so we may solve for one of the Ω's in terms of the others. Typically we choose the curvature, so

$$\Omega_k = 1 - (\Omega_{\rm m} + \Omega_{\rm r} + \Omega_\Lambda) \tag{2.92}$$

The Friedmann eq. can then be written as

$$E^2(t) = \frac{H^2(t)}{H_0^2} = \left[(1 - \Omega_{\rm m} - \Omega_{\rm r} - \Omega_\Lambda)\ a^{-2} + \Omega_{\rm m}\ a^{-3} + \Omega_{\rm r}\ a^{-4} + \Omega_\Lambda\right] \tag{2.93}$$

2.4 Solutions to the Friedmann Equation

Matter

For a Universe with only matter, we have

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= H_0^2 \Omega_{\rm m}\ a^{-3} \\ \left(\frac{da}{dt}\right)^2 &= H_0^2 \Omega_{\rm m}\ a^{-1} \\ a^{1/2}\ da &= H_0 \sqrt{\Omega_{\rm m}}\ dt \end{aligned} \tag{2.91}$$

integrating and setting imposing $a = 0$ at $t = 0$, we obtain

$$\frac{2}{3} a^{3/2} = H_0 \sqrt{\Omega_{\rm m}}\ t \tag{2.92}$$

or

$$a(t) = \left(\frac{3}{2}\sqrt{\Omega_{\rm m}} H_0 t\right)^{2/3} \qquad \text{(Matter Domination)} \tag{2.93}$$

For an Einstein-de Sitter (EdS) Universe, $\Omega_m = 1$ and the age of the Universe ($t =$today, with $a = 1$) is .

$$t = \frac{2}{3} H_0^{-1} \tag{2.94}$$

Radiation

For a Universe with only radiation, we have

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= H_0^2 \Omega_{\rm r}\ a^{-4} \\ \left(\frac{da}{dt}\right)^2 &= H_0^2 \Omega_{\rm r}\ a^{-2} \\ a\ da &= H_0 \sqrt{\Omega_{\rm r}}\ dt \end{aligned} \tag{2.92}$$

integrating, we obtain

$$\frac{1}{2} a^2 = H_0 \sqrt{\Omega_{\rm r}}\ t \tag{2.93}$$

or

$$a(t) = \left(2\sqrt{\Omega_r} H_0 t\right)^{1/2} \qquad \text{(Radiation Domination)} \tag{2.94}$$

Notice the Universe grows slower than in matter domination. For $\Omega_r = 1$ we have the age is

$$t = \frac{1}{2} H_0^{-1} \tag{2.95}$$

Cosmological Constant

For a Universe with only cosmological constant, we have

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= H_0^2 \Omega_\Lambda \\ \left(\frac{da}{dt}\right)^2 &= H_0^2 \Omega_\Lambda \; a^2 \\ \frac{da}{a} &= H_0 \sqrt{\Omega_\Lambda} \; dt \end{aligned} \tag{2.93}$$

integrating, we obtain

$$\ln(a) = H_0 \sqrt{\Omega_\Lambda} \; t + \text{const} \tag{2.94}$$

and we find

$$a(t) \propto \exp\left(\sqrt{\Omega_\Lambda} H_0 t\right) \qquad \text{(Cosmological Constant Domination)} \tag{2.95}$$

Notice the Universe grows exponentially fast in this case. In this case we cannot set $a = 0$ initially.

Curvature Dominated

For a Universe with only curvature, we have

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= H_0^2 \Omega_k a^{-2} \\ \left(\frac{da}{dt}\right)^2 &= H_0^2 \Omega_k \\ da &= H_0 \sqrt{\Omega_k} \; dt \end{aligned} \tag{2.93}$$

integrating, we obtain

$$a(t) = H_0 \sqrt{\Omega_k} \; t \qquad \text{(Curvature Domination)} \tag{2.94}$$

With $\Omega_k = 1$ the age is

$$t = H_0^{-1} \tag{2.95}$$

Matter + Curvature

For a Universe with both matter and non-zero curvature, we have

$$\begin{aligned}\frac{\dot{a}^2}{a^2} &= H_0^2\left[\Omega_{\rm m}\ a^{-3}+\Omega_k a^{-2}\right]\\ \left(\frac{da}{dt}\right)^2 &= H_0^2\left[\Omega_{\rm m}\ a^{-1}+\Omega_k\right]\end{aligned} \tag{2.94}$$

Therefore

$$\begin{aligned}\rightarrow H_0 dt &= \frac{da}{\sqrt{\Omega_{\rm m}a^{-1}+\Omega_k}}\\ &= \frac{1}{\sqrt{\Omega_{\rm m}}}\frac{a^{1/2}da}{\sqrt{1+(\Omega_k/\Omega_{\rm m})a}}\end{aligned} \tag{2.94}$$

It turns out that it is easier to first solve for the conformal time $d\eta = dt/a$. We have

$$\eta = \int d\eta = \int \frac{dt}{a} = \frac{1}{H_0\sqrt{\Omega_{\rm m}}}\int \frac{a^{-1/2}da}{\sqrt{1+(\Omega_k/\Omega_{\rm m})a}} \tag{2.95}$$

