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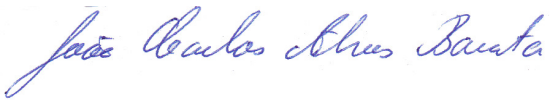
Sobre o Problema de Localizabilidade de sistemas quânticos relativísticos

Ivan Romualdo de Oliveira



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On the Localizability Problem of relativistic quantum systems

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“To myself I am only a child playing on the beach, while vast oceans of truth lie undiscovered before me”

— Isaac Newton

Abstract

In this work, we study the Localizability Problem for relativistic quantum systems. We do it on two fronts. In the first part of this thesis, we extend Newton-Wigner's localization approach to homogeneous globally hyperbolic spacetimes, defining generalized (local) Newton-Wigner position operators. We also give criteria to classify which unitary representations of the spacetime isometry group give origin to localizable representations, showing that the stabilizer group of the spatial isometry group plays a fundamental role.

In the second part, we present a novel approach to the Localizability Problem, utilizing techniques from the Modular Theory of Tomita-Takesaki. We argue that position measurements must follow logical principles, incorporated in a mathematical structure referred to as a *logic*. The core idea is to include the causality structure of Minkowski spacetime in this logic so that the causality problems inherent in Newton-Wigner's approach are solved. We do it through the Modular Localization map [1] for arbitrary massive representations of $\mathcal{P}_+$. Our main contribution is the construction of a (quasi-) probability measure on the logic structure of spacetime for each algebraic state (which we interpret as the probability of detection of the system in these spacetime regions), and of a position observable in the logic-theoretic sense. Additionally, we compare our new approach with Newton-Wigner's localization, showing they are approximate in certain regimes.

Keywords: Localizability Problem; Newton-Wigner Localization; Modular Localization; Newton-Wigner operators; Homogeneous globally hyperbolic spacetimes.

Resumo

Neste trabalho, estudamos o Problema de Localizabilidade para sistemas quânticos relativísticos, atacando o problema em duas frentes. Na primeira parte desta tese, estendemos a abordagem de localização de Newton-Wigner para espaços-tempo homogêneos globalmente hiperbólicos, definindo operadores generalizados de Newton-Wigner locais. Também fornecemos critérios para classificar quais representações unitárias do grupo de simetria do espaço-tempo dão origem a representações localizáveis, mostrando que o grupo estabilizador do grupo de simetria espacial desempenha um papel fundamental.

Na segunda parte, apresentamos uma nova abordagem para o Problema da Localizabilidade, baseada em técnicas provenientes da Teoria Modular de Tomita-Takesaki. Argumentamos que as medições de posição devem seguir princípios lógicos, incorporados em uma estrutura matemática chamada de *lógica*. A ideia principal é incluir a estrutura de causalidade do espaço-tempo de Minkowski nessa lógica para que os problemas de causalidade na abordagem de Newton-Wigner sejam resolvidos. Fazemos isso por meio do mapa de Localização Modular [1] para representações massivas arbitrárias de $\mathcal{P}_+$. Nossa principal contribuição é a construção de uma (quasi-) medida de probabilidade sobre a estrutura lógica do espaço-tempo para cada estado algébrico (que interpretamos como a probabilidade de detecção do sistema nessas regiões do espaço-tempo), e de um observável de posição no sentido lógico-teórico. Além disso, comparamos nossa nova abordagem com a localização de Newton-Wigner, mostrando que elas são aproximadas em certos regimes.

Palavras-chave: Problema da Localizabilidade; Localização de Newton-Wigner; Localização Modular; Operadores de Newton-Wigner; Espaços-temporais globalmente hiperbólicos homogêneos.

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Chapter 1

Introduction

Ihr naht euch wieder, schwankende Gestalten, die früh sich einst dem trüben Blick gezeigt. Versuch' ich wohl, euch diesmal festzuhalten? Fühl' ich mein Herz noch jenem Wahn geneigt? Ihr drängt euch zu! nun gut, so mögt ihr walten, wie ihr aus Dunst und Nebel um mich steigt.

J. W von Goethe

Quantum Field Theory and its offspring theory, modern Particle Physics, have been incredibly successful, both theoretically and experimentally. These theories were at the center of the scientific revolution of the past century; together, they redefined our paradigms of what is matter, and pushed the limits of human knowledge to a completely new level, by demanding unprecedentedly complex experiments and deep theoretical and mathematical foundations.

Quantum Field theory is probably among the most sophisticated and profound products of the human mind and has many faces and possible approaches, where none of them could be classified as better than the other, as they have very different goals. On one side we have a pragmatical perspective, which aims at computing certain observables, without entering many mathematical details, and measuring them in accelerators and particle detectors. This is the path of ordinary Particle Physics. In the other extreme, we have mathematically rigorous quantum field theories, which aspire to establish the foundations of the necessary mathematical structures, but lacking in predictive power when compared to the first approach.

By getting deep into the mathematical structure, it is a remarkable and mysterious fact that we simultaneously (and necessarily) deepen into the laws of Physics. At this trench, deep into the ocean of scientific knowledge, physical, mathematical, and philosophical questions can hardly be distinguished, and are often one and the same. One can think of any physical theory as having two components:

- **The experimental component** aims to develop techniques, instruments, and technologies

to probe the physical system. This involves preparing the system in a way that allows specific and objective questions to be answered, without ambiguity and free from external influences.

- **The theoretical component** aims to formulate a concise and as complete as possible theory that explains why the results of the experiments are as they are, and to formulate a *model of reality*.

The burden of fabricating our paradigmatic view of the universal laws is carried on the courageous hands of the second, while the solid feet of the first make contact with the real world. Let us start by giving a very broad perspective on physical measurements. In the measurement process, we can distinguish at least four parts of it:

1. The object on which the measurement is performed, that is, the *physical system*. We will use Greek letters as symbols for these objects (α_1, α_2, etc).

2. The measuring apparatus, that is, the instruments used to perform the measurement. We will use the symbols Q_1, Q_2, etc to refer to these objects.

3. The observer performing the measurement.

4. The environment in which the measurement is being done.

It is part of the scientific procedure to minimize the effects of 3 and 4 (although some argue that this cannot be done completely. See [2], for example), such that we can focus on the first two. When measuring a physical system α with an apparatus Q, the result is a real number. We use lowercase letters p, q, etc., to refer to these results. A measurement is usually repeated a large number of times, such that a statistical treatment is in place. If we measure a system α with the measurement apparatus Q, and the result p appears n_p times out of a total of N runs, we can obtain the ratio n_p/N. For a very large N, it is reasonable to assume that this ratio converges to a specific value:

$$w_\alpha^Q(p) = \lim_{N\to\infty} \frac{n_p}{N}.$$

The quantity $w_\alpha^Q(p)$ is then interpreted as the *probability* of obtaining the value p when measuring the system α with the apparatus Q.

In this broad perspective on physical measurements, we can define two essential concepts which appear in any area of Physics: *states* and *observables*. Let us define these concepts in this broad perspective on physical measurements. Let us denote the set of all objects $\alpha_1, \alpha_2, ...$

as Σ, and the set of all measuring instruments $Q_1, Q_2, ...$ as $\mathcal{A}$. We say that two objects α_1 and α_2 are *equivalent*, $\alpha_1 \sim \alpha_2$, if there are no instruments that can distinguish between them in the sense that:

$$w^Q_{\alpha_1}(p) = w^Q_{\alpha_2}(p)$$

for every $Q \in \mathcal{A}$ and every $p \in \mathbb{R}$. This is indeed an equivalence relation since it is clearly reflexive, symmetric, and transitive. Hence, Σ is partitioned into equivalence classes. We denote the equivalence class of an object α as $[\alpha]$. Likewise, we say that two measurement instruments Q_1 and Q_2 are equivalent, $Q_1 \sim Q_2$, if:

$$w^{Q_1}_{\alpha}(p) = w^{Q_2}_{\alpha}(p)$$

for every $\alpha \in \Sigma$ and every $p \in \mathbb{R}$. We denote the equivalence class of Q as $[Q]$. With these quantities, we follow Araki's [3] definition of states and observables.

Definition 1.1. We define a **(experimental) state** as an equivalence class $[\alpha]$, where $\alpha \in \Sigma$, and an **(experimental) observable** as an equivalence class $[Q]$, where $Q \in \mathcal{A}$.

We refer to these objects as experimental since they are defined exclusively in terms of the results of experiments. In contrast, there will be theoretical observables, which are defined in terms of the model of the theory, as we discuss below. Note that, so far, our analysis is completely general, and doesn't make reference to any specific theory or model. These concepts refer to the results of experiments, exclusively, and we are, therefore, only in the realm of the experimental component discussed above. When specific theories come into play, the theoretical component formulates a model of the theory, which includes a "translation" of the above concepts into the mathematical structure of the model. Let us attempt to give a heuristic view of what we mean by that. Suppose we have a specific theory (such as Classical Mechanics, Quantum Mechanics, General Relativity, etc), which we will refer to by $\mathfrak{T}$. We will denote the experimental component of this theory as Exp($\mathfrak{T}$), and the theoretical component as The($\mathfrak{T}$). The first is composed of triples $([\alpha], [Q], w^{[Q]}_{[\alpha]}) \in \text{Exp}(\mathfrak{T})$, while the later is written in mathematical language and is composed of corresponding triples in terms of the mathematical structures involved. The goal of a **model** is to construct The($\mathfrak{T}$) and a map $\mathcal{T} : \text{Exp}(\mathfrak{T}) \to \text{The}(\mathfrak{T})$:

$$\text{Exp}(\mathfrak{T}) \ni ([\alpha], [Q], w^{[Q]}_{[\alpha]}) \overset{\mathcal{T}}{\longmapsto} (\mathcal{O}_{[\alpha]}, \mathcal{O}_{[Q]}, \mu^{\mathcal{O}_{[Q]}}_{\mathcal{O}_{[\alpha]}}) \in \text{The}(\mathfrak{T}), \tag{1.1}$$

where $\mathcal{O}_{[\alpha]}, \mathcal{O}_{[Q]}$ are elements of the mathematical structure of the theory modeling $[\alpha]$ and

$[Q]$, which we refer to as the **theoretical states** and **theoretical observables**, and $\mu^{\mathcal{O}_{[Q]}}_{\mathcal{O}_{[\alpha]}}$ is a "probability measure", defined in some suitable structure in $\text{The}(\mathfrak{T})$. Of course, the theoretical part should not only contain the triples on the right-hand side, but also the way they relate and interfere with each other. A good model should also provide an "inverse" for the map $\mathcal{T}$, meaning that it is always consistent with experimental outcomes. Finally, a solid model must also determine its own *domain of validity*, which should be in agreement with other models and theories when their domains intersect.

For example, let $\mathfrak{T}_{QT} \coloneqq$ non-relativistic quantum theory. Then, the map $\mathcal{T}_{QT}$ is given by:

$$\Sigma \ni [\alpha] \mapsto \psi \in \mathcal{H} \tag{1.2}$$

$$\mathcal{A} \ni [Q] \mapsto A = A^* = \int_{\sigma(A)} \lambda dE(\lambda) \in \mathscr{L}(\mathcal{H}) \tag{1.3}$$

$$w^Q_\alpha \mapsto \mu^A_\psi(B) = \langle \psi, E(B)\psi \rangle, \quad B \in \mathcal{B}(\sigma(A)), \tag{1.4}$$

where $\mathcal{H}$ is a Hilbert space, $\mathscr{L}(\mathcal{H})$ is the set of linear operators acting on it, and μ^A_ψ is a probability measure on $\mathcal{B}(\sigma(A))$. A complete understanding of a model should provide a $\mathcal{T}$ that can map every triple in $\text{Exp}(\mathfrak{T})$ to triples in $\text{The}(\mathfrak{T})$. That **this is not the case for relativistic quantum theory** is the starting point of this thesis. More specifically, let $\mathfrak{T}_{RQT} \coloneqq$ relativistic quantum theory, which we think about as the intersection of special relativity theory and quantum theory. Let $[Q]_{\text{pos}}$ denote the observable consisting of instruments that measure the system's position. The main problem that we attack in this work is:

The Localizability Problem *is the absence of a map* $\mathcal{T}_{RQT}$ *such that* $\mathcal{T}_{RQT}\left([\alpha], [Q]_{pos}, w^{[Q]_{pos}}_{[\alpha]}\right) \in \text{The}(\mathfrak{T}_{RQT})$ *is well defined and fully compatible with relativistic principles.*

Observe that this is not a problem in non-relativistic quantum theory. For instance, consider a single quantum particle moving in one dimension. The Hilbert space is given by $L^2(\mathbb{R}, dx)$, and we have that:

$$\mathcal{T}_{QT}\left([\alpha], [Q]_{\text{pos}}, w^{[Q]_{\text{pos}}}_{[\alpha]}\right) = \left(\psi, Q, \mu^Q_\psi\right) \in \text{The}(\mathfrak{T}_{QT}),$$

where Q is the self-adjoint operator:

$$(Q\psi)(x) = x\psi(x)$$
$$D(Q) = \left\{ \psi \in L^2(\mathbb{R}, dx) \Big| \int_{\mathbb{R}} |(Q\psi)(x)|^2 dx < \infty \right\},$$

and the probability measure is:

$$\mu_\psi(B) \doteq \langle \psi, \chi_B \psi \rangle, \quad B \in \mathcal{B}(\sigma(Q)) = \mathcal{B}(\mathbb{R}),$$

where χ_B is the characteristic function of B. Analogously, the first natural idea is to find a self-adjoint operator for the position observable for relativistic quantum systems. However, when the causal structure of spacetime is included, things get way more complicated. Despite the many proposed solutions, none have been unanimously accepted by the scientific community. It is, from our point of view, a surprising fact that this problem has been so much overlooked by the majority of theoretical physicists, even though the position observable in non-relativistic Quantum Mechanics played such a central role. In Ruijsenaars words [4]: "This is so in spite of the fact that the approximate determination of the position of a physical system may be regarded as the most fundamental measurement of all". A possible explanation for that is the advent and the overwhelming success of Quantum Field Theory, where the quantum field operators became the protagonists, a change of paradigm took place, and a position operator (or something similar) was no longer seen as essential.

The first attempt to define a relativistic self-adjoint position operator was made in 1949 by Newton and Wigner in their seminal paper [5]. The main idea of this work is to give criteria to understand which states of an elementary system are *localized* in a given region of space. These localized states are then shown to be (generalized) eigenvectors of a self-adjoint operator Q^{NW}, nowadays known as the *Newton-Wigner position operator*. Hence, Newton-Wigner's proposed solution is:

$$\mathcal{T}_{RQT}\left([\alpha], [Q]_{\text{pos}}, w_{[\alpha]}^{[Q]_{\text{pos}}}\right) = \left(\psi, Q^{NW}, \mu_\psi^{NW}\right), \tag{1.5}$$

where ψ is now an element of a relativistic Hilbert space, and Q^{NW} is written as:

$$Q^{NW} = \int_{\mathbb{R}} x dP^{NW}(x),$$

where x is a spatial coordinate function, P^{NW} is a projection-valued measure coming from

Newton-Wigner's localization scheme (see Definition 3.2), and the probability measure is:

$$\mu_\psi^{NW}(B) = \langle \psi, P^{NW}(B)\psi \rangle, \quad B \in \mathcal{B}(\sigma(Q^{NW})) = \mathcal{B}(\mathbb{R}).$$

Therefore, Newton-Wigner's approach is equal in nature to the non-relativistic case but adapted to Hilbert spaces describing relativistic systems.

The underlying endeavor in this approach is the attempt to give a particle ontology to relativistic quantum systems. This enterprise is motivated by the experimentally observed property of elementary systems being observed as "particles", that is, individual, countable, extremely localized entities. Indeed, this raises the question of *what precisely is a particle?* or, in other words, *which are the states that we observe as particles?* This is a question that certainly has some connections with the Localizability Problem, but they are not the same, and we will not try to answer it directly in this work. For the particle ontology discussion, we refer the interested reader to [6–8]. For the notion of states describing particles in the context of Algebraic Quantum Field Theory, we refer to [9–17].

Following Newton-Wigner, Wightman published a paper in 1962 [18], where he emphasized the role of *systems of imprimitivity* (see Chapter 2) as the main mathematical object in the definition of the Newton-Wigner localization. With that, Newton-Wigner localization received a solid mathematical formulation. Nonetheless, in the following years, many realized that there were severe incompatibilities with the causal structure in this approach. Namely, if in a given time, a state is localized in a compact region of space, it can then propagate outside the future light cone of this region, as first realized by Fleming in [19], and later by Hegerfeldt in [20]. Many results in this direction appeared in the next decades, basically ruling out Newton-Wigner's approach as a solution to the Localizability Problem. Furthermore, numerous so-called *No-go Theorems* were also proven, excluding the possibility of using spectral measures and self-adjoint operators to define localizability. Among them, we have Malament's Theorem [21], Hegerfeldt's Theorem [20, 22], and strengthened versions of these theorems that followed, such as Halvorson's Theorem [6, 23] (see also Section 3.1 below).

A new approach was necessary. The most natural one was to weaken the definition of a spectral measure and require only a *positive operator-valued measure* (POVM) instead. In this case, *there is no position operator*, and the proposed solution is:

$$\mathcal{T}_{RQT}\left([\alpha], [Q]_{\text{pos}}, w_{[\alpha]}^{[Q]_{\text{pos}}}\right) = \left(\psi, E, \mu_\psi^E\right),$$

where ψ is an element of the relativistic Hilbert space, $\mathcal{B}(\mathbb{R}) \ni B \mapsto E(B)$ is a POVM, and the

probability measure is:

$$\mu_\psi^E(B) = \langle \psi, E(B)\psi \rangle, \quad B \in \mathcal{B}(\mathbb{R}).$$

Many works were done in this direction: Jauch [24, 25] pioneered this idea to describe position measurements of the photon, followed by Angelopoulos, Bayen, Flato [26], and Kraus [27]. For the localizability of massive particles within this approach, we refer to the many nice works by Castrigiano [28–30], Beck's book [31], and the recent papers by Moretti and De Rosa [32, 33]. Some forms of the mentioned No-go Theorems were also extended to this framework [23, 34].

Let us now explain the objectives and achievements of this work. The thesis is divided into Part I and Part II, with its respective goals:

1. **Part I**: In this part, the main goal is to formulate Newton-Wigner localization on homogeneous globally hyperbolic spacetimes. As previously mentioned, the primary mathematical object in Newton-Wigner localization on Minkowski spacetime is a system of imprimitivity, which is closely related to induced representations, as discussed in Chapter 2. However, the techniques used to construct and study these objects can be extended to much more general contexts, including curved spacetimes. The aim is to understand the essence of Newton-Wigner localization by giving up on the numerous symmetries of Minkowski spacetime and determining the bare minimal requirements on the background spacetime that allows this type of localization. Even though the map $\mathcal{T}_{RQT}$ is basically the same as in equation (1.5), and we don't expect the causality problems in Newton-Wigner's approach to be solved, we consider this endeavor to be important for the following reason: many of the new ideas to solve the Localizability Problem take inspiration in the Newton-Wigner approach, and they all somehow "orbit" around it, being approximate to Newton-Wigner in different ways. Hence, it is essential to have a complete understanding of this formalism. In Part II we propose a new approach to the problem, and also in our case, there is a close relation with Newton-Wigner, as exposed in Section 7.3.1.

   Our second objective here is to classify which representations of the spacetime isometry group give origin to a localizable system. The idea is to give a similar classification to what was done in Minkowski spacetime, where it was proved (see Theorem 3.4) that only massive representations are localizable on the spatial Cauchy surface. Our method only applies to induced representations of the spacetime isometry group, and we give criteria to determine when it is localizable. If this group is of the form of a regular semi-

direct product, then we are able to classify any unitary representation. We show that the stabilizer group of the spatial isometry group has a decisive role in this classification. Note that, in this context, there is no notion of massive representations, as this appeared as a consequence of the representation theory of the Poincaré group. We also investigate the problem of the uniqueness of the notions of localizability by dividing all possible ones into equivalence classes (the so-called Thompson components) and giving a physical interpretation of these.

