m,ta}} = \left(\frac{3\pi}{4}\right)^2 \sim 5.5517 \tag{C6}$$

At collapse, $\theta = 2\pi$, $y = 0$, $\tau_{\rm c} = \pi/\sqrt{\Omega_{\rm cluster,ta}}$, so we can get $x_{\rm c}$

$$\tau_{\rm c} = \frac{\pi}{\sqrt{\Omega_{\rm cluster,ta}}} = \frac{2}{3}\frac{x_{\rm c}^{3/2}}{\sqrt{\Omega_{\rm m,ta}}} \tag{C7}$$

so

$$\begin{aligned} x_{\rm c} &= \left(\frac{3\pi}{2}\right)^{2/3}\left(\frac{\Omega_{\rm m,ta}}{\Omega_{\rm cluster,ta}}\right)^{1/3} \\ &= \left(\frac{3\pi}{2}\right)^{2/3}\left(\frac{4}{3\pi}\right)^{2/3} = 2^{2/3} = 1.5874 \end{aligned} \tag{C8}$$

Expanding sines and cossines

$$\sin\theta = \theta - \theta^3/6 + \theta^5/120 - ... \tag{C9}$$
$$\cos\theta = 1 - \theta^2/2 + \theta^4/24 - ... \tag{C10}$$

we have

$$y = \frac{1}{2}\left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + ...\right) \tag{C11}$$
$$\tau = \frac{1}{2\sqrt{\Omega_{\rm cluster,ta}}}\left(\frac{\theta^3}{6} - \frac{\theta^5}{120} + ...\right) \tag{C12}$$

or

$$y = \frac{\theta^2}{4}\left(1 - \frac{\theta^2}{12} + ...\right) \tag{C13}$$
$$\tau = \frac{\theta^3}{12\sqrt{\Omega_{\rm cluster,ta}}}\left(1 - \frac{\theta^2}{20} + ...\right) \tag{C14}$$

To leading order $\theta \sim (12\sqrt{\Omega_{\rm cluster,ta}}\tau)^{1/3}$. Iterating on the equation for τ itself, we have

$$\tau = \frac{\theta^3}{12\sqrt{\Omega_{\rm cluster,ta}}}\left(1 - \frac{(12\sqrt{\Omega_{\rm cluster,ta}}\tau)^{2/3}}{20} + ...\right) \tag{C15}$$

or

$$\theta \sim (12\sqrt{\Omega_{\rm cluster,ta}}\tau)^{1/3}\left(1 + \frac{(12\sqrt{\Omega_{\rm cluster,ta}}\tau)^{2/3}}{60} - ...\right) \tag{C16}$$

and the solution for y can be approximated by

$$\begin{aligned} y &\sim \frac{\theta^2}{4}\left(1 - \frac{\theta^2}{12} + ...\right) \\ &\sim \frac{(12\sqrt{\Omega_{\rm cluster,ta}}\tau)^{2/3}}{4}\left(1 + \frac{(12\sqrt{\Omega_{\rm cluster,ta}}\tau)^{2/3}}{30} - ...\right) \\ &\quad \text{x}\left(1 - \frac{(12\sqrt{\Omega_{\rm cluster,ta}}\tau))^{2/3}}{12} + ...\right) \\ &\sim \frac{(12\sqrt{\Omega_{\rm cluster,ta}}\tau)^{2/3}}{4}\left(1 - \frac{(12\sqrt{\Omega_{\rm cluster,ta}}\tau)^{2/3}}{20} - ...\right) \\ &\sim \left(\frac{3\sqrt{\Omega_{\rm cluster,ta}}\tau}{2}\right)^{2/3}\left(1 - \frac{1}{5}\left(\frac{3\sqrt{\Omega_{\rm cluster,ta}}\tau}{2}\right)^{2/3}\right) \\ &\sim x\zeta^{1/3}\left(1 - \frac{1}{5}x\zeta^{1/3}\right) \end{aligned} \tag{C17}$$

The first term is the leading solution $y \sim x\zeta^{1/3}$ and the second term is the linear theory correction to the relation between y and x. The full solution is the whole series.

The density contrast at any given time is

$$\delta = \frac{\Omega_{\rm cluster} - \Omega_{\rm m}}{\Omega_{\rm m}} \tag{C18}$$
$$= \frac{\Omega_{\rm cluster}}{\Omega_{\rm m}} - 1 \tag{C19}$$
$$= \frac{\Omega_{\rm cluster,ta}y^{-3}}{\Omega_{\rm m,ta}x^{-3}} - 1 \tag{C20}$$
$$= \zeta\left(\frac{x}{y}\right)^3 - 1 \tag{C21}$$

We can evaluate the linear theory density constrast by using Eq.(B25)

$$\frac{x\zeta^{1/3}}{y}|_{\rm lin} = 1 + \frac{1}{5}x\zeta^{1/3} \tag{C22}$$

or

$$\zeta\left(\frac{x}{y}\right)^3|_{\rm lin} = 1 + \frac{3}{5}x\zeta^{1/3} \tag{C23}$$

We have

$$\delta^{\rm lin} = \frac{3}{5}x\zeta^{1/3} = \frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3}x \tag{C24}$$

At turn-around $x = y = 1$, and, whereas the true overdensity is $\delta_{\rm ta} = \zeta - 1 \sim 4.5517$, the linear prediction is $\delta_{\rm ta}^{\rm lin} = 3/5\zeta^{1/3} \sim 1.0624$.

At collapse, $x_{\rm c} = 2^{2/3}$ and $y_{\rm c} = 0$. Formally $\delta_{\rm c} = \infty$, but linear theory predicts

$$\delta_{\rm c}^{\rm lin} = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3} \sim 1.68647 \qquad \text{(C25)}$$