Let us assume we have a closed universe, i.e. $k > 0$ and therefore $\Omega_k = -k/H_0^2 < 0$. Then let $u^2 = -\Omega_k/\Omega_{\rm m}a$, so that $u = \sqrt{-\Omega_k/\Omega_{\rm m}}a^{1/2}$ and $du = 1/2\sqrt{-\Omega_k/\Omega_m}a^{-1/2}da$. We have

$$\begin{aligned}\eta &= \frac{1}{H_0\sqrt{\Omega_{\rm m}}}2\sqrt{\frac{\Omega_{\rm m}}{-\Omega_k}}\int\frac{du}{\sqrt{1-u^2}}\\ &= \frac{2}{H_0\sqrt{-k}}\sin^{-1}u\end{aligned} \tag{2.95}$$

or inverting

$$u = \sin(\theta/2) \tag{2.96}$$

$$\theta = H_0\sqrt{-\Omega_k}\eta \tag{2.97}$$

Under the same change of variables $(a \rightarrow u)$, $u^2du = 1/2(-\Omega_k/\Omega_{\rm m})^{3/2}a^{1/2}da$, and the equation for t becomes

$$\begin{aligned}t &= \frac{1}{H_0\sqrt{\Omega_{\rm m}}}\int\frac{a^{1/2}da}{\sqrt{1+(\Omega_k/\Omega_{\rm m})a}}\\ &= \frac{1}{H_0\sqrt{\Omega_{\rm m}}}2\left(\frac{\Omega_{\rm m}}{-\Omega_k}\right)^{3/2}\int\frac{u^2du}{\sqrt{1-u^2}}\\ &= \frac{2\Omega_{\rm m}}{H_0(-\Omega_{\rm k})^{3/2}}\int\frac{u^2du}{\sqrt{1-u^2}}\end{aligned}$$

Now changing $u = \sin(\theta/2)$, $du = \cos(\theta/2)d\theta/2$, we have

$$\begin{aligned}t &= \frac{2\Omega_{\rm m}}{H_0(-\Omega_{\rm k})^{3/2}}\int\frac{\sin^2(\theta/2)\cos(\theta/2)d\theta/2}{\sqrt{1-\sin(\theta/2)^2}}\\ &= \frac{\Omega_{\rm m}}{H_0(-\Omega_{\rm k})^{3/2}}\int\sin^2(\theta/2)d\theta\end{aligned} \tag{2.93}$$

Now using $\cos(\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta/2)$, we find

$$\begin{aligned} t &= \frac{\Omega_\mathrm{m}}{2H_0(-\Omega_\mathrm{k})^{3/2}} \int 1 - \cos(\theta) d\theta \\ &= \frac{\Omega_\mathrm{m}}{2H_0(-\Omega_\mathrm{k})^{3/2}} (\theta - \sin(\theta)) \end{aligned} \tag{2.92}$$

Finally, recall that $a = -(\Omega_\mathrm{m}/\Omega_k)u^2 = -(\Omega_\mathrm{m}/\Omega_k)\sin^2(\theta/2)$ so that we have a parametric solution for a cycloid

$$a = \frac{\Omega_\mathrm{m}}{-2\Omega_k} [1 - \cos(\theta)] \tag{2.93}$$

$$t = \frac{\Omega_\mathrm{m}}{2H_0(-\Omega_k)^{3/2}} [\theta - \sin(\theta)] \tag{2.94}$$

$$\theta = H_0 \sqrt{-\Omega_k} \eta \tag{2.95}$$

Notice that for small values of θ, we have

$$t \approx \frac{\Omega_\mathrm{m}}{12H_0(-\Omega_k)^{3/2}} \theta^3 \rightarrow \theta = \left(\frac{12H_0}{\Omega_\mathrm{m}}\right)^{1/3} (-\Omega_k)^{1/2} t^{1/3} \tag{2.96}$$

(2.97)

so that

$$a \approx \frac{\Omega_\mathrm{m}}{-4\Omega_k}\theta^2 = \frac{\Omega_\mathrm{m}}{-8^{2/3}\Omega_k}\left(\frac{12H_0}{\Omega_\mathrm{m}}\right)^{2/3} (-\Omega_k) t^{2/3} = \left(\frac{3}{2}\sqrt{\Omega_\mathrm{m}} H_0 t\right)^{2/3} \tag{2.98}$$

(2.99)

Matter + Cosmological Constant

For a Universe with both matter and cosmological constant, we have

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= H_0^2 \left[\Omega_\mathrm{m}\ a^{-3} + \Omega_\Lambda\right] \\ \left(\frac{da}{dt}\right)^2 &= H_0^2 \left[\Omega_\mathrm{m}\ a^{-1} + \Omega_\Lambda a^2\right] \end{aligned} \tag{2.98}$$