Finally, in the remaining sections of Chapter 4, we explore applications and consequences of our generalized notion of localization. We examine states that follow geodesics (Section 4.1), decompositions of the representation space induced by the position operator (Section 4.2), and the effects of perturbations on Minkowski spacetime on the Newton-Wigner operators (Section 4.3).

2. **Part II**: In this part, we propose a new approach to the Localizability Problem on Minkowski spacetime. The idea is that the way in which position measurements are performed, that is, the way in which the instruments $[Q]_{\text{pos}}$ are used, inherently follows a *logic*. The word logic here has a precise mathematical meaning (Definition 5.5). This structure appears in different contexts and is also hidden behind Newton-Wigner's localization approach. The spacetime itself has a logic structure (Section 5.4), and our main idea is to *incorporate this logic in the way the instruments* $[Q]_{pos}$ *are used*. This automatically solves all the causality problems in Newton-Wigner localization, as the causal structure is incorporated from the start. However, this implementation is not trivial, and we prove a No-go result in Theorem 7.4, showing how delicate things are. To achieve our goal, we use the mathematical tools of Modular Localization (Section 6.3), which are based on the mathematical structure of the Modular Theory of Tomita-Takesaki, a theory that has many profound applications in Quantum Field Theory [35]. In our new approach, the map $\mathcal{T}_{RQT}$ is:

$$\mathcal{T}_{RQT}\left([\alpha],[Q]_{\text{pos}},w_{[\alpha]}^{[Q]_{\text{pos}}}\right)=\left(\omega,x_q^{ML},\mu_\omega\right),$$

where the ingredients are the following:

- The mathematical structures in The$(\mathfrak{I}_{RQT})$ necessary to define the triple in the right-hand side are: the spacetime logic $\mathcal{L}_\Sigma$ (Definition 5.7), a massive representation U of $\mathcal{P}_+$ on a Hilbert space $\mathcal{H}$, and the modular localization map (equation (6.16)), which implements the spacetime logic in the representation space.

- ω is an algebraic state (Definition 6.10) on $B(\mathcal{H})$.
- x_q^{ML} is an observable (in the logic-theoretic sense 5.6) defined on the logic $\mathcal{L}_\Sigma$. Check equation (7.21).
- μ_ω is a (quasi)-probability measure (in the logic-theoretic sense 5.8) defined on the logic $\mathcal{L}_\Sigma$. Check Definition 7.12.

An interesting point is that the probability measures μ_ω can only be approximate (in a way explained in Corollary 7.11). We discuss the physical consequences of this fact in Section 7.4. Moreover, we give an explicit example in Section 7.3 for Minkowski spacetime in 1+1 dimensions, and in Subsection 7.3.1 we compare our new approach with Newton-Wigner localization.

The organization of the thesis is the following: in both Part I and Part II, the first two chapters give background material, and no new results are presented: in Chapter 2 we present the main definitions and result in the theory of induced representations and systems of imprimitivity, which are the central mathematical objects in the definition of Newton-Wigner localization; in Chapter 3, we provide a review on Newton-Wigner localization in flat spacetime, presenting the results in a directed way, preparing the reader for the more abstract constructions in the following chapter; in Chapter 5 we present the mathematical theory of logics and its connections with measurements in Physics; in Chapter 6, we give background material in Quantum Field Theory and its connections with the Modular Theory of Tomita-Takesaki. In the third chapter in each part, we present our new results. Finally, in Appendix A we present basic material in Functional Analysis, that may serve as an auxiliary tool for some readers.

Part I

Newton-Wigner Localization on Homogeneous Globally-Hyperbolic Spacetimes

Chapter 2

Induced Representations and Systems of Imprimitivity

Mathematics has sort of inevitable structure which unfolds as one studies it perceptively. It is as though it were already there and one had only to uncover it.
G. W. Mackey

As discussed in the introduction 1, the central mathematical object in the definition of localizability in the sense of Newton-Wigner is a *System of Imprimitivity* (SI): (E, U). This object is composed of a spectral measure E (Definition A.10) and a unitary representation (Definition 2.14) U of some relevant group in such a way that the spectral measure transforms covariantly with respect to this representation (see Definition 2.34 for a precise definition). It turns out that this apparently simple mathematical object carries a very rich theory behind it, with many applications in mathematical physics. In particular, there is a deep connection between SI's and induced representations, which is made precise by a Theorem due to Mackey (Theorem 2.36).

One of the first and most important applications of induced representations in Physics is due to Wigner [36]. In this work, he classified all unitary, irreducible representations of the Poincaré group using the (at that time, heuristic) method of induction. It was Mackey [37–39] who later realized that the heuristic methods of Wigner could be made mathematically precise and not just for the Poincaré group, but rather for a large class of groups, namely, locally compact topological groups.

As pointed out in the Introduction, when it comes to the Localizability Problem in Newton-Wigner's approach, it was Wightman who noticed that their whole program could be synthesized in this unique mathematical object (for a relativistic quantum system defined on Minkowski spacetime). The starting point of the work described in Part I of this Thesis is to realize that the general techniques developed by Mackey allow us to extend Wightman's approach to ho-

mogeneous globally hyperbolic spacetimes, where U is a unitary representation of the (spatial) isometry group.

The goal of this chapter is to introduce the necessary mathematical objects and prepare the reader for the results shown in chapter 4. For that, we need to introduce Mackey's theory in its full generality. We start studying basic facts and definitions about homogeneous G-spaces and representations of locally compact topological groups. Next, we study induced representations and their connections with SI's.

2.1 Topological Groups and Homogeneous spaces

Let us start with some basic definitions. The goal of these first pages is to prepare the reader for the core concepts of topological groups and homogeneous spaces, which constitute the first basic mathematical structures necessary for our work. We try to give a comprehensive exposition in terms of definitions and basic structural results, without delving into proofs, since these can be found in good books such as [40–42].

Definition 2.1. Let X be an arbitrary set and τ a collection of subsets of X satisfying:

1. $\emptyset \in \tau$.
2. If $A \in \tau$ and $B \in \tau$, then $A \cap B \in \tau$.
3. If $A_i \in \tau$ for an arbitrary index set I, then $\bigcup_{i \in I} A_i \in \tau$.

Then, we call the pair $\{X, \tau\}$ a **topological space**. Every element in τ is called an **open set**, and a **neighborhood** of an element $x \in X$ is an arbitrary set that contains an open set containing x.

The definition of a topology on a set is critical because it defines two fundamental notions, namely, *convergence* and *continuity*.

Definition 2.2. Let $\{x_n\}$, $x_n \in X$, be a sequence. We say that this sequence **converges** to an element $x \in X$ if, for each open set $A \in \tau$ containing x, there is an integer N such that for $n \geq N$, $x_n \in A$. A mapping $f : X \to Y$ from a topological space $\{X, \tau\}$ into another topological space $\{Y, \tau'\}$ is **continuous** if for each open set $A \in Y$ it follows that $f^{-1}(A)$ is an open set in X. A continuous one-to-one mapping is an **homeomorphism** if f^{-1} is also continuous.

For the same arbitrary set X, changing the topology could change completely these two above-defined concepts. For example, in the *trivial topology* defined as $\tau = \{\emptyset, X\}$ every

sequence is convergent to each point $x \in X$, while in the *discrete topology*, defined as the set of all subsets of X, a sequence converges to x if, and only if, $x_n = x$ for every $n \geq N$, for some $N \in \mathbb{N}$. In particular, a space where the limiting point of a convergent sequence is not unique could be very hard to handle. To solve this, we have the following definition.

Definition 2.3. Let $\{X, \tau\}$ be a topological space. We say that this space is a **Hausdorff space** if for every pair of distinct points x_1 and x_2 there exists neighborhoods A_1 and A_2 such that $x_1 \in A_1$, $x_2 \in A_2$, and such that $A_1 \cap A_2 = \emptyset$.

It follows that in every Hausdorff space, every convergent sequence has a unique limit (see [43]). The next important concept, which will be fundamental in the representation theory of topological groups developed in the next sections, is that of compactness and local compactness.

Definition 2.4. We say that a Hausdorff space X is **compact** if every collection of open sets, whose union covers X, contains a finite subcollection, whose union covers X. We say that it is **locally compact** if each point has a compact neighborhood.

As an example, it is easy to see that a topological space with the discrete topology is compact if, and only if, it is finite: if it is infinite, the cover $X = \bigcup_{x \in X} \{x_i\}$ does not contain a finite subcover. However, every discrete space is locally compact. The compactness of a topological Hausdorff space is clearly preserved under homeomorphisms, but also under continuous transformations: if X is compact and Y is Hausdorff, and if there is a continuous transformation between these spaces, then Y is also compact.

Moving forward, we now merge the two very important notions of topological spaces and abstract groups into the concept of topological groups. First, let us recall what is an abstract group.

Definition 2.5. A **group** G is a non-empty set with a binary operation $\cdot : G \times G \to G$ called *product*, and a bijective unary operation "$^{-1}$" called *inverse* such that:

1. *Associativity:* For every $a, b, c \in G$, it is true that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

2. *Neutral element:* There exists an unique element $e \in G$, called the *identity*, such that $e \cdot g = g \cdot e = g$ for all $g \in G$.

3. For each $g \in G$, there exists an unique $h \in G$, called the *inverse*, such that $h \cdot g = g \cdot h = e$.

A topological group is nothing more than a topological space such that the algebraic group operations are compatible with the topological ones. More precisely.

Definition 2.6. A **topological group** is a set G such that:

1. G is an abstract group
2. G is topological space.
3. the function $a \mapsto a^{-1}$ is a continuous map from $G \to G$ and the function $(a, b) \mapsto a.b$ is a continuous function from $G \times G \to G$.

We say that G is a compact (resp. locally compact) topological group if G is compact (resp. locally compact) as a topological space.

Accordingly, operations on a topological group should be compatible with both algebraic and topological structures. For example, a *topological subgroup* $H \subseteq G$ should not only be a subgroup of G, but it must also be a closed subset (otherwise, it would not be a topological space). Neither the algebraic nor the topological properties alone are sufficient to characterize completely a topological group. For instance, the group consisting of all matrices of the form:

$$\begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}, \quad a, b \in \mathbb{R}$$

and the group of all matrices of the form:

$$\begin{bmatrix} e^a & b \\ 0 & e^{-a} \end{bmatrix}, \quad a, b \in \mathbb{R}$$

are both homeomorphic to $\mathbb{R}^2$ and, hence, topologically equivalent. However, the first group is abelian while the second is not. On the other hand, we could construct two different topological groups by giving distinct topologies to one single group. Hence, we can only say that two topological groups are equivalent or, more precisely, *isomorphic*, when there is a one-to-one correspondence which is both a group isomorphism and a homomorphism.

It is easy to be convinced that the notion of a topological group is very important in Physics since it includes all matrix groups (with the topology coming from $\mathbb{R}^n$) and all Lie groups. It might even look like this is a too broad and abstract definition since it includes so many examples, and it wouldn't influence so much in the actual application to problems in Physics. Nonetheless, for the specific problem that we are aiming to contribute (namely, the localization of relativistic quantum systems), these properties will be especially important, as will be clear in later sections. Our next important concept is that of a homogeneous space.

Definition 2.7. A topological space $\{X, \tau\}$ is **homogeneous** if for any pair of elements $x, y \in X$ there exists a homeomorphism f of $\{X, \tau\}$ onto itself such that $f(x) = y$.

As a first example, we notice that every topological group is homogeneous. Let $x, y \in G$ be any two elements. Then:

$$y = ax, \quad a \doteq yx^{-1}$$

and, because of the uniqueness of the inverse and continuity of the multiplication, this map is a homeomorphism. It is clear that a homogeneous space has a much simpler structure when compared to a general topological space since we can study the space in the vicinity of a point and this can be carried to the rest by homeomorphisms. In the above example, the homeomorphisms are given by the left translations which are, in some sense, the group "acting" on itself. We can generalize this notion to topological groups acting on general topological spaces as follows.

Definition 2.8. Let Γ be a topological space and G a topological group. Then, we say that G **acts by the left** on Γ if

1. For each $g \in G$ there is associated a homeomorphism $\gamma \mapsto g\gamma$ of Γ to Γ.
2. The identity $e \in G$ is associated with the identity homeomorphism of Γ.
3. The mapping $(g, \gamma) \mapsto g\gamma$ of $G \times \Gamma$ into Γ is continuous.
4. $(g_1 g_2)\gamma = g_1(g_2\gamma)$ for every $g_1, g_2 \in G$ and $\gamma \in \Gamma$.

The topological space Γ is then called a G**-space**.

We could, similarly, define the **right group action**. The difference would be in the order in which an element gh, for $g, h \in G$, acts on an element $\gamma \in \Gamma$: for the left action, h acts first, while for the right action, g acts first. Let us distinguish some important types of actions: we say that G acts **transitively** on Γ if for every pair of points $\gamma_1, \gamma_2 \in \Gamma$ there exists at least one element $g \in G$ such that $\gamma_2 = g\gamma_1$; the action is **simply transitive** if there exists a unique element in G such that the previous statement is true; if e is the only element of G which leaves each $\gamma \in \Gamma$ fixed, the action is called **effective**; the action is **free** if e is the only element with fixed points (that is if there exists one $\gamma \in \Gamma$ such that $g\gamma = \gamma$, then $g = e$). An action is simply transitive if, and only if, it is transitive and free. Given any point $\gamma \in \Gamma$, we call the subset $\mathcal{O}_\gamma \doteq \{\xi \in \Gamma | \xi = g\gamma, g \in G\}$ the **orbit of** γ under the action of G. Clearly, the space Γ is split into orbits and the action of G is transitive in each of them.

Let us note that if G acts transitively on Γ, then by Definition 2.7, Γ is a homogeneous space, where the homeomorphisms are given by the group action (according to property 1 in Definition 2.8). Homogeneous spaces under a group action form one of the basic structures necessary for Part I of this thesis, and in the following, we try to give a better characterization of these spaces. It turns out that the structure of a homogeneous space is highly dependent on the **stability group** (in Physics literature, sometimes also called the little group) of $\gamma \in \Gamma$, which is defined as the subgroup of G which leaves γ fixed (see Theorem 2.9 below). On homogeneous spaces, we can talk about *the* stability subgroup since any two points can be connected by a homeomorphism, and hence the stability groups of any two points are isomorphic.

Let us start to show the importance of the stability group by constructing a homogeneous space with it: let G be a topological group, and $H \subset G$ a closed subgroup. We denote by G/H the collection of **left cosets**, that is, the collection of all subsets of the form xH, $x \in G$ (note that, unless H is a normal subgroup, the space G/H will not be a group). The **right cosets** are similarly defined as the collection of all subsets of the form Hx, $x \in G$. We define the topology on G/H with the canonical projection $\pi : G \ni x \mapsto xH \in G/H$. More precisely, we say that a set $A \subset G/H$ is open if $\pi^{-1}(A)$ is open in G. We define the (left) action of G on G/H by assigning to each $g \in G$ the map $g : xH \mapsto gxH$. With these definitions, it is clear that G/H is a homogeneous space under this action, and that H is the stability group of this space. It turns out that *every homogeneous space under the action of a topological group is of this form*, as is shown in the following theorem.

Theorem 2.9. *Let G be a locally compact topological group with a countable basis acting transitively on a locally compact Hausdorff space Γ. Let γ be any point of Γ and H its stability subgroup. Then:*

1. *H is closed.*

2. *The map*

$$gH \mapsto g\gamma$$

is a homeomorphism of G/H onto Γ.

Proof. See Theorem 3.2 in [44]. □

2.2 Invariant and Quasi-invariant measures

Homogeneous spaces appear frequently in Physics and are often associated with space/spacetime regions. For reasons that will become more clear in the next sections, it is very important to understand if/when we can define measures on these spaces that are *invariant* (or, slightly less stringent, *quasi-invariant*) under the action of the associated group. For instance, in the Euclidean space example, the Lebesgue measure is the (unique!) invariant measure with respect to the action of the Euclidean group, while in the Poincaré group example, the Lorentz invariant measure is the (also unique) invariant measure, but this time defined on the mass-shell in the momentum space. Do these measures always exist in general homogeneous spaces? If they exist, are they unique? How much can they differ from each other? These are the questions we will answer in this subsection. From this section on, we will always *assume that the group G acting on Γ is a topological locally compact group*.

We start with a particular case: let us consider a homogeneous space Γ under the action of a group G with a trivial stability group, that is, $H = \{e\}$. In this case, Theorem 2.9 tells us that Γ is homeomorphic with G itself, and hence we are looking for invariant measures on locally compact groups. In this case, the answers to all the above questions are given in terms of the *Haar measures*.

Definition 2.10. Let G be a locally compact group, and let $C_0(G)$ and $C_0^+(G)$ denote the space of continuous and continuous non-negative functions on G with compact support, respectively. A **positive Radon measure** is a positive linear map μ on $C_0(G)$ which is non-negative on $C_0^+(G)$. If, in addition, this measure is left-invariant, that is,

$$\mu(T_g^L f) = \mu(f), \quad \text{where } (T_g^L f)(x) \doteq f(g^{-1}x), \quad x, g \in G, \quad \text{and } f \in C_0(G),$$

then it is called a **left Haar measure**.

We could, similarly, define a right Haar measure, by changing T_g^L with T_g^R (right translation) in the above definition. The fundamental result is the following.

Theorem 2.11. *Every locally compact group has a left Haar measure μ. If ν is any other non-zero left Haar measure, then $\nu = c\mu$ for some positive number c.*

Proof. See Chapter 2 in [41]. □

It can be easily shown that the existence of a left Haar measure implies the existence of a right Haar measure, although it does not necessarily coincide with the left Haar measure. A

measure that is both left and right invariant is called an **invariant measure**. So far the measures are defined on the above-described function spaces. However, by *Riesz theorem*, there is a Borel measure on the measurable sets in G (which we also denote by μ) such that:

$$\mu(f) = \int_G f(g) d\mu(g),$$

and the left-invariance of the Haar measure implies that:

$$\mu(gX) = \mu(X),$$

for all measurable $X \subset G$, and all $g, x \in G$.

When the stability group is not trivial, the above theorem does not apply and it is not guaranteed that invariant measures exist. In fact, it can be shown that they, in general, do not exist (see counter-example in Chapter 4 in [41]). This leads us to define the following, more general, kind of measure.

Definition 2.12. A positive measure μ on $X \doteq G/H$ is called **quasi-invariant** if it is equivalent to the measure $\mu_g(A) \doteq \mu(gA)$, $A \subseteq X$ measurable, for every $g \in G$.

Hence, while an invariant measure μ equals μ_g for every $g \in G$, a quasi-invariant measure is only required to be equivalent. With this relaxation, we can guarantee existence.

Theorem 2.13. *Let G be a locally compact separable group, H a closed subgroup, and $X = G/H$. Then*

1. *There exists a quasi-invariant measure on X. In addition, any two quasi-invariant measures are equivalent.*

2. *If an invariant measure exists, it is unique up to a multiplicative constant.*

Proof. See Chapter 4, Theorem 1 in [41]. □

2.3 Representation Theory basics

Groups and their representations play a fundamental whole in Theoretical Physics. They often appear as groups of symmetries, and if the quantum system being analyzed is symmetric "enough", then a lot can be said about the systems just by analyzing the group and its representations. Of course, what precisely we mean by enough will depend on the specific problem we are trying to solve. For Localizability Problem within the Newton-Wigner approach, enough

means a homogeneous space, where the topological space Γ represents a globally hyperbolic spacetime, and G is its group of isometries. As we will see, the homogeneity assumption is enough to formulate Newton-Wigner localization only using representation theory techniques. For this reason, we include this short subsection on the representation theory of groups, even though this topic is extremely broad. Throughout this section, G will be a locally compact, separable, and unimodular (that is, the right and left Haar measures coincide) topological group, $\mathcal{H}$ will be a separable, complex Hilbert space, and $B(\mathcal{H})$ will denote the set of bounded operators acting on $\mathcal{H}$.

Definition 2.14. A map $G \ni x \mapsto T(x) \in B(\mathcal{H})$ is a **representation of G in** $\mathcal{H}$ if

1. $T(xy) = T(x)T(y)$.
2. $T(e) = \mathbb{I}$.

The first condition says that the map defining the representation is a homomorphism of G into the bounded, linear operators in $\mathcal{H}$. The second condition says that these operators are invertible:

$$T(x)T(x^{-1}) = T(x^{-1})T(x) = T(e) = \mathbb{I} \Longrightarrow T^{-1}(x) = T(x^{-1}).$$

For technical reasons, it is important to impose continuity conditions on the representations as well. We will always require that the representation is **strongly continuous**, which means that for any $y \in G$:

$$\|T(x)u - T(y)u\| \to 0, \quad \text{as } x \to y,$$

for every $u \in \mathcal{H}$. The easiest example of a representation is the *trivial* representation, defined as $T(x) = \mathbb{I}$ for all $x \in G$. A representation is **unitary** if each $T(x)$, $x \in G$, is a unitary operator.

Let $T_1(x)$ be a representation of G in $\mathcal{H}_1$. If $S : \mathcal{H}_1 \to \mathcal{H}_2$ is any bounded isomorphism, then we can easily see that $T_2(x) \doteq ST_1(x)S^{-1}$ defines a representation of G in $\mathcal{H}_2$. However, these representations are, for most uses in representation theory, essentially the same. Hence, it is useful to partition all possible representations into *equivalence classes*.

Definition 2.15. Let G be a group and T_1, T_2 two representations acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. A linear operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$UT_1(x) = T_2(x)U$$

is valid for all $x \in G$ is called an **intertwiner**. The set of all intertwiners between two representations forms a linear space, which we denote by $R(T_1, T_2)$. Two representations are **equivalent** if there exists an inversible intertwiner between them. They are **unitarily equivalent** if, in addition, the intertwiner is a unitary operator.

The next very important concept in representation theory is that of reducibility of representations.

Definition 2.16. Let T be a representation of a group G in $\mathcal{H}$. A subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ is called **invariant** if $u \in \mathcal{H}_1$ implies $T(x)u \in \mathcal{H}_1$ for every $x \in G$. The representation is said to be **irreducible** if it has no proper closed invariant subspaces. If a representation is not irreducible, then it is called **reducible**.

If there is an invariant subspace $\mathcal{H}_1$ for a representation T in $\mathcal{H}$, then we can form a new representation just by restricting T to $\mathcal{H}_1$. This representation is called a *subrepresentation* of T. Let us analyze another example of a reducible representation. Let T_1 and T_2 be two representations of the group G, acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then, we can define a representation of their direct sum as:

$$(T_1 \oplus T_2)(x)(u_1, u_2) \doteq (T_1(x)u_1, T_2(x)U_2), \quad x \in G,$$

where $u_i \in \mathcal{H}_i$. Clearly, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are proper invariant subspaces of $\mathcal{H}_1 \oplus \mathcal{H}_2$, and hence the representations is reducible. On the other way around, suppose we have a representation T acting on $\mathcal{H}$, and subrepresentations T_i acting on invariant subspaces $\mathcal{H}_i$ such that $\mathcal{H} = \bigoplus_i \mathcal{H}_i$. Then, we write:

$$T = \bigoplus_i T_i.$$

If each T_i in the decomposition above is irreducible, we say that T is **completely reducible**. Note, however, that it might not always be possible to decompose a reducible representation in such a way. For finite-dimensional (meaning that $\mathcal{H}$ is finite-dimensional) unitary representations, though, this is always possible. In Quantum Mechanics and Quantum Field Theory, however, we are mostly working with infinite-dimensional Hilbert spaces. In addition, as we will discuss in more detail later, there is an intrinsic connection between irreducible representations of symmetry groups and elementary quantum systems. Therefore, it is very important to understand the relation between a given unitary representation and its irreducible components. For the general case, the direct sum is not enough and is substituted by its generalized version

named *direct integral* (Definition A.23). The following theorem is quite general and is valid even for non-unitary representations.

Theorem 2.17. *Let T be a representation of a locally compact group G acting on a Hilbert space $\mathcal{H}$. Then, there is a measure space (Λ, μ) and a direct integral decomposition $\mathcal{H} = \int_\Lambda^\oplus \mathcal{H}(\lambda) d\mu(\lambda)$ such that:*

$$T = \int_\Lambda T(\lambda) d\mu(\lambda),$$

where each $T(\lambda)$ is an irreducible representation acting on $\mathcal{H}(\lambda)$.

Proof. See Section 2.6 in [39]. □

Up to now, we have studied representations of locally compact groups and their decompositions, but we haven't addressed a fundamental question: given an arbitrary topological group G, does it always have (nontrivial) irreducible representations? The following theorem answers this question positively, and there are even irreducible representations that are nontrivial enough to separate points.

Theorem 2.18 (Gelfand-Raikov Theorem). *Let G be a separable topological group. Then for every two elements $x_1, x_2 \in G$, with $x_1 \neq x_2$, there exists an irreducible representation T such that $T(x_1) \neq T(x_2)$.*

Proof. See Chapter 6, Theorem 2, in [41]. □

To close this section, we study the representations of a very important class of groups for Physics and our future applications to the localization problem, namely, compact groups. Let G be a compact group, T a representation acting on a separable, complex Hilbert space $\mathcal{H}$, where an inner product $\langle , \rangle$ is defined. The first important fact that we notice is that on every compact G we have an invariant measure μ, that is, the left and right Haar measures coincide (see Chapter 2, Proposition 2 in [41]). We can use this measure to define a new inner product on $\mathcal{H}$:

$$\langle u, v \rangle_G \doteq \int_G \langle T(x)u, T(x)v \rangle d\mu(x),$$

where $u, v \in \mathcal{H}$, and $x \in G$. Let $\|.\|_G$ be the norm defined from this inner product. Then we have:

Theorem 2.19. *Let G be a compact group, and T a representation acting on a separable, complex Hilbert space $\mathcal{H}$. Then*

1. *The norm $\|.\|_G$ and the original norm $\|.\|$ on $\mathcal{H}$ are equivalent. This means that $\mathcal{H}$ is also a Hilbert space with the inner product $\langle ., . \rangle_G$.*

2. *With respect to $\langle , \rangle_G$, the representation T is unitary.*

Proof. See Chapter 7 in [41]. □

The theorem above is basically saying that every representation of a compact group is equivalent to a unitary representation. This theorem shows that representations of compact groups are very special. In fact, even more can be said about them.

Theorem 2.20. *Let G be a compact group and T a unitary representation acting on a separable, complex Hilbert space $\mathcal{H}$. Then*

1. *If T is irreducible, then it is a finite-dimensional representation.*

2. *If T is reducible, then it is a countable direct sum of finite-dimensional unitary representations.*

Proof. See Chapter 7 in [41]. □

2.4 Induced Representations

Now that the basic definitions and results about the existence and decomposition of representations have been studied, we explore the next obvious question: how do we obtain the representations of a given group? The answer to this question can vary greatly, and depend highly on the structure of the group we are interested in. For the applications we have in mind, the most appropriate technique is that of induction. The idea is the following: given a locally compact group G (whose representations we are interested in), we obtain (in a *constructive* way) some unitary representations of this group from unitary representations of a closed subgroup $K \subseteq G$. In this sense, we are *inducing* a representation of the larger group from a representation of the smaller. This method is very useful because the smaller group often has a simpler structure. For instance, the subgroup K could be compact, even if G is not, and as we saw in the last section, those are much easier to work with. The method of induction (for finite groups) goes back to Frobenius, and it was used in a heuristic way by Wigner [36] in his celebrated paper on the irreducible, unitary representations of the Poincaré group. Nonetheless, it was Mackey who formalized the method and generalized it to locally compact groups. In some cases, the technique of induction is so powerful that it allows us to obtain *all* unitary, irreducible representations of a given group (Theorems 2.46 and 2.47). In this section, we define and analyze the basic properties of induced representations for a locally compact, separable group G.

2.4.1 The carrier Space

Let K be a closed subgroup of G and $K \ni k \mapsto L(k)$ a unitary representation of K in a separable Hilbert space $\mathcal{H}$. The first step in the induction method is to construct the Hilbert space (also called the *carrier space*) where the induced representation of G will act. This space is dependent on L and we will denote it by $\mathcal{H}^L$. Let μ be any quasi-invariant measure on the homogeneous space of right cosets $X \doteq K \setminus G = \{Kg | g \in G\}$ (remember Theorem 2.13), and $\pi : G \to X$ the canonical projection. All the following constructions and results also work for the space of left cosets with minor changes (see Section 2.7). We define the carrier space as follows.

Definition 2.21. The **carrier space** $\mathcal{H}^L$ is defined as the set of all (vector-valued) functions $f : G \to \mathcal{H}$ satisfying the following conditions:

1. The function $g \mapsto \langle f(g), v\rangle_{\mathcal{H}}$ is measurable for all $v \in \mathcal{H}$.
2. $f(kg) = L(k)f(g)$, for all $k \in K$ and all $g \in G$.
3. $\int_X \|f(g)\|^2_{\mathcal{H}} d\mu(\pi(g)) < \infty$

Notice that condition 3 is well-defined because condition 2, together with the fact that L is unitary, ensures that:

$$\|f(kg)\|_{\mathcal{H}} = \|L(k)f(g)\|_{\mathcal{H}} = \|f(g)\|_{\mathcal{H}}.$$

Hence, $\|f(g)\|_{\mathcal{H}}$ is a function that depends only on the right cosets. The space $\mathcal{H}^L$ is clearly a complex-linear vector space. The next step is to transform it into a Hilbert space. We do it by defining the following inner product.

Proposition 2.22. *The vector space $\mathcal{H}^L$ is a Hilbert space with the following inner product:*

$$\langle f_1, f_2\rangle_{\mathcal{H}^L} \doteq \int_X \langle f_1(g), f_2(g)\rangle_{\mathcal{H}} d\mu(\pi(g)).$$

Proof. See Chapter 16, Lemma 1 in [41]. □

Due to condition 1 in Definition 2.21, the function $\langle f_1(g), f_2(g)\rangle_{\mathcal{H}}$ is measurable, and due to condition 2, it depends only on the right cosets. This Hilbert space, as we will show, is the appropriate space to construct the (induced) representation of G. Before we construct this representation, let us remember the following basic fact about equivalent measures. If two

positive measures, $d\mu_1$ and $d\mu_2$, are equivalent, then according to the *Radon-Nykodym Theorem*, there exists a function $\rho(x) \geq 0$ such that:

$$d\mu_1(x) = \rho(x) d\mu_2(x).$$

The function $\rho(x) \doteq d\mu_1(x)/d\mu_2(x)$ is called the **Radon- Nikodym derivatie**. Let μ be any of the quasi-invariant measures defined on X. Then, $\mu_g(A) \doteq \mu(Ag)$ and μ are equivalent and there is a Radon-Nikodym derivative $\rho_g(x)$ for every $g \in G$ in such a way that the composition rule:

$$\rho_{g_1 g_2} = \rho_{g_1}(x)\rho_{g_2}(xg_1) \tag{2.1}$$

is satisfied. We are now in a position to define a representation of G on $\mathcal{H}^L$.

Lemma 2.23. *The map $G \ni s \mapsto U^L(s)$ given by:*

$$(U^L(s)f)(g) \doteq (\rho_s(g))^{1/2} f(gs), \quad s, g \in G, \quad f \in \mathcal{H}^L, \tag{2.2}$$

*defines a unitary representation of G in $\mathcal{H}^L$, which we denote as the **induced representation** of G by L.*

Proof. See Chapter 16, Lemma 2 in [41]. □

Notice that, because of condition 2 in Definition 2.21, the right-hand side in (2.2) can also be written as:

$$(\rho_s(g))^{1/2} f(gs) = (\rho_s(g))^{1/2} L(g) f(s),$$

which makes the dependence on L more explicit. Finally, it can be proven that the space $\mathcal{H}^L$ is non-trivial, that is, there are indeed non-zero functions that satisfy all the requirements. The following theorem not only proves that this space is non-trivial, but it shows how to construct a dense set of functions in this space.

Theorem 2.24. *Let $G \ni\mapsto w(g) \in \mathcal{H}$ be an arbitrary, continuous functions and define:*

$$\hat{w}(g) \doteq \int_K L^{-1}(k) w(kg) dk,$$

where dk is the right Haar measure in K. Then:

1. *$\hat{w}(g)$ is a continuous function on G with compact support on $K \setminus G$.*

2. $\hat{w}(g) \in \mathcal{H}^L$.

3. *The set* $C_0^L = \{\hat{w}(g) | w(g) = \lambda(g)v,$ *where* $\lambda(g) \in C_0(G), v \in \mathcal{H}\}$ *is dense in* $\mathcal{H}^L$.

Proof. See Chapter 16, Proposition 3 in [41]. □

It turns out that there is another way of constructing the carrier space of the induced representation, in terms of a function space of square-integrable functions, which will be more useful and intuitive for the applications we have in mind. For that, we need a technical decomposition theorem, the **Mackey decomposition**.

Theorem 2.25. *Let* G *be a separable, locally compact group and* K *a closed subgroup. Then there exists a Borel set* S *in* G *such that every element* $g \in G$ *can be uniquely represented as:*

$$g = k_g l_g, \tag{2.3}$$

where $k \in K$ *and* $l \in S$.

Proof. See Chapter 2 in [41]. □

We are now ready for the alternative construction of the carrier space.

Theorem 2.26. *The space* $\mathcal{H}^L$ *constructed above is isomorphic to the Hilbert space of square-integrable vector-valued functions* $L^2(X, \mu, \mathcal{H})$*, where* $X = K \setminus G$*. The isomorphism is given by the map:*

$$f(g) = L(k_g)\tilde{f}(\pi(g)),$$

where $\tilde{f} \in L^2(X, \mu, \mathcal{H})$*, and* k_g *is the factor of* g *in the Mackey decomposition given in* (2.3)*.*

Proof. See Chapter 16, Lemma 1 in [41]. □

Having constructed a new carrier space, we now define the action of the induced representation on this space. For simplicity, we will also denote this representation by U^L, and we will drop the tilda over the functions in $L^2(X, \mu, \mathcal{H})$, since it will always be clear from the context.

Lemma 2.27. *The map* $G \ni s \mapsto U^L(s)$ *given by*

$$(U^L(s)f)(x) = (\rho_s(x))^{1/2} L(k_{l_g s}) f(xs), \quad s, g \in G, \quad f \in L^2(X, \mu, \mathcal{H}), \tag{2.4}$$

is a unitary representation of G *in* $L^2(X, \mu, \mathcal{H})$*, where* $x = \pi(g) \in X$*, and* $k_{l_g s}$ *is the unique element in the Mackey decomposition* (2.3) *of the element* $l_g s$ *(and* l_g *is the unique element in the decomposition of* g*).*

Proof. See Chapter 16, Proposition 5 in [41]. □

Even though this second method of construction of the carrier space and the induced representation might look at first as being more complicated, it can actually be more intuitive for problems in Physics, as will be the case in Newton-Wigner localization. One natural question that arises at this point is about the dependence of the results with the choice of quasi-invariant measure μ: what if we choose another quasi-invariant measure ν? The following theorem guarantees that the corresponding constructions are (unitarily) equivalent.

Theorem 2.28. *Let μ and ν be two quasi-invariant measures in $X = K \setminus G$. Denote by $\mathcal{H}_\mu^L$ and $\mathcal{H}_\nu^L$ the corresponding carrier spaces and by U_μ^L and U_ν^L the corresponding induced representations. Then, there exists a unitary transformation $V : \mathcal{H}_\mu^L \to \mathcal{H}_\nu^L$ such that:*

$$VU_\mu^L(g)V^{-1} = U_\nu^L(g)$$

for all $g \in G$.

Proof. See Chapter 16, Proposition 4 in [41]. □

2.4.2 Fundamental Theorems of Induced Representations

In this subsection, we collect some technical results on the properties of induced representations. These will be useful for our future applications.

The first property that we want to study is the interplay between induced representations and direct sums, or direct integrals. Consider two unitary representations L_1 and L_2 of a closed subgroup $K \subseteq G$, acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We can construct the direct sum representation as the operator:

$$(L_1 \oplus L_2)(u_1, u_2) \doteq (L_1 u_1, L_2 u_2),$$

where $u_i \in \mathcal{H}_i$, $i = \{1, 2\}$, which acts on the Hilbert space $\mathcal{H}_1 \bigoplus \mathcal{H}_2$. Since the inner product in the direct sum space is given by $\langle , \rangle_{\mathcal{H}_1 \bigoplus \mathcal{H}_2} = \langle , \rangle_{\mathcal{H}_1} + \langle , \rangle_{\mathcal{H}_2}$, it is clear that a vector (u_1, u_2) will be in $\mathcal{H}^{L_1 \oplus L_2}$ if, and only if, each vector $u_i \in \mathcal{H}_i$ satisfy, individually, the conditions in Definition 2.21. Hence, the operations of induction and direct sum are interchangeable. More generally, we have the following theorem.

Theorem 2.29. *Let $K \subseteq$ be a closed subgroup of a locally compact separable group G. Let L be a unitary representation of K, which can be decomposed into a direct integral*

$$L = \int_\Lambda L(\lambda) d\mu(\lambda),$$

acting on the Hilbert space:

$$\mathcal{H} = \int_\Lambda^\oplus \mathcal{H}(\lambda) d\mu(\lambda).$$

Then the representation U^L is unitarily equivalent to $\int_\Lambda U^{L(\lambda)} d\mu(\lambda)$.

Proof. See Chapter 16, part B, Theorem 1 in [41]. □

Corollary 2.30. *If the representation U^L of G is irreducible, then the representation L is also irreducible.*

We emphasize that the converse of this corollary is not true. The next important property of induced representations that we consider is the so-called *induction in stages*. Let $N \subseteq K \subseteq G$ be two closed subgroups of G. Starting from a representation L of N, we can construct an induced representation of G in two ways: we can induce directly a representation of G, or we can first induce a representation of K and, with this representation, induce a representation of G. What is the difference between the resulting representations of G? This question is answered by the following theorem.

Theorem 2.31. *Consider the groups N, K, G as above. Let L be a representation of N, and $U^{L,K}$ and $U^{L,G}$ be the induced representation of K and G, respectively. Then the representations $U^{L,G}$ and $U^{U^{L,K}}$ of G are unitarily equivalent.*

Proof. See Chapter 16, part C, Theorem 2 in [41]. □

The last important result on induced representations that we want to consider is the so-called *Induction- Reduction Theorem*. The idea is the following. Let N, K be any two subgroups of a group G, not necessarily one contained in the other. It is often very important to understand the restriction of a given representation to subgroups. For example, if U is a unitary representation of G, then $U|_N$ is a unitary representation of N. Is there a method to decompose this restricted representation in terms of irreducible representations of N? With this generality, this is a very hard problem. However, if U is an induced representation (say, from K), then we do have a method to determine this decomposition. This is the content of the Induction-Reduction Theorem.