Therefore

$$\begin{aligned} \rightarrow H_0 dt &= \frac{da}{\sqrt{\Omega_\mathrm{m} a^{-1} + \Omega_\Lambda a^2}} \\ &= \frac{1}{\sqrt{\Omega_\mathrm{m}}} \frac{a^{1/2} da}{\sqrt{1 + (\Omega_\Lambda/\Omega_\mathrm{m}) a^3}} \qquad (\text{ Let } u^2 = \Omega_\Lambda/\Omega_\mathrm{m} a^3) \\ &= \frac{2/3}{\sqrt{\Omega_\Lambda}} \frac{du}{\sqrt{1+u^2}} \end{aligned} \tag{2.96}$$

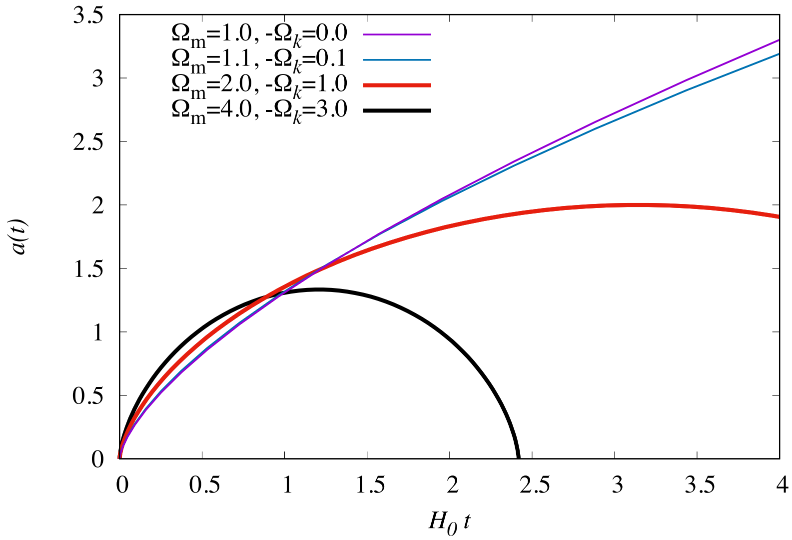

Figure 2.3: Scale factor $a(t)$ as a function of H_0t (cosmic time normalized by the Hubble time H_0^{-1}) for a universe with only matter and curvature, with different values of Ω_m and $\Omega_k = 1 - \Omega_m$. Since $\Omega_k = -k/H_0$, $\Omega_k < 0$ corresponds to a closed Universe ($k > 0$), which reaches a maximum turn-around scale factor $a_{\rm ta} = \Omega_{\rm m}/(-\Omega_k)$ at time $H_0t_{\rm ta} = (\pi/2)\Omega_{\rm m}/(-\Omega_k^{3/2})$. As $\Omega_k \to 0$, both $a_{\rm ta}, t_{\rm ta} \to \infty$ and the solution approaches that of a flat Universe without turn-around, i.e. $a(t) = (3/2\sqrt{\Omega_{\rm m}}H_0t)^{2/3}$.

integrating, we obtain

$$H_0t = \frac{2/3}{\sqrt{\Omega_\Lambda}}\sinh^{-1}(u) = \frac{2/3}{\sqrt{\Omega_\Lambda}}\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_{\rm m}}}a^{3/2}\right) \tag{2.97}$$

and

$$a(t) = \left(\frac{\Omega_{\rm m}}{\Omega_\Lambda}\right)^{1/3}\sinh^{2/3}\left(\frac{3\sqrt{\Omega_\Lambda}H_0}{2}t\right) \qquad \text{(Matter + Cosmological Constant)} \tag{2.98}$$

which reduces to the matter dominated solution for small t:

$$a(t) \approx \left(\frac{\Omega_{\rm m}}{\Omega_\Lambda}\right)^{1/3}\left(\frac{3\sqrt{\Omega_\Lambda}H_0}{2}\right)^{2/3}t^{2/3} = \left(\frac{3}{2}\sqrt{\Omega_{\rm m}}H_0t\right)^{2/3} \tag{2.99}$$

and recovers the cosmological constant solution for large t [$\sinh(at) = (e^{at} - e^{-at})/2 \to e^{at}/2$]

$$a(t) \propto \left[\exp\left(\frac{3\sqrt{\Omega_\Lambda}H_0}{2}t\right)\right]^{2/3} = \exp\left(\sqrt{\Omega_\Lambda}H_0t\right) \tag{2.100}$$

2.5 Photon Geodesics and Energy

Recall we defined the 4-momentum $P^\alpha = (E, \mathbf{p})$ for a massive particle as

$$P^\alpha = m\frac{dx^\alpha}{d\tau} \tag{2.101}$$

But for a massless particle (e.g. a photon), both $m = d\tau = 0$, so we need an alternative definition. We define it then with respect to a general implicit parameter λ along the particle trajectory:

$$P^\alpha = \frac{dx^\alpha}{d\lambda} = \left(\frac{dE}{d\lambda}, \frac{d\mathbf{p}}{d\lambda}\right) \tag{2.102}$$