Let us start with an illustration of the idea of the method. Let U^L be a representation of G induced by a representation L of K, and $X \doteq K \setminus G$. Of course, G acts transitively on X. However, the subgroup N does not in general act transitively on X. Suppose there are two subsets, X_1 and X_2, which are invariant under N, and such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. Denote by $\mathcal{H}_1$ and $\mathcal{H}_2$ the subspaces of $\mathcal{H}^L$ consisting of functions which vanish outside X_1 and X_2, respectively. Then, it is clear that these spaces are orthogonal complements of each other since they are invariant under $U^L|_N$. In this way, the carrier space can be written as:

$$\mathcal{H}^L = \mathcal{H}_1 \bigoplus \mathcal{H}_2,$$

and the restriction of the induced representation as:

$$U^L|_N = U_1 \oplus U_2,$$

where U_1 and U_2 acts on the respective invariant subspaces. Note, however, that the subspaces X_i, $i \in \{1, 2\}$, are not necessarily homogeneous with respect to N. That means that we can decompose each of them into orbits relative to N. Let us assume for the moment that each invariant subset can be decomposed into a countable set of orbits X_i^j, such that $X_i = \cup_{j \in \mathbb{N}} X_i^j$. These are elements of the double-coset space $\mathcal{D} \doteq K \backslash G / N$ (that is, the set of elements of the form KgN, $g \in G$). Then, we have:

$$\mathcal{H}^L = \bigoplus_{i,j} \mathcal{H}_{i,j}, \quad U^L|_N = \oplus_{i,j} U_{i,j}.$$

This double summation above is clearly a sum of all elements of $\mathcal{D}$. The Induction-Reduction Theorem extends this construction for the case when the orbits of the double-coset are not necessarily countable and in addition it tells us how to obtain the representations $U_{i,j}$ in terms of induction from a smaller subgroup.

Before we get to the theorem, we need first some technicalities. An **admissible measure** on $\mathcal{D}$ is any measure constructed in the following way. Let $\tilde{\nu}$ be any finite measure in G with the same sets o measure zero as the Haar measure. Define the map $s : g \mapsto KgN$ which associates to each $g \in G$ its double-coset in $\mathcal{D}$. Then we define a measure ν in $\mathcal{D}$ as $\nu \doteq \tilde{\nu}(s^{-1}(D))$, for $D \in \mathcal{D}$. We also need the following regularity conditions on the subgroups K and N.

Definition 2.32. Let K, N, G be as above. We say that K **and** N **act regularly in** G, by $g(k, n) = k^{-1}gn$, if there exists a sequence of Borel sets $Z_i \subset G$ such that

1. $\tilde{\nu}(Z_0)$, and $Z_i(k, n) = Z_i$, for each $(k, n) \in K \times N$ and all i.

2. Every orbit O not contained in Z_0 relative to the action of $K \otimes N$, is an intersection of sets Z_i containing the orbit O.

Theorem 2.33 (**Induction-Reduction**). *Let G be a separable, locally compact group, and K and N be any closed subgroups of G acting regularly in G. Let U^L be a representation of G induced by the representation L of K. Then*

1.

$$U^L|_N \simeq \int_{\mathcal{D}} U_N(D) d\nu(D),$$

where $D \in \mathcal{D} \doteq K\backslash G/N$, $U_N(D)$ is a unitary representation of N, and ν is any admissible measure on $\mathcal{D}$.

2. *The representations $U_N(D)$ in the decomposition above are determined (within unitary equivalence) by a double-coset D. These can be chosen as induced representations in the following way. For every $g \in D$, the subgroup $N \cap g^{-1}Kg$ of N depends on the double-coset D only. Hence, the representation $U_N(D)$ can be chosen as an induced representation from this subgroup.*

3. *Let $x_D = x_0 g_D$ for some $g_D \in G$, and where $x_0 = e = K$. Then $N \cap g_D^{-1} K g_D$ is the stabilizer group of x_D under the action of N.*

Proof. See Chapter 18, Theorem 1 in [41]. □

2.5 Systems of Imprimitivity

In this section, we introduce a new mathematical object, namely, systems of imprimitivity (SI). This is the single most important mathematical object in the study of Newton-Wigner localization. As we will see in the next chapters, it encompasses mathematically all the minimum (and most important) physical requirements of localization. In this section, though, we will restrict our attention to the mathematical understanding of these objects, while the connection with Physics will be postponed for later. Systems of imprimitivity are deeply connected, in a non-obvious way, with induced representations. Hence, the comprehension of the first implies a better understanding of the latter. For some classes of groups (which include important groups for Physics, like the Poincaré or Euclidean groups), this connection between SI's and induced representations is so important that its understanding allows us to obtain *all* irreducible, unitary representations of the group.

Definition 2.34. Let G be a separable, locally compact group, and U a unitary representation of G in a separable Hilbert space $\mathcal{H}$. A **System of Imprimitivity (SI) based on** X is a pair (E, U) where:

1. $E : X \to \mathcal{P}(\mathcal{H})$ is a **spectral measure**, that is, it associates to each Borel set $Z \subseteq X$ an orthogonal projection $E(Z) \in \mathcal{P}(\mathcal{H})$ and satisfies the following relations:
   - $E(X) = \mathbb{I}, \quad E(\emptyset) = 0.$
   - $E(Z_1 \cap Z_2) = E(Z_1)E(Z_2).$
   - $E(\bigcup_{n\in\mathbb{N}} Z_n) = \text{s-}\lim_{k\to\infty} \sum_{n=1}^{k} E(Z_n).$
2. The pair (E, U) satisfies the covariance relation:

$$U(g)E(Z)U(g^{-1}) = E(Zg^{-1}). \tag{2.5}$$

If a representation U admits the existence of a spectral measure E such that (E, U) is a SI, then we say that the representation U is *imprimitive*. We start our analysis of SI's by showing that every induced representation is imprimitive. Let K be a closed subgroup of a separable, locally compact group G, L a unitary representation of K in $\mathcal{H}$, and U^L the induced representation of G in $\mathcal{H}^L$. Let $X \doteq K\backslash G$ denote the homogeneous space of right-cosets, $Z \subseteq X$ be an arbitrary Borel set, and χ_Z its characteristic function. Let us define an operator $E(Z)$ acting on $\mathcal{H}^L$ as:

$$(E(Z)f)(g) \doteq \chi_Z(\pi(g))f(g), \quad g \in G, \tag{2.6}$$

where $\pi(g) = Kg$ is the canonical projection on X. The first thing we need to check is that the right-hand side of this equation is again an element of $\mathcal{H}^L$. Since χ_Z and π are measurable functions, it follows that this function is weakly measurable. Moreover, we have that:

$$E(Z)f(kg) = \chi_Z(\pi(g))f(kg) = L(k)(\chi_Z(\pi(g))f(g)) = L(K)E(Z)f(g),$$

and hence condition 2 in Definition 2.21 is also satisfied. Finally, we also have

$$\begin{aligned}\int_Z \|\chi_Z(\pi(g))f(g))\|^2 d\mu(\pi(g)) &= \int_Z \chi_Z(\pi(g))\|f(g))\|^2 d\mu(\pi(g)) \\ &= \int_Z \|f(g))\|^2 d\mu(\pi(g)) < \infty,\end{aligned}$$

which concludes the proof that (2.6) is an element of $\mathcal{H}^L$. We now prove that (E, U^L) is always a SI. It is easy to see that this map also defines a spectral measure, that is, the conditions in item 1 in Definition 2.34 are satisfied. Due to the importance of this particular spectral measure, we will call it from now on the **canonical spectral measure**. The covariance relation (2.5) can readily be verified:

$$\begin{aligned}(U^L(h)E(Z)U^L(h^{-1})f)(g) &= [\rho_h(g)]^{1/2}(E(Z)U^L(h^{-1})f)(gh)\\ &= [\rho_h(g)]^{1/2}\chi_Z(\pi(gh))(U^L(h^{-1})f)(gh)\\ &= [\rho_h(g)]^{1/2}[\rho_{h^{-1}}(gh)]^{1/2}\chi_Z(\pi(gh))f(g)\\ &= [\rho_h(g)]^{1/2}[\rho_{h^{-1}}(gh)]^{1/2}\chi_{Zh^{-1}}(\pi(g))f(g)\\ &= E(Zh^{-1})f(g),\end{aligned}$$

where in the last step we used the composition law (2.1) for the Radon-Nikodym derivatives. Hence, we conclude that every induced representation is imprimitive with respect to the canonical spectral measure. We denote the corresponding SI the **canonical system of imprimitivity**, which is naturally based on X. Similar to what is done to group representations, we can attribute to SI's the notion of irreducibility and equivalence, as follows.

Definition 2.35. Let (P, U) be an SI based on a homogeneous space X, and acting on $\mathcal{H}$.

1. We say that (P, U) is **irreducible**, if for an arbitrary $V \in B(\mathcal{H})$, $g \in G$, and a Borel subset $Z \subseteq X$:

$$VU(g) = U(g)V, \quad \text{and} \quad VP(Z) = P(Z)V,$$

   implies that $V = \lambda\mathbb{I}, \lambda \in \mathbb{C}$.

2. Let $(\tilde{P}, \tilde{U})$ be another SI, also based on X, and acting on $\tilde{\mathcal{H}}$. We say that (P, U) and $(\tilde{P}, \tilde{U})$ are **equivalent** (in symbols, $(P, U) \simeq (\tilde{P}, \tilde{U})$) if there exist a unitary operator $V : \mathcal{H} \to \tilde{\mathcal{H}}$ such that

$$VU(g) = \tilde{U}(g)V, \quad \text{and} \quad VP(Z) = \tilde{P}(Z)V,$$

   for all $g \in G$, and all Borel sets $Z \subseteq X$.

We showed in the previous paragraphs that to every induced representation there is always a system of imrpimitivity associated with it, namely, the canonical SI. A natural question that

arises is if the converse is true, that is: can we associate to every system of imprimitivity an induced representation? The answer is yes, and it goes by the name of *Imprimitivity Theorem*.

Theorem 2.36 (**Mackey's Imprimitivity Theorem**). *Let K be a closed subgroup of a separable, locally compact group G, and (P, U) a system of imprimitivity based on the homogeneous space $X = K\backslash G$ and acting on a separable Hilbert space $\mathcal{H}$. Then, there exists a unique (up to unitary equivalence) representation L of K and a unitary operator $W : \mathcal{H} \to \mathcal{H}^L$ such that:*

$$WU(g)W^{-1} = U^L(g), \quad \textit{for all } g \in G,$$
$$WP(Z)W^{-1} = E(Z), \quad \textit{for every Borel set } Z \subseteq X,$$

where (E, U^L) is the canonical system of imprimitivity acting on $\mathcal{H}^L$ obtained by induction from L.

Proof. See Chapter 16, paragraph 3, Theorem 1 in [41]. For an alternative proof, check Section 6.5 in [45]. □

Hence, we see that every system of imprimitivity is unitarily equivalent to an induced representation induced by a representation of the stabilizer subgroup. This theorem shows the close connection between induced representations and SI's. In addition, it shows that the stabilizer K in a homogeneous space $X = K\backslash G$ encodes much of the structure of systems of imprimitivity based on this space. In fact, representations of the stabilizer subgroup determine completely all possible SI's based on X. This is shown by the following theorems.

Theorem 2.37. *Let (E, U^L) and $(\tilde{E}, U^{\tilde{L}})$ be two canonical systems of imprimitivity of a group G, based on $X \doteq K\backslash G$, and acting on the carrier spaces $\mathcal{H}^L$ and $\mathcal{H}^{\tilde{L}}$, where L and $\tilde{L}$ are unitary representations of K. Let $R(L, \tilde{L})$ be the set of all intertwining operators (see Definition 2.15) and S be the set of all linear operators $V \in \mathscr{L}(\mathcal{H}^L, \mathcal{H}^{\tilde{L}})$ from $\mathcal{H}^L$ to $\mathcal{H}^{\tilde{L}}$ such that:*

1. *$U^{\tilde{L}}(g)V = VU^L(g)$, for all $g \in G$.*
2. *$\tilde{E}(Z)V = VE(Z)$, for all Borel sets $Z \subseteq X$.*

Then, the spaces $R(L, \tilde{L})$ and S are isomorphic. In particular, a linear operator $U \in R(L, \tilde{L})$ is unitary if, and only if, the corresponding operator $V \in S$ is also unitary.

Proof. See Chapter 16, part B, Theorem 3 in [41]. □

The sets $R(L, \tilde{L})$ and S contain all the information about the equivalence classes (with their respective meanings) of the representations of K and the SI's on X, respectively. Therefore,

the above isomorphism shows that one is completely determined by the other. In particular, choosing $L = \tilde{L}$ and L irreducible (which means that $R(L, L) = \{\lambda\mathbb{I}|\lambda \in \mathbb{C}\}$), it follows from this theorem that $S = \{\lambda\mathbb{I}|\lambda \in \mathbb{C}\}$, and we have the following corollary.

Corollary 2.38. *Let U^L be a representation of a locally compact, separable group G induced by a representation L of $K \subseteq G$. Then, the canonical system of imprimitivity (E, U^L) is irreducible if, and only if, the representation L is irreducible.*

Therefore, even though the irreducibility of L is not enough (in general) to guarantee the irreducibility of U^L, it is enough to guarantee the irreducibility of the canonical system of imprimitivity. It also follows immediately from the last assertion of the theorem the following result.

Corollary 2.39. *The systems of imprimitivity (E, U^L) and $(\tilde{E}, U^{\tilde{L}})$ are unitarily equivalent if, and only if, L and $\tilde{L}$ are unitarily equivalent.*

2.6 Induced Representations of Regular Semi-direct Products

So far, the method of induced representations proved to be a powerful tool to obtain unitary representations of a vast class of groups (namely, separable, locally compact topological groups) in terms of representations of closed subgroups. The power of the method relies on the fact that it is constructive, that is, we construct explicitly the induced representation and the corresponding carrier space. Since groups and their representations appear very frequently in Physics, this technique finds there an extensive field for applications. Still, it is interesting to know when we can go one step further and determine the *irreducible* unitary representations of the group. When the group under consideration is an isometry group of a given spacetime, these are related, according to Wigner's interpretation, to elementary systems. Nonetheless, as we saw in the last section, it is not possible with such a level of generality, to guarantee that the induced representation U^L will be irreducible, even if L is.

Hence, the most natural strategy is to put more conditions on the structure of the groups, such that we can guarantee that U^L is irreducible (for irreducible L). This was done by Mackey, and the most general class of groups such that this property is guaranteed to be valid is that of (regular) semi-direct products, where the normal group is also abelian. Fortunately, this class is still large enough to include many important groups for Physics, such as the Poincaré, the Euclidean, and the Galilean groups. The goal of this section is to show that, for this type of

group, not only we can obtain unitary, irreducible representations by induction, but *all* such representations can be obtained by this method. Let us start by defining precisely what is a regular semi-direct product of groups.

The idea is to construct, from two given groups N and H, a third group $G = N \rtimes H$, where $\rtimes$ denotes the semi-direct product. However, the two groups cannot be completely unrelated. The group H must be homomorphic with a subgroup of the group of all automorphisms of N, $Aut(N)$. There are two ways of constructing a semi-direct product: the first, called *inner semi-direct product*, we start with a group G, a normal subgroup N, and an arbitrary subgroup H. If some conditions between these groups are satisfied (see Definition 2.40 below), we say that G is the inner semi-direct product of the two; the second, called *outer semi-direct product*, we start with any two groups N and H (with the condition that H is homomorphic with a subgroup of $Aut(N)$) and construct a third one from the Cartesian product of the two, which will be the outer semi-direct product of N and H. Both constructions are equivalent and we will later on only say semi-direct product when referring to $G = N \rtimes H$.

Definition 2.40. Let G be an arbitrary group, N a normal subgroup, and H an arbitrary subgroup. If any element $g \in G$ can be written as $g = nh$, where $n \in N$ and $h \in H$, and in addition $N \cap H = \{e\}$, we say that $G = N \rtimes H$ is the **inner semi-direct product** of N and H.

Note that, if $g_1 = n_1 h_1$ and $g_2 = n_2 h_2$, then

$$g_1 g_2 = n_1 h_1 n_2 h_2 = n_1 h_1 n_2 h_1^{-1} h_1 h_2 = n_1 (h_1 n_2 h_1^{-1}) h_1 h_2,$$

and $(h_1 n_2 h_1^{-1}) \in N$, since N is normal. Note further that the map $\phi_h : N \ni n \mapsto \phi_h(n) = (h_1 n_2 h_1^{-1}) \in N$ is an automorphism of N, and $H \ni h \mapsto \phi_h(.) \in Aut(N)$ is an homomorphism from H to $Aut(N)$. With that in mind, the connection with the next definition of semi-direct product is clear.

Definition 2.41. Let N and H be any two groups and $\phi : H \to Aut(N)$ a homomorphism. Then the **outer semi-direct product** $G = N \rtimes H$ is defined as the set $G \doteq N \times H$, together with the group multiplication law:

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \phi_{h_1}(n_2), h_1 h_2). \tag{2.7}$$

Some properties of semi-direct products will later be useful:

- If e_N and e_H denotes the identities in N and H, then the subsets of elements of the form (n, e_H) and (e_N, h), with $n \in N$ and $h \in H$, are isomorphic with the groups N and H,

respectively. Furthermore, N is a normal subgroup.

- The space of left cosets G/N is isomorphic with H.
- If ϕ is the trivial homomorphism ($\phi_h = e_N, \forall h \in H$), then the semi-direct product coincides with the *direct product*.
- The *Levi-Malcev Theorem* (see Chapter 1, paragraph 3, Theorem 5 in [41]) says that every connected Lie group is locally isomorphic with a semi-direct product.

The representation theory of semi-direct products also presents special characteristics. From now on, we will assume that N and H are topological groups. Equipping $G = N \rtimes H$ with the product topology, it also becomes a topological group. Let T be a representation of G, and lets denote by $U \doteq T|_N$ and $V \doteq T|_H$ its restrictions to N and H, respectively. Since $(n, h) = (n, e_H) \cdot (e_n, h)$ for all $n \in N$ and $h \in H$, we have that:

$$T(g) = T(n, h) = U(n)V(h), \tag{2.8}$$

which means that a representation of a semi-direct product $G = N \rtimes H$ is completely determined by its restrictions. Note, however, that due to the special multiplication law (2.7), not every representation U and V are acceptable: they need to be consistent with the multiplication law. More explicitly, let $g_1 = (n_1, h_1)$ and $g_2 = (n_2, h_2)$ be any two elements of G. Then, since $T(g_1)T(g_2) = T(g_1 \cdot g_2)$, and $g_1 \cdot g_2 = (n_1\phi_{h_1}(n_2), h_1h_2)$, we have that:

$$\begin{aligned} T(g_1)T(g_2) &= U(n_1)V(h_1)U(n_2)V(h_2) \\ &= U(n_1\phi_{h_1}(n_2))V(h_1h_2), \end{aligned}$$

which implies that:

$$V(h_1)U(n_2) = U(\phi_{h_1}(n_2))V(h_1).$$

Since this relation must be true for any elements in N and H, we can just drop the indices and write $V(h)U(n) = U(\phi_h(n))V(h)$, which can be put in the form:

$$V(h)U(n)V^{-1}(h) = U(\phi_h(n)). \tag{2.9}$$

This equation is surprisingly similar to the covariance equation defining a system of imprimitivity (2.5), the only difference being that in this last equation, we don't have a spectral mea-

sure. Nonetheless, we can get to an exact equality if we *assume further that N is abelian*. In this case, we can try to use the spectral measure on the dual space of N which comes from the decomposition of U in the famous SNAG Theorem (check, for instance, Chapter 6, paragraph 2, in [41]). Let us be more precise. Let $\hat{N}$ denote the group of all characters of N. We want to define an action of H on $\hat{N}$ and transform it into an H-space, in the sense of Definition 2.8. If $\hat{n} : N \ni n \mapsto \hat{n}(n) = \langle n, \hat{n} \rangle \in \mathbb{C}$ is a character, and $h \in H$, then $\hat{\phi}_h(\hat{n}) : N \ni n \mapsto \langle \phi_h(n), \hat{n} \rangle \in \mathbb{C}$ is also a character. Furthermore, the map $\hat{N} \ni \hat{n} \mapsto \hat{\phi}_h(\hat{n}) \in \hat{N}$ is an automorphism of $\hat{N}$, and $H \ni h \mapsto \hat{\phi}_h \in Aut(\hat{N})$ is a homomorphism. Thus, we can define the right action of H on $\hat{N}$ by:

$$\hat{n}h \doteq \hat{\phi}_h(\hat{n}),$$

or with a different notation by:

$$\langle n, \hat{\phi}_h(\hat{n}) \rangle = \langle \phi_h(n), \hat{n} \rangle. \tag{2.10}$$

With this action, $\hat{N}$ is now a H-space. We can even extend this action to the whole group G by defining the action of N on $\hat{N}$ as the trivial action, that is:

$$\hat{n}g \doteq \hat{n}h = \hat{\phi}_h(\hat{n}), \tag{2.11}$$

where $g = (n, h) \in G$. Next, applying the SNAG Theorem, and from equation (2.9), we have that:

$$\begin{aligned} V(h)U(n)V^{-1}(h) &= U(\phi_h(n)) \\ &= \int_{\hat{N}} \langle \phi_h(n), \hat{n} \rangle dP(\hat{n}) \\ &= \int_{\hat{N}} \langle n, \hat{n} \rangle dP(\phi_h^{-1}(\hat{n})), \end{aligned}$$

where P is a spectral measure on $\hat{N}$, and where we used (2.10) in the last step. On the other hand, the left-hand side of (2.9) is:

$$\int_{\hat{N}} \langle n, \hat{n} \rangle d(V(h)P(\hat{n})V(h)^{-1}),$$

which gives:

$$\int_{\hat{N}} \langle n, \hat{n} \rangle d(V(h)P(\hat{n})V(h)^{-1}) = \int_{\hat{N}} \langle n, \hat{n} \rangle dP(\phi_h^{-1}(\hat{n})).$$

This equation, together with the fact that characters separate points in N, allow us to conclude that the spectral measures must be the same and:

$$V(h)P(Z)V(h)^{-1} = P(Zh^{-1}),$$

for every Borel set $Z \subseteq \hat{N}$. That is, the pair (P, V) is a system of imprimitivity based on $\hat{N}$. Furthermore, since $U(n)P(Z)U(n)^{-1} = P(Z)$ for all Z, it follows that $T(g)P(Z)T^{-1}(g) = P(Zg^{-1})$: in other words, *every* unitary representation of a semi-direct product (with N abelian) is imprimitive.

This analysis shows that the problem of finding a pair of representations U and V of N and H such that $T(n, h) \doteq U(n)V(h)$ is a representation of $G = N \rtimes H$ is equivalent to finding a system of imprimitivity (P, V) based on $\hat{N}$. We will show that there is a procedure to obtain the irreducible representations of G. The first step is to note that H does not have, necessarily, a transitive action on $\hat{N}$. Thus, we split $\hat{N}$ into orbits:

$$\hat{N} = \bigcup_{\hat{n} \in \hat{N}} \mathcal{O}_{\hat{n}}, \tag{2.12}$$

where $\mathcal{O}_{\hat{n}}$ is the set of all $\hat{n}h$ for a given character $\hat{n}$ and for all $h \in H$. Since any orbit is a homogeneous space, it follows from Theorem 2.9 that each orbit is homeomorphic with the right-coset space $\mathcal{O}_{\hat{n}} \simeq K_{\hat{n}} \backslash H$, where $K_{\hat{n}}$ is the stabilizer of $\hat{n}$. From Theorem 2.13, we know that there exists a quasi-invariant (with respect to H) measure μ on each orbit. For our next step, it is important to understand if these quasi-invariant measures are concentrated in an orbit or not. In general, this is not true (see Chapter 17, Example 1, in [41] for a counter-example). Therefore, to guarantee that this pathological case does not occur, we make the further assumption that the semi-direct product is *regular*.

Definition 2.42. We say that $G = N \rtimes H$ is a **regular semi-direct product** if $\hat{N}$ contains a countable family $Z_1, Z_2, ...$ of Borel subsets, each a union of G orbits, such that every orbit in $\hat{N}$ is the intersection of the members of a subfamily $Z_{n_1}, Z_{n_2}, ...$ containing that orbit.

The importance of this property appears in the following proposition.

Proposition 2.43. *Let T be a unitary representation of a regular semi-direct product $G = N \rtimes H$, and let $E(.)$ be the spectral measure associated with the restriction U of the representation*

T to N. Then, if T is irreducible there exists an orbit $\mathcal{O}_{\hat{n}}$ where E is concentrated, that is, $E(\mathcal{O}_{\hat{n}}) = \mathbb{I}$ and $E(\hat{N} - \mathcal{O}_{\hat{n}}) = 0$.

Proof. See Chapter 17, Proposition 1 in [41]. □

Hence, let T be an irreducible, unitary representation, $E(.)$ the associated spectral measure of $U = T|_N$, $\mathcal{O}^T$ the orbit associated to T by the above proposition, and K^T the stability group of $\mathcal{O}^T$ such that $\mathcal{O}^T \simeq K^T \backslash H$. Because the action of N on $\hat{N}$ is trivial (equation (2.11)), it is also true that $\mathcal{O}^T \simeq N \rtimes K^T \backslash G$. Then, we can write:

$$T(g)E(Z)T(g)^{-1} = E(Zg^{-1}),$$

where $Z \subseteq \mathcal{O}^T$ is a Borel set. We can now apply the Imrpimitivity Theorem 2.36 to conclude that every irreducible representation T of G is equivalent to a representation U^L which is induced from a representation L of $N \rtimes K^T$. The representation U^L is realized in the Hilbert space $H^L = L^2(\mathcal{O}^T, \mu, \mathcal{H})$, where $\mathcal{H}$ is the Hilbert space where the representation L of $N \rtimes K^T$ is acting. Consider the set S defined in Theorem 2.37. If $V \in S$, this implies that $V \in R(U^L, U^L)$. Since T and U^L are unitarily equivalent, this also implies that $V \in R(T, T)$. Hence, $R(T, T) = S$, and according to Theorem 2.37 the set S is isomorphic with $R(L, L)$. Since T is by assumption irreducible, it follows that L is also irreducible.

The following Lemma give us an useful description of the reduction of L to N, understood as a subgroup of $N \rtimes K^T$.

Lemma 2.44. *Let L be as above, $\mathcal{H}$ be the Hilbert space where L is acting, and $L|_N$ be the restriction of L to N. Then:*

$$L|_N u = \langle n, \hat{n} \rangle u,$$

where $u \in \mathcal{H}$ and $\hat{n} \in \mathcal{O}^T$.

Proof. See Chapter 17, Lemma 2 in [41]. □

From this lemma and the above discussion, one could ask how much the irreducible representation L is determined by a representation M of K^T. Our next lemma answers this question.

Lemma 2.45. *Every irreducible unitary representation L of $N \rtimes K^T$ is determined by and determines an irreducible unitary representation M of K^T.*

Proof. See Chapter 17, Lemma 3 in [41]. □

We summarize the above discussion in the following theorem.

Theorem 2.46. *Let $G = N \rtimes H$ be a regular, semi-direct product of separable, locally compact groups N and H, and let N be abelian. Let T be an irreducible, unitary representation of G. Then:*

1. *We can associate with T an orbit $\mathcal{O}^T$ in $\hat{N}$ with stabilizer group K^T.*

2. *T is unitarily equivalent to an induced representation U^L, where L is an irreducible, unitary representation of $N \rtimes K^T$, which is completely determined by an irreducible, unitary representation M of K^T. The representation U^L is realized in the Hilbert space $\mathcal{H}^L = L^2(\mathcal{O}^T, \mu, \mathcal{H})$.*

Hence, we have shown that every irreducible representation of G is unitarily equivalent to an induced representation. The next theorem proves the converse, namely, that every induced representation is irreducible.

Theorem 2.47. *Let G be as in the previous theorem. Then:*

1. *With each orbit $\mathcal{O}_{\hat{n}}$ in $\hat{N}$ (with stabilizer $K_{\hat{n}}$), and with each irreducible representation L of $N \rtimes K_{\hat{n}}$ (which comes from a irreducible representation M of $K_{\hat{n}}$) we can construct the induced representation U^L which is irreducible. Because a representation of a semi-direct product is of the form* (2.8)*, and because the restriction to N is determined by Lemma 2.44, the representation U^L acts on $\mathcal{H}^L = L^2(\mathcal{O}, \mu, \mathcal{H})$ as:*

$$(U^L(g)f)(\hat{n}) = \langle n, \hat{n} \rangle U^M(h) f(\hat{n}), \tag{2.13}$$

   where $g = (n, h) \in G$, and where U^M is a representation of H induced by $K_{\hat{n}}$.

2. *The spectral measure $E(.)$ coming from the restriction of U^L to N is concentrated on the orbit $\mathcal{O}$.*

Proof. See Chapter 17, Theorem 5 in [41]. □

Combining the last two theorems, we have a complete understanding of all the irreducible, unitary representations of G in terms of induced representations. We finish this section with an overview of an application of this theory to a very important group: the Poincaré group.

Example 2.48. The construction of all irreducible, unitary representations of the Poincaré group was first done by Wigner in his seminal paper [36]. Many of his steps, however, were "heuristic". Nonetheless, they served as an inspiration to Mackey to construct rigorously the mathematical machinery of induced representations presented in the last chapters. Here, we only give

an overview of the construction of these representations, since a full presentation would be very lengthy. We refer to [41, 46] for a full analysis.

We will focus our attention on the universal cover of the proper orthocronus Poincaré group $\widetilde{\mathcal{P}_+^\uparrow}$, that is, the identity connected, time orientation preserving component of $\widetilde{\mathcal{P}}$. This group can be written as the regular semi-direct product $\widetilde{\mathcal{P}_+^\uparrow} = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$. Following the steps described in this section, we define the action of $SL(2,\mathbb{C})$ on $\widehat{\mathbb{R}}^4$ as in equation (2.10). This action partitions $\widehat{\mathbb{R}^4}$ into orbits, like in equation (2.12). The next step is to understand these orbits. We have the following possibilities:

- $\mathcal{O}_m^\pm$: Choose the vector $\widehat{\mathbb{R}^4} \ni \hat{n} = (\pm m, 0, 0, 0)$, $m \in \mathbb{R}$. The action of $SL(2,\mathbb{C})$ generates the *mass hyperboloid* $\Omega_m = \{\hat{n} \in \widehat{\mathbb{R}^4} | \hat{n}^\mu \hat{n}_\mu \doteq \hat{n}^0\hat{n}_0 - \hat{n}^1\hat{n}_1 - \hat{n}^2\hat{n}_2 - \hat{n}^3\hat{n}_3 = m^2\}$. The stability group of this orbit is the (cover of) the rotation group, $SU(2)$. The irreducible representations of this group are parametrized by $j = 0, 1/2, 1, 3/2, ...$ with dimension $2j + 1$. Hence, the irreducible representations arising from this orbit are parametrized by a pair (m, j), interpreted as the mass and spin of an elementary system.

- $\mathcal{O}_{im}$: Choose the vector $\widehat{\mathbb{R}^4} \ni \hat{n} = (0, m, 0, 0)$. The stability group for this orbit is $SL(2,\mathbb{R})$. This group has three series of irreducible representations, denoted by *principal series, discrete series*, and the *supplementary series*.

- $\mathcal{O}_0^\pm$: We take the representative $\widehat{\mathbb{R}^4} \ni \hat{n} = (1/2, 0, 0, 1/2)$. The stability group of this orbit is $\widehat{\mathcal{E}}(2) = \mathbb{R}^2 \rtimes S^1$, the cover of the two-dimensional Euclidean group. As this group has the form of a regular semi-direct product, its irreducible representations are, again, obtained by induction. In complete analogy, we define the action of S^1 on $\widehat{\mathbb{R}^2}$, which splits this space into orbits given by circles with radius r. If $r = 0$, the stability group is S^1, whose irreducible representations are parametrized by $j = 0, \pm 1/2, \pm 1, ...$ The obtained representations of $\widetilde{\mathcal{P}_+^\uparrow}$ correspond to massless particles with finite helicity j. If $r > 0$, the stability group is $\{\mathbb{I}, -\mathbb{I}\}$, which possess only the trivial and the adjoint representations, labeled by $\tau = 0$ or $\tau = 1$. Hence, the obtained representations of $\widetilde{\mathcal{P}_+^\uparrow}$ have two parameters (τ, r). These are the so-called *infinity spin representations*.

- $\mathcal{O}_0^0$: In this case, the only representative is $\widehat{\mathbb{R}^4} \ni \hat{n} = (0, 0, 0, 0)$, and the stability group is the whole $SL(2,\mathbb{C})$, the Lorentz group. This group has two series of irreducible representations: the principal series, and the supplementary series. We refer to [41] for more details.

These are all the orbits and, according to the theorems of this section, they give rise to all unitary, irreducible representations of $\widetilde{\mathcal{P}_+^\uparrow}$. The carrier space of each of these representations is

$L^2(\mathcal{O}, \mu, \mathcal{K}_\mathcal{O})$, where μ is a quasi-invariant measure, and $\mathcal{K}_\mathcal{O}$ is the representation space of the representations of the stabilizer of each orbit. For example, for the massive representaions this space is given by $L^2(\Omega, \mu_m, \mathbb{C}^{2j+1})$, where μ_m is the Lorentz-invariant measure $\mu_m = dp^3/\omega$, $\omega = \sqrt{p^2 + m^2}$.

2.7 Formulation for left-action

In this chapter, most of our constructions were done assuming the right action of a group on the homogeneous space. All of the results, however, can be equivalently written for the left action. In the following, we summarize how this is done.

Condition 2 in the Definition 2.21 of H^L now reads:

$$f(gk) = L(k^{-1})f(g),$$

for all $k \in K$ and $g \in G$, and the action of U^L on $\mathcal{H}^L$, previously given by equation (2.2) is now:

$$(U^L(s)f)(g) \doteq (\rho_s(g))^{1/2} f(s^{-1}g), \quad s, g \in G, \quad f \in \mathcal{H}^L.$$

For the formulation on the carrier space $L^2(X, \mu, \mathcal{H})$, the action of U^L, previously given by equation (2.4), is now:

$$(U^L(s)f)(x) = (\rho_s(x))^{1/2} L(k_{l_{gs}}) f(s^{-1}x), \quad s, g \in G, \quad f \in L^2(X, \mu, \mathcal{H}).$$

Finally, for the regular semi-direct products, equation (2.10) is substituted by:

$$\langle n, \hat{\phi}_h(\hat{n}) \rangle = \langle \phi_h^{-1}(n), \hat{n} \rangle.$$

Chapter 3

Newton-Wigner Localization on Minkowski Spacetime

> *I venture to say that any notion of localizability in three-dimensional space which does not satisfy (the Newton-Wigner axioms) will represent a radical departure from present physical ideas.*
>
> *A. S. Wightman*

As we will see below, the Newton-Wigner localization formalism is based on very few, weak assumptions, which led Wightman to state the above quote in his seminal paper [18] on the localizability of quantum mechanical systems. In a very natural sense, it is the most straightforward generalization of the successful notion of the position observable in non-relativistic Quantum Mechanics. It relies on the foundational axioms of Quantum Mechanics, namely, that every observable (in the sense of Definition 1.1) of a quantum system is implemented in the theory's mathematical structure by a self-adjoint operator, whose spectrum corresponds to the possible outcomes of experiments (remember the map $\mathcal{T}_{QT}$ in equation (1.2)). The most fundamental theories we presently have, describe the universe in different aspects: on one side the quantum mechanical description of the structure of matter, and on the other the relativistic description of space and time. In the attempt to join both of these theories a "back reaction" is expected: quantum mechanical principles should change the way we understand the spacetime structure, while the causality constraints of spacetime should be incorporated into the mathematics of Quantum Mechanics. While a complete understanding of this intersection remains in distant dreams, we can only hope to take small steps in this direction.

The Newton-Wigner localization is an attempt to incorporate the causality structure of spacetime while retaining the axioms of Quantum Mechanics intact. That this was a failed enterprise, is both a surprise and an indication that the introduction of new mathematical ob-

jects in the theory of relativistic quantum systems ought to be considered. The traditional idea of an observable as a self-adjoint operator might have to be replaced by more general mathematical structures. This idea is already present in nowadays research, where the spectral measure is replaced by positive operator-valued measures (POVM). This is used, for example, in attempts to define a time operator [47, 48]. In the second part of this thesis, we will provide an analogous abstraction of the idea of an observable, introducing a notion of an observable associated with a logic. Before venturing into this endeavor, however, it is necessary to have a deep understanding of why the natural proposal of Newton and Wigner fails. For that we dedicate both this and the next chapter: first, we concentrate on the full description of Newton-Wigner localization in Minkowski spacetime, showing its successes and failures. To get to the bottom of this localization scheme, we get rid of the special features of Minwkoski spacetime and generalize its construction to general homogeneous globally hyperbolic spacetimes in Chapter 4.

The organization of this chapter is as follows. We start by giving a straightforward derivation of the Newton-Wigner operator and its most immediate properties. We then proceed to give an abstract definition, based on the theory of systems of imprimitivity developed in the last chapter. Finally, we close the chapter by displaying some of the many No-go Theorems on the Localizability Problem.

We start with the position operator in non-relativistic Quantum Mechanics. Let $\mathcal{H} = L^2(\mathbb{R}^3, dx^3)$ be the Hilbert space of a free particle moving in the three-dimensional space. Then, the position operators are defined as:

$$\begin{aligned}(Q_i\psi)(x) &= x_i\psi(x)\\ D(Q_i) &= \left\{\psi \in L^2(\mathbb{R}^3, dx^3)\Big| \int_{\mathbb{R}^3} |(Q_i\psi)(x)|^2 dx^3 < \infty\right\},\end{aligned}$$

where $i = 1, 2, 3$. These are unbouded, self-adjoint operators with $\sigma(Q_i) = \mathbb{R}$. Furthermore, these operators can be obtained through a spectral integral with respect to the canonical spectral measure in this space:

$$Q_i = \int_{\mathbb{R}} x_i dE(x_i),$$

where

$$(E(B)\psi)(x) = \chi_B(x)\psi(x),\ B \in \mathcal{B}(\mathbb{R}^3). \tag{3.1}$$

Let $\mathcal{F} : L^2(\mathbb{R}^3, dx^3) \to L^2(\mathbb{R}^3, dp^3)$ denote the Fourier transform into the momentum space.

In this space, the position operators have the form:

$$\mathcal{F}Q_i\mathcal{F}^{-1} = i\frac{\partial}{\partial p_i}, \tag{3.2}$$

and they are hermitian with respect to the inner product in $L^2(\mathbb{R}^3, dp^3)$. On the other hand, the Hilbert space of a relativistic particle is $L^2(\Omega_m, \mu_m)$, where Ω_m is the mass hyperboloid, and μ_m is the Lorentz-invariant measure. In this space, the inner product is given by:

$$\langle \psi, \phi \rangle = \int_{\Omega_m} \overline{\psi}(p)\phi(p)\frac{dp^3}{\omega}, \tag{3.3}$$

where $\omega = \sqrt{p^2 + m^2}$. An immediate question that arises is: can we also define (3.2) as a position operator acting on the relativistic Hilbert space? Although this is still a well-defined operator, we cannot interpret it as modeling an observable anymore, simply because it is not a hermitian operator with respect to the inner product (3.3):

$$\begin{aligned}\left\langle \psi, i\frac{\partial}{\partial p_i}\phi \right\rangle &= i\int_{\mathbb{R}^3} \overline{\psi}(p)\left(\frac{\partial}{\partial p_i}\phi(p)\right)\frac{dp^3}{\omega} \\ &= \int_{\mathbb{R}^3} \phi(p)\left[\left(-i\frac{\partial}{\partial p_i} + i\frac{p_i}{p^2+m^2}\right)\overline{\psi}\right](p)\frac{dp^3}{\omega} \\ &\neq \left\langle i\frac{\partial}{\partial p_i}\psi, \phi \right\rangle,\end{aligned}$$

where we did a partial integration in the second step.

Hence, we see that only the Fourier transform is not enough to carry successfully the position operator from $L^2(\mathbb{R}^3, dx^3)$ to $L^2(\Omega_m, \mu_m)$. We can, however, try to "correct" the unitary map between these spaces, such that we have a self-adjoint operator. Let us define the map $W : L^2(\Omega_m, \mu_m) \to L^2(\mathbb{R}^3, dx^3)$ as:

$$(W\psi)(x) \doteq \left[\mathcal{F}^{-1}\left(\frac{\psi}{\omega^{1/2}}\right)\right](x). \tag{3.4}$$

This map can be shown to be unitary (see Chapter 20 in [41]). We can now define the following operators:

$$Q_i^{NW} \doteq W^{-1}Q_iW = i\left(\frac{\partial}{\partial p_i} - \frac{p_i}{2\omega^2}\right).$$

These are the so-called **Newton-Wigner position operators**. We can immediately see that this operator is hermitian (and self-adjoint) with respect to the inner product (3.3) since the ω term in the definition of W compensates for the ω term in the inner product. Also, these operators

can similarly be obtained by a spectral integral:

$$Q_i^{NW} = \int_{\mathbb{R}} x_i dP^{NW}(x_i),$$

where $P^{NW} = W^{-1}EW$ is the spectral measure.

Let us now have a closer look at the properties of these operators. Let g be the metric tensor in Minkowski spacetime $\mathbb{M}$, with signature $(+---)$, where we assume a time orientation is given. Let V_+ denote a preferred half of the open cone of time-like vectors representing the future-directed time-like vectors. Define:

$$T_+ \doteq \{v \in V_+ | v \text{ unit, time-like, and future-oriented}\}.$$

Every $n \in T_+$ defines a **reference frame** in $\mathbb{M}$. We denote by $\Sigma_{n,t}$ the three-dimensional **Cauchy-surface** to which n is normal, where t denotes the proper time of n. The **Euclidean group** $\mathcal{E}(3)$ corresponds to the subgroup of $\mathcal{P}_+^\uparrow$ that preserves n. We denote by (x_0, x_1, x_2, x_3) a set of coordinates on $\mathbb{M}$ which is co-moving with n. With this choice of structure, we put extra labels "t" and "n" in the Newton-Wigner operator $Q_{i,n,t}^{NW}$ and on the spectral measure $P_{n,t}^{NW}$ such that the choice of a reference frame, a co-moving set of coordinates, and a foliation into Cauchy- surfaces $\Sigma_{n,t}$ is implicit. If the context permits, we also drop the label "n" to simplify the notation.

Proposition 3.1. *Let U be a unitary, irreducible, massive, spinless representation of $\mathcal{P}_+^\uparrow$ acting on $\mathcal{H}$, and consider the Newton-Wigner operator $Q_{i,n,t}^{NW}$ and spectral measure $P_{n,t}^{NW}$, as above. Then:*

1. *Poincaré covariance:*

$$U(h)P_{n,t}^{NW}(B)U^{-1}(h) = P_{\Lambda n, t_h}^{NW}(hB), \quad h = (\Lambda, a) \in \mathcal{P}_+^\uparrow,\ B \in \mathcal{B}(\Sigma_{n,t}).$$

2. *Euclidean covariance:*

$$U(g)P_{n,t}^{NW}(B)U^{-1}(g) = P_{n,t}^{NW}(gB), \quad g \in \mathcal{E}(3),\ B \in \mathcal{B}(\Sigma_{n,t}).$$

3. *Heisenberg commutation relations:*

$$[Q_{i,n,t}^{NW}, Q_{j,n,t}^{NW}] = 0 = [P_{i,n}, P_{j,n}], \quad [Q_{i,n,t}^{NW}, P_{j,n}] = i\delta_{i,j},$$

where the P_i, n's are the momentum operators.

4. *Heisenberg uncertainty principle:*

$$\Delta_\psi(Q^{NW}_{i,n,t})\Delta_\psi(P_{i,n}) \geq \frac{\hbar}{2},$$

where Δ_ψ denotes the standard deviation with respect to $\psi \in \mathcal{H}$.

5. *Time evolution:*

$$U((\mathbb{I},\tau))^{-1} Q^{NW}_{i,n,0} U((\mathbb{I},\tau)) = Q^{NW}_{i,n,\tau},$$

where τ denotes a time translation vector.

6. *Time-like worldline of position expectation values: the four-vector*

$$\left(t, \langle\psi, Q^{NW}_{1,n,t}\psi\rangle, \langle\psi, Q^{NW}_{2,n,t}\psi\rangle, \langle\psi, Q^{NW}_{3,n,t}\psi\rangle\right) \in \mathbb{M}$$

is time-like for every $t \in \mathbb{R}$ and every $\psi \in \mathcal{H}$.

Proof. See Proposition 8, Proposition 13, and Corollary 14 in [32]. □

The above proposition shows that Newton-Wigner localization has many appealing features similar to the position observable in non-relativistic Quantum Mechanics. Additionally, items 1 and 6 in the proposition demonstrate some agreement with relativistic principles (though see the No-Go theorems in the next section). The Euclidean covariance item is a direct consequence of the Poincaré covariance, which might seem redundant. However, we listed them separately to emphasize an important point: the Euclidean group acts on $\Sigma_{n,t}$, and the spectral measure transforms covariantly with this group's action on Borel spatial subsets. On the other hand, Poincaré covariance would not hold if we considered a Borel subset of spacetime $B \in \mathcal{B}(\mathbb{M})$ instead of a Borel subset B of the spatial slice.

Finally, the Newton-Wigner approach offers a candidate solution to the (spatial) Localizability Problem since we can construct a family of probability measures for each $\psi \in L^2(\Omega_m, \mu_m)$ with it:

$$\mu^{NW}_{\psi,t}(B) \doteq \langle\psi, P^{NW}_{n,t}(B)\psi\rangle, \quad B \in \mathcal{B}(\Sigma_{n,t}), \tag{3.5}$$

and then $\mathcal{T}_{RQT}\left([\alpha], [Q]_{\text{pos}}, w^{[Q]_{\text{pos}}}_{[\alpha]}\right) = \left(\psi, Q^{NW}, \mu^{NW}_{\psi,t}\right)$ as discussed in the Introduction 1.

Up to this point, we constructed the Newton-Wigner program concretely, starting with an irreducible, massive, spinless, unitary representation of the proper, orthochronus Poincaré group acting on $\mathcal{H} = L^2(\Omega_m, \mu_m)$. However, we can also approach it more abstractly using the tools from the previous chapter. We now rederive the Newton-Wigner program in the abstract language of systems of imprimitivity. This prepares us for the next chapter, where we will use this language for all our constructions.

Note that we have two fundamental ingredients in the construction we made above:

1. **A spectral measure:** With this object we can construct a probability measure for each vector in the Hilbert space and, through a spectral integral of the co-moving coordinates, self-adjoint operators with the properties of position observables.

2. **Covariance of the spectral measure with respect to the spatial isometry group:** The spectral measure takes values on the Borel sets in $\Sigma_{n,t}$, which means that it must be covariant with respect to a unitary representation V of the Euclidean group $\mathcal{E}(3)$.

The interesting point is that a pair (P, V) of a spectral measure and a unitary representation V of $\mathcal{E}(3)$ such as above is *the precise definition of a system of imprimitivity* (check Definition 2.34) which is based on $\Sigma_{n,t}$. We can then give the following definition.

Definition 3.2. A **Newton-Wigner localization scheme at time** t in a Hilbert space $\mathcal{H}$ is a system of imprimitivity (P, V) based on $\Sigma_{n,t}$, where V is a unitary representation of the Euclidean group.

The action of the Euclidean group on $\Sigma_{n,t}$ is transitive, meaning that this is a homogeneous space, and a $\mathcal{E}(3)$-space, in the sense of Definitions 2.7 and 2.8, respectively. Hence, from Theorem 2.9 it is homeomorphic to a space of left cosets with the stability group, which is $SO(3)$ in this case, that is, $\Sigma_{n,t} \simeq \mathcal{E}(3)/SO(3)$. Note that, in this chapter, we are using the left-action convention, which is the standard one in Physics literature. In contrast, we used the right-action convention in the previous chapter, in accordance with the Mathematics literature. This, however, poses no difficulty. The results and definitions in one convention can be easily translated into the other, as is done in Section 2.7.

We now come to an important point: we can directly apply Mackey's Imprimitivity Theorem 2.36 to conclude that *any* Newton-Wigner localization scheme is unitarily equivalent to the canonical one in the induced Hilbert space. More precisely:

- If (P, V) is a Newton-Wigner scheme on $\mathcal{H}$ and based on $\Sigma_{n,t} \simeq \mathcal{E}(3)/SO(3)$, then there exists a unique (up to unitary equivalence) representation L of $SO(3)$ and a unitary

operator $W : \mathcal{H} \to \mathcal{H}^L$ such that (P, V) is unitarily equivalent to (E, U^L), the canonical system of imprimitivity.

- The canonical spectral measure is defined on $\Sigma_{n,t} \simeq \mathcal{E}(3)/SO(3)$ and is defined as in equation (2.6).

- The induced representation U^L of $\mathcal{E}(3)$, giving by equation (2.2), acts on the carrier space $\mathcal{H}^L$ (see Definition 2.21). Equivalently, the carrier space is giving by $L^2(\mathcal{E}(3)/SO(3), dx^3, \mathcal{K})$, where dx^3 is the (invariant) Lebesgue measure, and $\mathcal{K}$ is the representation space of L. The action of the induced representation in this space is given in equation (2.4).

By the third item, we see that a Newton-Wigner localization scheme naturally leads to a space of (vector-valued) square-integrable functions on the spatial section $\Sigma_{n,t} \simeq \mathcal{E}(3)/SO(3)$, whose functions give a spatial probability distribution. Let us exemplify the abstract characterization with an important case. Consider that L is the trivial representation. In this case, the carrier space of the representation of $\mathcal{P}_+^\uparrow$ is given by $L^2(\Sigma_{n,t}, dx^3)$, and the canonical spectral measure is exactly the one given in equation (3.1). The unitary map (3.4) is an example of the unitary map between $\mathcal{H}$ and $\mathcal{H}^L$ that appears in Mackey's Imprimitivity Theorem.

Let us now think more carefully about the Hilbert space $\mathcal{H}$ where we define a Newton-Wigner localization scheme. If $\mathcal{H} = L^2(\mathbb{R}^3, dx^3)$, as in the non-relativistic case described at the beginning of this chapter, a Newton-Wigner localization scheme corresponds to the usual localization in non-relativistic Quantum Mechanics for a massive, spinless particle, and the position operator to the usual one. In our language, this Hilbert space is the carrier space of a representation of the Galilean group, obtained by induction from the trivial representation of $SO(3)$. Likewise, by choosing nontrivial representations of the stability groups of other orbits, we obtain by induction all irreducible representations of the Galilean group (as a consequence of it being a semi-direct product and Theorems 2.46 and 2.47) corresponding to massive or massless particles with arbitrary spin, and a Newton-Wigner localization scheme is again the usual one for non-relativistic systems. Hence, when restricted to Hilbert spaces of non-relativistic elementary systems, the Newton-Wigner localization coincides with the standard localization in Quantum Mechanics.

The interesting point is that the same procedure applies when $\mathcal{H}$ describes a relativistic system. Remember that, similar to the Galilean group, all irreducible representations of the proper, orthochronus Poincaré group can be obtained by induction (Example 2.48) since both of these groups are semi-direct products. These irreducible representations all act on Hilbert spaces with the form $\mathcal{H} = L^2(\mathcal{O}, \mu, \mathcal{K}_\mathcal{O})$, where $\mathcal{O}$ is some orbit which determines the representation, μ is

some quasi-invariant measure, and $\mathcal{K}_{\mathcal{O}}$ is the representation space of the stability group of the orbit $\mathcal{O}$. In complete analogy with the non-relativistic case, any Newton-Wigner localization scheme (P, V) defined on these relativistic spaces will be unitarily equivalent to the canonical system of imprimitivity on $L^2(\Sigma_{n,t}, dx^3, \mathcal{K})$ for some representation L of $SO(3)$. A first strategy to construct a Newton-Wigner localization scheme is to start with one of these irreducible representations, restrict it to the Euclidean group, and check if there is any spectral measure such that this pair is a system of imprimitivity based on $\Sigma_{n,t}$. Since each of the irreducible representations of $\mathcal{P}_+^\uparrow$ carry the important interpretation of representing an elementary particle, we can classify which irreducible representations are localizable, in the sense of giving rise or not to a Newton-Wigner localization scheme, and likewise for general unitary representations. This is the idea of the following definition.

Definition 3.3. Let U be a unitary representation of $\mathcal{P}_+^\uparrow$. We say that U **defines a localizable system on** $\Sigma_{n,t}$, or that it is a **localizable representation**, if there is a spectral measure P such that $(P, U|_{\mathcal{E}(3)})$ is a Newton-Wigner localization scheme.

Next, we show that every massive, unitary representation of $\mathcal{P}_+^\uparrow$ is localizable. Observe that, due to the Imprimitivity Theorem, a representation is localizable if, and only if, $U|_{\mathcal{E}(3)}$ is an induced representation of $SO(3)$. In this case, the canonical system of imprimitivity is based on $\mathcal{E}(3)/SO(3) \simeq \mathbb{R}^3$ and the canonical spectral measure on the induced representation, together with this representation, defines a Newton-Wigner localization scheme.

Theorem 3.4. *Let U be a unitary representation of $\mathcal{P}_+^\uparrow$. Then U defines a localizable system on $\Sigma_{n,t} \simeq \mathbb{R}^3$ if, and only if, it is a direct integral of massive, irreducible (arbitrary spin), unitary representations.*

Proof. This proof is an adaptation and an extension of Proposition 1, Chapter 20, in [41]. Let us start by classifying the irreducible representations of $\mathcal{P}_+^\uparrow$ that are localizable. Note that direct sums or direct integral representations of the localizable irreducible representations will also be localizable due to Theorem 2.29. The main tool for this goal is the important Induction-Reduction Theorem 2.33. In the present case, we choose the groups G, N, and K appearing in the formulation of this theorem to be $\mathcal{P}_+^\uparrow$, $\mathcal{E}(3)$, and $\mathbb{R}^4 \rtimes SO(3)$, respectively. This means that the induced representation U^L of G coming from a representation L of $K = \mathbb{R}^4 \rtimes SO(3)$ is an irreducible, massive representation with arbitrary spin. Then, the Induction-Reduction theorem says that:

$$U^L|_{\mathcal{E}(3)} \simeq \int_{\mathcal{D}} U_{\mathcal{E}(3)}(D) d\nu(D), \tag{3.6}$$

where $D \in \mathcal{D} = \mathbb{R}^4 \rtimes SO(3)\backslash\mathcal{P}_+^\uparrow/\mathcal{E}(3)$, $U_{\mathcal{E}(3)}$ is a unitary representation of $\mathcal{E}(3)$, and ν is any admissible measure on $\mathcal{D}$. By our previous argument, $U^L|_{\mathcal{E}(3)}$ will be localizable if, and only if, all the representations of the Euclidean group appearing in the direct integral (3.6) are induced from $SO(3)$. Next, let us understand better the space of double cosets $\mathcal{D}$. The space $\mathbb{R}^4 \rtimes SO(3)\backslash\mathcal{P}_+^\uparrow$ of right-cosets is well known to be homeomorphic to the mass hyperboloid Ω_m. Then, the space $\mathcal{D}$ is formed by the orbits of Ω_m under the action of $\mathcal{E}(3)$. From items 2 and 3 in the Induction-Reduction Theorem, it follows that the representations $U_{\mathcal{E}(3)}$ are induced representations from the stabilizer subgroups of each of these orbits. However, each point in Ω_m is invariant under the action of $\mathcal{E}(3)$, meaning that each point in the hyperboloid is an orbit with the same stabilizer, namely, $SO(3)$. We conclude that all irreducible massive representations U^L induced from $\mathbb{R}^4 \rtimes SO(3)$ are localizable in $\mathbb{R}^3$. Furthermore, these are the only irreducible representations of $\mathcal{P}_+^\uparrow$ that are localizable, since the stabilizer groups would be different for other irreducible representations. Finally, using Theorem 2.29 we conclude that any unitary representation which is a direct integral of massive (arbitrary spin) irreducible, unitary representations of $\mathcal{P}_+^\uparrow$ is also localizable. □

Hence, we see that the only localizable systems on a Cauchy surface in Minkowski spacetime are massive. We could extend our definition of localizability to include different homogeneous subregions of Minkowski spacetime rather than just $\Sigma_{n,t}$. In this case, different symmetry groups under which the spectral measures are covariant would be involved, but the general approach is just the same. There are proposals in this direction for massless particles [24, 26] but, since we are focusing on systems that are localizable in a Cauchy surface, we don't get into further details.

So far, we have found a way to construct Newton-Wigner localization schemes by starting with massive representations. Fixing one such representation U, can we guarantee the uniqueness of the system of imprimitivity? Note that, in principle, there can be unitary maps $Y : \mathcal{H} \to \mathcal{H}$ that commute with $U|_{\mathcal{E}(3)}$ but that do not commute with the spectral measure. Therefore, there is some arbitrariness in this process. Let us understand how this can affect our notion of localizability. Let $(P, U|_{\mathcal{E}(3)})$ and $(Q, U|_{\mathcal{E}(3)})$ be Newton-Wigner localization schemes for the same representation U acting on $\mathcal{H}$. Then, according to the Imprimitivity Theorem, there exist unitary maps $W, Z : \mathcal{H} \to L^2(\Sigma_{n,t}, dx^3, \mathcal{K})$ such that both these systems of imprimitivity are equivalent to the canonical one in $L^2(\Sigma_{n,t}, dx^3, \mathcal{K})$, respectively. For a given $\psi \in \mathcal{H}$, let $f(x) \doteq (W\psi)(x)$ and $g(x) \doteq (Z\psi)(x)$ denote the corresponding functions in $L^2(\Sigma_{n,t}, dx^3, \mathcal{K})$.

Then, the probability measures are defined as:

$$\mu_\psi^P(B) = \langle \psi, P(B)\psi \rangle = \langle f, \chi_B f \rangle$$
$$\mu_\psi^Q(B) = \langle \psi, Q(B)\psi \rangle = \langle g, \chi_B g \rangle,$$

for $B \in \mathcal{B}(\Sigma_{n,t})$. Obviously, these probability measures do not necessarily coincide, and they can predict the same state $\psi \in \mathcal{H}$ to be located in completely different regions. The question of uniqueness was answered by Wightman in [18]. Uniqueness holds for elementary systems (when U is an irreducible massive representation) and under further technical regularity requirements.

Finally, we can extend the probability measure defined in equation (3.5) to include general algebraic states on $B(\mathcal{H})$ (see Definition 6.10).

Definition 3.5. Let U be an arbitrary massive, unitary representation of $\mathcal{P}_+^\uparrow$ acting on $\mathcal{H}$, and let $\omega : B(\mathcal{H}) \to \mathbb{C}$ be an arbitrary algebraic state. Consider a Newton-Wigner localization scheme $(P, U_{\mathcal{E}(3)})$ at time t. Then, we define the **Newton-Wigner probability measures** on $\mathcal{B}(\Sigma_{n,t})$ as:

$$\mu_{\omega,t}^{NW}(B) \doteq \omega(P_{n,t}^{NW}(B)), \quad B \in \mathcal{B}(\Sigma_{n,t}).$$

A fair question to ask at this point is: what is the probability measure associated with the vacuum state? Recall that the vacuum state is invariant under the action of the Poincaré group, meaning that it arises from the trivial representation of this group. However, this representation is not included in the localizable representations according to Theorem 3.4. Hence, the vacuum representation is non-localizable with respect to the Newton-Wigner localization approach, and there is no probability measure associated with the vacuum state.

3.1 Problems with Newton-Wigner and No-go Theorems

The Newton-Wigner approach, despite the many good features displayed in the last section, is not considered to be the solution to the Localizability Problem. We will explain the reason for that in this section. The main issue is that, even though the starting point of its construction is a representation of the Poincaré group, the Newton-Wigner localization is not entirely compatible with the causal structure of special relativity. This is expressed in the failure to satisfy a causality condition called "Castrigiano's condition" [29], to be explained below. Furthermore, the problem is shown to be difficult by many so-called No-go Theorems that have been proven in the last decades. These theorems, which have Newton-Wigner as the first victim, also put

severe constraints on alternative approaches to the Localizability Problem. We will refrain from an extensive exposition on the topic of these theorems, as it would be too lengthy. Instead, we will focus on a theorem proved by Malament [21], as we will present an analogous result in Section 7.1. In this section, for simplicity, we restrict our attention to unitary, irreducible, massive, spinless representations of $\mathcal{P}_+^{\uparrow}$.

Suppose we know that a relativistic quantum system in a one-particle state $\psi \in \mathcal{H} = L^2(\Omega_m, \mu_m)$ is localized in a compact region $\Delta \subset \Sigma_{n,t}$ at time t, meaning that $\mu_{\psi,t}^{NW}(\Delta) = 1$. Let Δ' be the intersection of the causal future of this region (denoted by $J^+(\Delta)$) with another Cauchy surface $\Sigma_{n,t'}$, that is, $\Delta' \doteq J^+(\Delta) \cap \Sigma_{n,t'}$, as in the figure below.

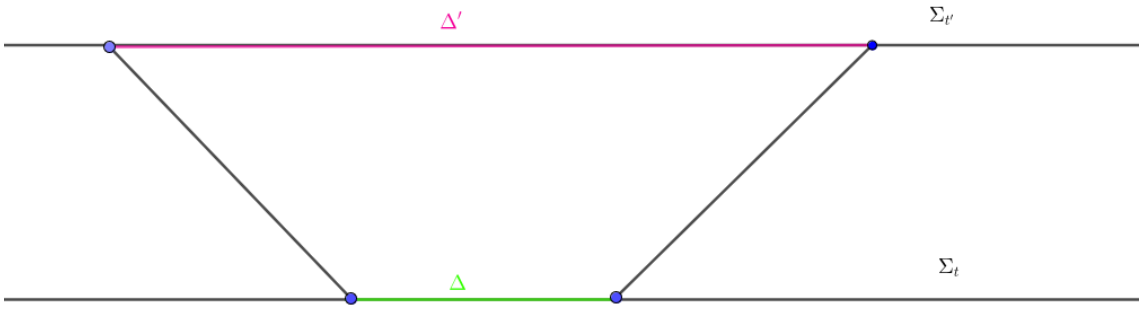


Figure 3.1: Causal future of a compact region Δ.

The idea of Castrigiano's causality condition is to guarantee that $\mu_{\psi,t'}^{NW}(A) = 0$ for any $A \in \mathcal{B}(\Sigma_{n,t} \backslash \Delta')$, meaning that there is no faster than light propagation. The precise definition of this condition, as formulated in [32], is the following.

Definition 3.6 (Castrigiano's Causality Condition). Let $\mu_\psi \doteq \{\mu_{\psi,n,t} | n \in T_+, t \in \mathbb{R}\}$ denote an arbitrary family of probability measures on $\mathscr{L}(\Sigma_{n,t})$ (Lebesgue measurable sets), for each $\psi \in \mathcal{H}$. Then, we say that μ_ψ defines a **causal family of probability measures** if for every $\Delta \in \mathscr{L}(\Sigma_{n,t})$:

$$\mu_{\psi,n,t}(\Delta) \leq \mu_{\psi,n',t'}(\Delta'), \qquad \Delta' \doteq (J^+(\Delta) \cup J^-(\Delta)) \cap \Sigma_{n',t'}$$

for all $n, n' \in T_+$, and for all $t, t' \in \mathbb{R}$.

The need to use the Lebesgue measurable sets $\mathscr{L}(\Sigma_{n,t})$ instead of the Borel sets is that, if $\Delta \in \mathcal{B}(\Sigma_{n,t})$, it may happen that $\Delta' \notin \mathcal{B}(\Sigma_{n,t})$, as proved in Lemma 16 in [29].

Proposition 3.7. *The Newton-Wigner family of probability measures* $\mu_\psi^{NW} \doteq \{\mu_{\psi,n,t}^{NW} | n \in T_+, t \in \mathbb{R}\}$ *is not causal.*

Proof. See Corollary 20 in [32]. □

This result represents a drastic strike on Newton-Wigner's localization, as it implies that a system localized in a compact region can later be detected outside its causal future. This probability, however, is extremely low: check [4] for a numerical estimate. Due to this fact, one could argue that Newton-Wigner is correct up to a very good precision and that we should not worry about its limitations. However, this conflict with the causal structure indicates that something deep is hidden behind all this, and that the pursuit of a solution to this problem might lead to a new understanding and a change in our perspective on how we should interpret the position observable in the relativistic context. After this fatal blow to the Newton-Wigner approach, physicists have been trying to come up with new definitions of something similar to a Newton-Wigner localization scheme. The problem is difficult, though, because Newton-Wigner's approach is already very minimal: we only require a spectral measure covariant under a representation of the spatial symmetry group. What conditions in this definition could be weakened such that we have a hope to counter the causality issues? Let us try the following, even more minimal approach.

Consider any family of orthogonal projections (not necessarily a spectral measure) $\{P(B)|B \in \mathcal{B}(\Sigma_{n,t})$ and bounded$\}$ on a Hilbert space $\mathcal{H}$ with the following properties:

1. *Localizability:* If $A, B \in \mathcal{B}(\Sigma_{n,t})$ are disjoint, then $P(A)P(B) = 0$.

2. *Translation covariance:* Let U be a representation of the translation group $\mathbb{R}^4$. For any $B \in \mathcal{B}(\Sigma_{n,t})$ and for any translation $a \in \mathbb{R}^4$, $U(a)P(B)U^{-1}(a) = P(B + a)$.

3. *Energy bounded below:* For any time-like translation a, the generator $H(a)$ of the one parameter group $\{U(ta)|t \in \mathbb{R}\}$ has a spectrum bounded from below.

4. *Microcausality:* Let $A, B \in \mathcal{B}(\Sigma_{n,t})$ be disjoint sets with a non-zero distance between them. Then, for any time-like translation a, there is an $\epsilon > 0$ such that $[P(A), P(B + ta)] = 0$ for all $|t| < \epsilon$.

Note that these are way more general conditions since there are no spectral measures involved, no Poincaré group or Euclidean group representations, and no systems of imprimitivity. Nonetheless, we have the following no-go result.

Theorem 3.8 (Malament's No-go Theorem). *Let $\{P(B)|B \in \mathcal{B}(\Sigma_{n,t})$ and bounded$\}$ be a family of orthogonal projections satisfying conditions 1 to 4 above. Then, $P(B) = 0$ for all $B \in \mathcal{B}(\Sigma_{n,t})$.*

Proof. See [21]. □

This theorem does not apply to the Newton-Wigner localization since condition 4 is not satisfied. This is an example among many No-go theorems on the localizability of relativistic systems. We refer to [23] for a more complete exposition of the topic. Even more minimal approaches appeared in the last decades. The most popular so far is to substitute the orthogonal projections with *positive operator-valued measures* (POVM's). Many interesting works have been done with this approach [29, 30, 32, 33]. However, some of these No-go theorems were also extended to this context [23, 34], showing how difficult the problem is. In the second part of this thesis, we prove our own version of this theorem in the context of orthocomplemented lattices (see Theorem 7.4).

Chapter 4

Newton-Wigner Localization on Homogeneous Globally Hyperbolic Spacetimes

I'm enough of an artist to draw freely on my imagination.

A. Einstein

In this chapter, we present our extension of the Newton-Wigner formalism from Minkowski spacetime to homogeneous globally hyperbolic spacetimes. Let us set our goals very clearly:

1. First and foremost, our goal is to formulate Newton-Wigner localization in this context. More precisely, we want to define an analogous definition of a Newton-Wigner localization scheme (Definition 3.2).

2. With this definition in place, we want to construct (as in the flat spacetime case) a position observable, as well as a family of probability measures, one for each state in the relevant Hilbert space. We also discuss the lack of uniqueness of the Newton-Wigner localization scheme, and we provide a further classification in terms of Thompson components (see below).

3. We aim to classify what are the representations of the spacetime isometry group that gives origin to a notion of localization, in the spirit of Theorem 3.4.

Besides these three main points, we also provide, in the next sections, some applications and decompositions of the Hilbert space that are induced by the position operators.

We start by defining our framework: we assume that (M, g) is a connected, time-oriented, lorentzian, globally hyperbolic smooth manifold of dimension $\dim(M) = m$ and metric g,

which is homogeneous under the (right) action of its spacetime isometry group. We will refer to an M with such a mathematical structure as our "spacetime". We distinguish two groups of symmetries: G^{ST} **is the isometry group of the spacetime**, and G^S **is the subgroup of spatial isometries**, namely, the subgroup of G^{ST} that preserves Σ. We *assume that both of these groups are separable and locally compact*. These manifolds allow for a "separation" of the spatial component. This is the content of the following theorem.

Theorem 4.1. *Consider a connected, time-oriented, globally hyperbolic spacetime* (M, g). *Then,* M *is isometric to* $\mathbb{R} \times \Sigma$ *with metric* $-\beta dt^2 + g_t$ *where* β *is a smooth positive function,* g_t *is a Riemannian metric on* Σ *depending smoothly on* $t \in \mathbb{R}$ *and each* $\{t\} \times \Sigma$ *is a smooth spacelike Cauchy hypersurface in M.*

Proof: See [49].

□

Hence, if $\dim(M) = m$, then Σ is a submanifold with dimension $\dim(\Sigma) \equiv n = m - 1$, and the pair (Σ_t, g_t), where $\Sigma_t \doteq \{t\} \times \Sigma$, is a Riemannian manifold for any t. We will define the notion of Newton-Wigner localization for a fixed, arbitrary time, say $t = 0$. The choice of t is unimportant since all Σ_t are diffeomorphic (see Lemma 2.2 in [49]). With this notation, the group G^{ST} corresponds to all diffeomorphisms on M that preserves the metric g, and G^S is the subgroup of diffeomorphisms preserving g_0. Due to the assumption that G^{ST} acts transitively on M (and as a consequence, G^S acts transitively on Σ), we can write (remember Theorem 2.9):

$$M \simeq K^{ST} \backslash G^{ST}$$
$$\Sigma \simeq K^S \backslash G^S,$$

where K^{ST} and K^S are the stability groups under the actions of G^{ST} and G^S, respectively. For instance, if M is the Minkowski spacetime, G^{ST} is the Poincaré group, K^{ST} is the Lorentz group, while G^S is the Euclidean group, and K^S is $SO(3)$.

The next important ingredient in our construction is the Hilbert space where our notion of localizability will be established. Recall that in flat spacetime, this Hilbert space was derived as a representation of the spacetime isometry group, obtained through induction. This symmetry group was the central object, being "big enough", such that the whole construction could rely entirely on the information provided by this group. As a consequence of Theorems 2.46 and 2.47, all irreducible representations can be obtained by the induction process. However, in the framework we are currently working on, *we cannot expect the isometry group to provide*

so much information and allow such a complete construction. Nonetheless, we claim that the assumption of an isometry group acting transitively on M provides the minimal and sufficient framework for our purposes, namely, the construction of localizability:

- As the reader might already have guessed, the central mathematical object for a notion of localizability will again be a system of imprimitivity. Mackey's Imprimitivity Theorem 2.36 provides a direct connection between this object and induced representations. Therefore, we need to work in a framework where induction is possible.

- With the assumption that M is homogeneous, the induction method remains applicable, allowing us to study systems of imprimitivity. However, we can no longer guarantee that all representations can be obtained in this manner, nor that the ones obtained will be irreducible.

- Whatever the method used to construct the Hilbert space describing a quantum system on M, it will have to carry a representation of G^{ST}. We want to study and possibly classify, which of these representations are localizable. As we will see, if this representation is one obtained by induction, we can classify it. Otherwise, there is nothing we can say about it.

The following definitions are very natural extensions of Definitions 3.2 and 3.3.

Definition 4.2. A **(generalized) Newton-Wigner localization scheme at time** t in a Hilbert space $\mathcal{H}$ is a system of imprimitivity (P, V) based on Σ_t, where V is a unitary representation of G^S.

Definition 4.3. Let U be a unitary representation of G^{ST}. We say that U **defines a localizable system on** Σ_t, or that it is a **localizable representation**, if there is a spectral measure P such that $(P, U|_{G^S})$ is a Newton-Wigner localization scheme.

From Definition 4.2 alone, we can already obtain strong conclusions, very similar to those in the previous chapter. It follows from Mackey's Imprimitivity Theorem that:

- If (P, V) is a Newton-Wigner scheme on $\mathcal{H}$ and based on $\Sigma_t \simeq K^S \backslash G^S$, then there exists a unique (up to unitary equivalence) representation L of K^S and a unitary operator $W : \mathcal{H} \to \mathcal{H}^L$ such that (P, V) is unitarily equivalent to (E_c, U^L), the canonical system of imprimitivity.

- The canonical spectral measure E_c is defined on Σ_t and given in equation (2.6).

- The induced representation U^L of G^S, given by equation (2.2), acts on the carrier space $\mathcal{H}^L$ (see Definition 2.21). Equivalently, the carrier space is given by $L^2(K^S\backslash G^S, \mu, \mathcal{K})$, where μ is a quasi-invariant measure, and $\mathcal{K}$ is the representation space of L. The action of the induced representation in this space is given in equation (2.4).

- The measure μ can be determined more precisely: note that the volume measure ν_t (locally defined as $\nu_t = \sqrt{g_t}dx^1 \wedge ... \wedge dx^n$, for local coordinates $(x^1, ..., x^n)$) on $\mathcal{B}(\Sigma_t)$ is invariant under the action of the spatial isometry group G^S. Hence, as a consequence of item 2 in Theorem 2.13, this is the unique (up to a multiplicative constant) measure on $K^S\backslash G^S$ and the carrier space can be written as $L^2(K^S\backslash G^S, \nu_t, \mathcal{K})$.

The general carrier space $L^2(K^S\backslash G^S, \nu_t, \mathcal{K})$ has an interesting resemblance with the Minkowski spacetime case. In flat spacetime, the carrier space is given by $L^2(\mathbb{R}^3, dx^3, \mathcal{K})$, where $\mathcal{K}$ is the representation space of $SO(3)$. The irreducible representations of $SO(3)$ are finite-dimensional and are interpreted as the spin degrees of freedom of the quantum system. For irreducible representations L, the carrier space can then be written as $L^2(\mathbb{R}^3, dx^3, \mathbb{C}^N) \simeq L^2(\mathbb{R}^3, dx^3) \otimes \mathbb{C}^N$. It is a profound and fascinating fact, that the quantum property of spin, representing internal degrees of freedom, arises as representations of the stabilizer group $SO(3)$, a subgroup of the spatial isometry group. Even more intriguing: the same holds true for the general case, where $\mathcal{K}$ is the representation space of the stabilizer group K^S. Because (Σ_t, g_t) is a Riemannian manifold, its stabilizer group under the action of G^S is necessarily compact (see Corollary 1.78 in [50]). Therefore, since every irreducible representation of a compact group is finite-dimensional (Theorem 2.20), it also follows that in this case the general carrier space has the form:

$$L^2(K^S\backslash G^S, \nu_t, \mathcal{K}) \simeq L^2(K^S\backslash G^S, \nu_t) \otimes \mathbb{C}^N. \tag{4.1}$$

Nonetheless, it is not clear how to physically interpret K^S and the internal degrees of freedom coming from its irreducible representations when compared to the spin. Mathematically, however, they are perfectly analogous.

Given a Newton-Wigner localization scheme, we can follow a directly analogous construction to what we did in the flat spacetime case. Let us start by defining a position operator on $\mathcal{H}$. Recall that we constructed it in the flat spacetime case by performing a spectral integral of a coordinate function which, in that case, was a global coordinate function. In general, in homogeneous manifolds, curvature might be present, such that we can only define local coordinates. We then have the following definition of a position operator.

Definition 4.4. Let (P, V) be a Newton-Wigner localization scheme in a Hilbert space $\mathcal{H}$, based

on Σ_t. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be an atlas on Σ_t, and define the following extension of the local coordinate function:

$$h^i_{U_\alpha}(s) \doteq \begin{cases} 0 & , \quad \text{if } s \notin U_\alpha \\ \phi^i_\alpha(s) & , \quad \text{if } s \in U_\alpha. \end{cases}$$

Then, we define the **(generalized) Newton-Wigner operators** as the spectral integral:

$$Q^{NW}_{i,\alpha} \doteq \int_{\Sigma_t} h^i_{U_\alpha}(s) dP(s),$$

where the sub-index α is a reference to the local chart (U_α, ϕ_α).

Due to the equivalence between (P, V) and (E_c, U^L) through a unitary map $W : \mathcal{H} \to L^2(\Sigma_t, \nu_t, \mathcal{K})$, the Newton-Wigner operators are unitarily equivalent to the operator:

$$M_{i,\alpha} \doteq \int_{\Sigma_t} h^i_{U_\alpha}(s) dE_c(s),$$

which acts on $L^2(\Sigma_t, \nu_t, \mathcal{K})$. The operators $Q^{NW}_{i,\alpha}$ and $M_{i,\alpha}$ have the following immediate properties, as can be seen from Proposition A.18:

1. It is self-adjoint, since $h^i_{U_\alpha}$ is real-valued.

2. It is bounded if, and only if, $\phi^i_\alpha(s) < C \; \forall s \in U_\alpha$, for some $C < \infty$.

3. Spectrum $\sigma(Q^{NW}_{i,\alpha}) = \sigma(M_{i,\alpha}) = \overline{h^i_{U_\alpha}(\Sigma_t)} = \{0\} \cup \overline{\phi^i_\alpha(U_\alpha)}$. Note that 0 is the unique eigenvalue. The associated eigenvectors have interesting localization properties. We see that:

$$(M_{i,\alpha} - 0\mathbb{I}) f(s) = 0 \tag{4.2}$$

$$(M_{i,\alpha} f)(s) = 0 \tag{4.3}$$

   Thus, $f \in L^2(\Sigma_t, \nu_t, \mathcal{K})$ is an eigenvector with eigenvalue 0 if, and only if, its support lies outside U_α. If 0 is not in the image of the local coordinate function, then $\overline{\phi^i_\alpha(U_\alpha)}$ is purely continuous. Otherwise, it is a purely continuous spectrum with a single eigenvalue in the middle of the continuous spectra.

Also in complete analogy with the Newton-Wigner probability measures in the flat spacetime framework (Definition 3.5), we can define:

Definition 4.5. Let (P, V) be a Newton-Wigner localization scheme at time t in a Hilbert space $\mathcal{H}$, based on Σ_t. Let $\omega : B(\mathcal{H}) \to \mathbb{C}$ be an arbitrary algebraic state. Then, we define the **(generalized) Newton-Wigner probability measures** on $\mathcal{B}(\Sigma_t)$ as:

$$\mu^{NW}_{\omega,t}(B) \doteq \omega(P(B)), \quad B \in \mathcal{B}(\Sigma_t).$$

Hence, we have a well-defined notion of Newton-Wigner localizability on M. Our next task is, in the spirit of Theorem 3.4, to classify the unitary representations U of G^{ST} that are localizable.

Theorem 4.6. *Let G^{ST}, G^S, and K^S be as above, and Z an arbitrary closed subgroup of G^{ST}. Assume further that G^S and Z act regularly (Definition 2.32) on G^{ST}. Let U^L denote a unitary representation of G^{ST} induced by a unitary representation L of Z. Then:*

1. $$U^L|_{G^S} \simeq \int_{\mathcal{D}} U|_{G^S}(D) d\nu(D),$$

   where $\mathcal{D} = Z\backslash G^{ST}/G^S$, $U|_{G^S}$ is a unitary representation of G^S, and ν is any admissible measure.

2. *The representation U^L is localizable if, and only if, ν has measure zero on orbits $D \in \mathcal{D}$ that doesn't have K^S as the stabilizer group under the action of G^S.*

3. *Let $Z = K^S$. If K^S is a normal subgroup of G^{ST}, then all representations U^L are localizable.*

Proof. This proof follows similar steps to the proof of Theorem 3.4. The first item in the theorem is a direct application of the Induction-Reduction Theorem 2.33. Let us prove the second. Note that if the representation U_{G^S} is induced from K^S, then the canonical system of imprimitivity originating from the induction is based on $K^S\backslash G^S \simeq \Sigma_t$. Hence, if the only representations appearing in the direct integral are of this kind, the representation $U^L|_{G^S}$ is localizable. On the other direction, if $U^L|_{G^S}$ is localizable, then due to the Imprimitivity Theorem it is unitarily equivalent to a representation induced from K^S. Finally, to prove the last assertion, we use item 3 in the Induction-Reduction Theorem, which asserts that for an arbitrary $x_D = K^S g_D \in D$, $g_D \in G^{ST}$, its stability group is given by $G^S \cap g_D^{-1} K^S g_D$. From the normality condition of K^S, and the fact that it is contained in G^S, it follows that K^S is the stability group of an arbitrary x_D. □

Therefore, we see that we have a partial classification of localizable representations. However, observe that, when compared to the flat spacetime scenario, this classification has limitations: since it is not guaranteed that all (if any) irreducible representations of G^{ST} can be obtained by induction, our classification method does not apply to all representations, only those obtained by induction. If U is not of this form, there is nothing we can say about it. Notwithstanding, remember from Section 2.6 that if G^{ST} is in the form of a regular semi-direct product, then *all* irreducible representations can be obtained by induction. Hence, in this case, Theorem 4.6 allows us to classify all representations of G^{ST}.

The next point we want to address is that of the uniqueness of a Newton-Wigner Localization scheme. In Minkowski spacetime, it was proved that for a fixed representation V of $\mathcal{E}(3)$, there is a unique Newton-Wigner localization scheme for each elementary system [18]. However, this result requires further regularity conditions, which are dependent on the symmetries of Minkowski spacetime, and we cannot expect the same result in such a general framework. Nonetheless, we propose a new classification method. Let us fix a representation V of G^S and consider a Newton-Wigner localization scheme (P, V). Then, all other systems of imprimitivity with this same representation V are obtained by unitary maps $Y : \mathcal{H} \to \mathcal{H}$ that commute with V but not with P. Our goal is to further classify these unitarily equivalent spectral measures. For that, we use the following partition of $B(\mathcal{H})^+$ (the set of positive bounded operators) into equivalence classes, first proposed by Thompson in [51].

Definition 4.7. Let $A, B \in B(\mathcal{H})^+$. We say that A and B are **equivalent** $A \sim B$ if, and only if, there exists positive numbers α, β such that:

$$A \leq \alpha B \text{ and } B \leq \beta A.$$

We name each equivalence class as a **Thompson component**.

These Thompson components have interesting geometrical structures and are well studied in the literature [52–54]. Let (P, V) and (Q, V) be two Newton-Wigner localization schemes for the same representation V. Our objective is to use the partition into Thompson components to understand what is the physical difference between them. For the projections in the spectral measures, and for an arbitrary $B \in \mathcal{B}(\Sigma_t)$, the equivalence condition above can be written as the existence of $\alpha, \beta > 0$ such that:

$$\langle \psi, P(B)\psi \rangle - \alpha \langle \psi, Q(B)\psi \rangle \leq 0 \quad \text{and} \quad \langle \psi, Q(B)\psi \rangle - \beta \langle \psi, P(B)\psi \rangle \leq 0, \, \forall \psi \in \mathcal{H}.$$

Equivalently, we can write:

$$\mu^{NW}_{\psi,t}(B) - \alpha\lambda^{NW}_{\psi,t}(B) \leq 0 \quad \text{and} \quad \lambda^{NW}_{\psi,t}(B) - \beta\mu^{NW}_{\psi,t}(B) \leq 0, \forall\psi \in \mathcal{H},$$

where $\mu^{NW}_{\psi,t}(B) \doteq \langle\psi, P(B)\psi\rangle$ and $\lambda^{NW}_{\psi,t}(B) \doteq \langle\psi, Q(B)\psi\rangle$. The important point here is that these equations can only be satisfied if for every ψ given a non-zero probability measure $\mu^{NW}_{\psi,t}(B) \neq 0$ on B it is also true that $\lambda^{NW}_{\psi,t}(B) \neq 0$ and vice versa. The physical interpretation is that if the Newton-Wigner localization scheme (P, V) predicts a non-zero probability of detecting the quantum system in the state ψ in B, then (Q, V) is (locally) equivalent to (P, V) if it also predicts a non-zero probability. We give the following defiition.

Definition 4.8. Let (P, V) and (Q, V) be two Newton-Wigner localization schemes at time t. We say that they are **locally equivalent** in $B \in \mathcal{B}(\Sigma_t)$ if $P(B) \sim Q(B)$. We say that (P, V) and (Q, V) are **totally equivalent** $(P, V) \sim (Q, V)$ if they are locally equivalent for every $B \in \mathcal{B}(\Sigma_t)$.

It follows that if two Newton-Wigner localization schemes are totally equivalent then they cannot be too different: if one says that there is a non-zero probability of detection in an arbitrary region, then the other says the same.

4.1 States following geodesics

A problem related to the Localizability Problem is determining which states of a quantum system are particle states. An even deeper question would be: what exactly are particles? We will not get into this discussion in this work (we refer the interested reader to [6–8]), but in this section we would like to explore one possible criterion. In experiments probing the position of a quantum system, we often observe particles following (classical) trajectories in spacetime, for instance, in Bubble Chambers [55]. From item 6 in Proposition 3.1, we see that, for irreducible, massive, spinless representations of the Poincaré group, these trajectories can be obtained from the expectation values of the Newton-Wigner operators. These are causal curves on Minkowski spacetime for every $\psi \in L^2(\Omega_m, \mu_m)$ (see also [56]). We would like to formulate a similar idea in our present framework. However, since we are in a possibly curved background, instead of straight lines, we need to look for *geodesics*. For this section only, *we make the further assumption that there is a one-parameter group describing time evolution* such that we can time evolve the Newton-Wigner operators $t \mapsto Q^{NW}_i(t)$ through a unitary representation of this group, and that *the operators $Q^{NW}_{i,\alpha}$ are bounded*, meaning that $\phi^i_\alpha(U_\alpha) \subset \mathbb{R}^n$ is bounded for all $i = 1, ..., n$.

Recall that the spectrum of any $Q^{NW}_{i,\alpha}$ is given by:

$$\sigma(Q^{NW}_{i,\alpha}) = \{0\} \cup \overline{\phi^i_\alpha(U_\alpha)}.$$

For simplicity and without loss of generality, suppose that $\phi^i_\alpha(U_\alpha) > 0$ for all $i = 1, ..., n$. Hence, according to our Lemma A.9, we have:

$$\overline{N(Q^{NW}_{i,\alpha})} = \text{co}(\sigma(Q^{NW}_{i,\alpha})).$$

That is, the possible values of the expectation values of $Q^{NW}_{i,\alpha}$ are given by the closed interval in the real line, ranging from 0 to the maximum value of ϕ^i_α.

Next, let us define the following function on the manifold M (defined in terms of the local coordinates of U_α) for each $\psi \in \mathcal{H}$:

$$\gamma_\psi(t) = \left(t, \langle\psi, Q^{NW}_{1,\alpha}(t)\psi\rangle, ..., \langle\psi, Q^{NW}_{n,\alpha}(t)\psi\rangle\right). \tag{4.4}$$

If $\langle\psi, Q^{NW}_{i,\alpha}(t)\psi\rangle \in \phi^i_\alpha(U_\alpha)$ for every $i = 1, ..., n$, then $\gamma_\psi(t)$ defines a curve on M expressed in local coordinates. This will be the case if $\psi \in P(U_\alpha)\mathcal{H}$, since $\sigma(Q^{NW}_{i,\alpha}|_{P(U_\alpha)\mathcal{H}}) = \overline{\phi^i_\alpha(U_\alpha)} = \overline{N(Q^{NW}_{i,\alpha}|_{P(U_\alpha)\mathcal{H}})}$. The relevant question is then: for which states $\psi \in \mathcal{H}$ is the geodesic equation:

$$\frac{d^2\gamma^\lambda_\psi}{dt^2}(t) + \Gamma^\lambda_{\mu\nu}\frac{d\gamma^\mu_\psi}{dt}(t)\frac{d\gamma^\nu_\psi}{dt}(t) = 0 \qquad (\Gamma^\lambda_{\mu\nu} : \text{Christoffel symbols})$$

satisfied? It is not clear that, like in Minkowski spacetime, all these curves will be causal geodesics.

4.2 Decompositions of $L^2(\Sigma_t, \nu_t)$ induced by $M_{i,\alpha}$

The spectral theorem in the direct integral form (Theorem A.25) says that any bounded, self-adjoint operator induces a direct integral such that the operator acts as multiplication by elements of the spectrum on this space. In this section, we will obtain this decomposition for the specific case when the carrier space in equation (4.1) is $L^2(K^S\backslash G^S, \nu_t) \simeq L^2(\Sigma_t, \nu_t)$ (that is, the representation L of K^S is the trivial representation), and the operator $M_{i,\alpha}$ is bounded ($\phi^i_\alpha(U_\alpha)$ has finite volume). The space $L^2(\Sigma_t, \nu_t)$ will usually have non-trivial invariant subspaces under the action of $M_{i,\alpha}$. We will also find a countable decomposition of $L^2(\Sigma_t, \nu_t)$ into a countable direct sum of invariant subspaces where each of them possesses a cyclic vector. We

start with this second decomposition.

Definition 4.9. Let A be a (possibly unbounded) self-adjoint operator. A vector $\psi \in \cap_{n=0}^{\infty}\mathcal{D}(A^n)$ is called a **cyclic vector** if span$\{A^n\psi | n \in \mathbb{N}\}$ is dense in $\mathcal{H}$. If A has a cyclic vector, then we say that it has a **simple spectrum**.

Alternatively, an operator A has a cyclic vector if, and only if, the linear span of vectors $E_A(B)\psi$ is dense in $\mathcal{H}$, where E_A is the spectral measure associated with A, and B varies over all Borel subsets of the spectrum (see [57] for proof of this equivalence).

Example 4.10. Consider the Hilbert space $L^2(\mathbb{R}, dx)$, where dx is the Lebesgue measure, and consider the global position operator on this space given by

$$\begin{aligned}(M_x f)(x) &= xf(x)\\ \mathcal{D}(M_x) &= \{f \in L^2(\mathbb{R}, dx) | xf \in L^2(\mathbb{R}, dx)\}.\end{aligned}$$

This operator has a cyclic vector. Recall that the spectral measure associated with this operator is the multiplication by the characteristic functions of the Borel subsets of $\mathbb{R}$. Define the vector

$$\psi(x) = \sum_{k=-\infty}^{\infty} 2^{-|k|}\chi_{[k,k+1)}(x).$$

Then, $\psi \in L^2(\mathbb{R}, dx)$ and the linear span of vectors of the form $\chi_B\psi$, $B \in \mathcal{B}(\mathbb{R})$, contains the characteristic functions of all bounded Borel sets of the real line. Since this space is dense in $L^2(\mathbb{R}, dx)$, ψ is a cyclic vector.

However, local position operators do not posses cyclic vectors. Let $O \in \mathbb{R}$ be a bounded interval. Then, the local position operator is given by

$$\begin{aligned}(M_O f)(x) &= \chi_O(x)xf(x)\\ \mathcal{D}(M_O) &= \{f \in L^2(\mathbb{R}, dx) | \chi_O xf \in L^2(\mathbb{R}, dx)\}.\end{aligned}$$

Clearly, the image of this operator is in the closed subspace $\chi_O L^2(\mathbb{R}, dx)$, and it can not be equal to the whole space if $O \neq \mathbb{R}$.

The second part of this example teaches us that the local position operators, in general, possess non-trivial invariant subspaces. The identification of all these invariant subspaces is our first goal in this section. In addition, if an operator has a simple spectrum, then the associated direct integral has a simple expression.

Lemma 4.11. *Let $A \in B(\mathcal{H})$ be a self-adjoint operator with cyclic vector ψ. Let μ_ψ be the unique measure such that*

$$\langle \psi, f(A)\psi \rangle = \int_{\sigma(A)} f(\lambda) d\mu_\psi(\lambda),$$

for all continuous, real valued functions f defined on $\sigma(A)$. Then, there exists a unitary map $U : \mathcal{H} \to L^2(\sigma(A), \mu_\psi)$ such that

$$[UAU^{-1}\phi](\lambda) = \lambda\phi(\lambda)$$

for all $\phi \in L^2(\sigma(A), \mu_\psi)$

Proof. See [58]. □

Hence, for position operators with simple spectrum, the two decompositions proposed in this section have simpler forms. However, as illustrated in the above example, this is not the case in general. The following lemma guarantees that the decomposition into direct sums is possible.

Lemma 4.12. *Suppose $A \in B(\mathcal{H})$ is a self-adjoint operator. Then, $\mathcal{H}$ can be decomposed as an orthogonal direct sum*

$$\mathcal{H} = \bigoplus_{j=1} W_j,$$

where each W_j is a closed, non-zero, subspace invariant under A, and such that the restriction of A to each W_j has a cyclic vector ψ_j. The number of W_j's is either finite or countably infinite.

Proof. See [58]. □

Our first goal is to find the decomposition of $L^2(\Sigma_t, \nu_t)$ in terms of the direct sum of these invariant (under a fixed $M_{i,\alpha}$) subspaces. For concreteness and simplicity, let us fix $i = 1$ and perform our analysis for the operator $M_{1,\alpha}$. Unfortunately, there is no algorithm to find these invariant subspaces and the respective cyclic vectors that work for all operators. A good place to start is the following. Consider the closed subspace of functions with support in U_α. By the orthogonal decomposition theorem, we have:

$$L^2(\Sigma_t, \nu_t) = L^2(U_\alpha, \nu_t) \oplus (L^2(U_\alpha, \nu_t))^\perp.$$

Its is clear that $L^2(U_\alpha, \nu_t)$ is invariant under $M_{1,\alpha}$. It can be proven that, if V is an invariant subspace for a bounded operator A, then $V^\perp$ is invariant under A^* (see [58]). Hence, $(L^2(U_\alpha, \nu_t))^\perp$ is also invariant under $M_{1,\alpha}$. Unfortunately, these subspaces do not possess cyclic vectors for the restriction of the position operator, and we need to search if these subspaces possess proper invariant subspaces. Let us begin with the following proposition.

Proposition 4.13. $(L^2(U_\alpha, \nu_t))^\perp = L^2(U_\alpha^c, \nu_t)$.

Proof. It is obvious that $L^2(U_\alpha^c, \nu_t) \subset (L^2(U_\alpha, \nu_t))^\perp$. We want to prove the converse relation. Let $f \in (L^2(U_\alpha, \nu_t))^\perp$. Hence

$$\langle \chi_{U_\alpha}, f \rangle = \int_{U_\alpha} f(s) d\nu_t(s) = 0.$$

The function f can be decomposed into real and imaginary parts, f_r and f_i, respectively, such that

$$f(s) = f_r(s) + i f_i(s),$$

where f_r and f_i are real-valued, measurable functions. Next, define the positive and negative parts of a real-valued, measurable function g as $g^+(s) \equiv \max\{0, g(s)\}$ and $g^-(s) \equiv \max\{0, -g(s)\}$, respectively. These are positive, measurable components, and we can write

$$\begin{aligned} f_r(s) &= f_r^+(s) - f_r^-(s) \\ f_i(s) &= f_i^+(s) - f_i^-(s). \end{aligned}$$

It follows that

$$\begin{aligned} 0 = \langle \chi_{U_\alpha}, f \rangle &= \int_{U_\alpha} f(s) d\nu_t(s) \\ &= \int_{U_\alpha} f_r^+(s) d\nu_t(s) - \int_{U_\alpha} f_r^-(s) d\nu_t(s) + i \int_{U_\alpha} f_i^+(s) d\nu_t(s) - i \int_{U_\alpha} f_i^-(s) d\nu_t(s). \end{aligned}$$

Let $\text{supp}(f_{r,i}^{+,-})$ denote the supports of each component. Then

$$\begin{aligned} \int_{U_\alpha} f(s) d\nu_t(s) = &\int_{U_\alpha \cap \text{supp} f_r^+} f_r^+(s) d\nu_t(s) - \int_{U_\alpha \cap \text{supp} f_r^-} f_r^-(s) d\nu_t(s) \\ &+ i \int_{U_\alpha \cap \text{supp} f_i^+} f_i^+(s) d\nu_t(s) - i \int_{U_\alpha \cap \text{supp} f_i^-} f_i^-(s) d\nu_t(s). \end{aligned}$$

However, f must be orthogonal to *all* functions in $L^2(U_\alpha, \nu_t)$ and, in particular, it must be

orthogonal to all characteristic functions of the type $\chi_{U_\alpha \cap \text{supp} f^{+;-}_{r,i}} \in L^2(U_\alpha, \nu_t)$. This implies, for example, that $\int_{U_\alpha \cap \text{supp} f^+_r} f^+_r(s) d\nu_t(s) = 0$. Since f^+_r is a non-negative function, then it is zero almost-everywhere in U_α. This also implies that f^-_r is zero almost-everywhere in U_α. Repeating this same argument for the imaginary part, we conclude that the function f is null almost-everywhere on U_α and the proposition is proven. □

Therefore, up to now, we have the decomposition in invariant subspaces

$$L^2(\Sigma_t, \nu_t) = L^2(U_\alpha, \nu_t) \oplus L^2(U^c_\alpha, \nu_t). \tag{4.5}$$

The second component of this direct sum corresponds to the space of eigenvectors with eigenvalue 0 (see equation (4.2)). Since it is a subspace of a separable Hilbert space, it is also separable. Then, there is a countable orthonormal basis, call it $\{h_i\}_{i\in\mathbb{N}}$, that span $L^2(U^c_\alpha, \nu_t)$ in the sense that

$$L^2(U^c_\alpha, \nu_t) = \text{span}\{h_1\} \oplus \text{span}\{h_2\} \oplus .... \tag{4.6}$$

In any component of this sum, the operator $M_{1,\alpha}$ acts as multiplication by zero. Hence, each subspace span$\{h_i\}$ is invariant under $M_{1,\alpha}$ with cyclic vector h_i, because the set:

$$\{h_i, M_{1,\alpha}h_i, (M_{1,\alpha})^2 h_i, ...\} = \{h_i, 0, 0, ...\}$$

is dense in span$\{h_i\}$. We have thus found a decomposition of the second component in (4.5) into invariant subspaces with cyclic vectors. Let us do the same for the first component.

We identify the first component with the space:

$$L^2(U_\alpha, \nu_t) \simeq L^2(\phi_\alpha(U_\alpha), dx),$$

where $dx = dx_1 dx_2 ... dx_n$ is the product of the Lebesgue measures in each component of $\mathbb{R}^n$. By its turn, this last space can be seen as an L^2-space constructed from the product σ-algebra of the Lebesgue σ-algebras on each $\phi^i_\alpha(U_\alpha)$, with the product measure $dx = dx_1 \otimes ... \otimes dx_n$ (see [59]). That is, we have

$$L^2(\phi_\alpha(U_\alpha), dx) = L^2(\phi^1_\alpha(U_\alpha), dx_1) \otimes ... \otimes L^2(\phi^n_\alpha(U_\alpha), dx_n),$$

where

$$(f^1 \otimes ... \otimes f^n)(x_1, ..., x_n) = f^1(x_1)...f^n(x_n),$$

for $f^i \in L^2(\phi^i_\alpha(U_\alpha))$. On this space, the operator $M_{1,\alpha}$ acts as multiplication by x_1 on the first component and as identity on the others, that is, $M_{1,\alpha} = M_{x_1} \otimes \mathbb{I} \otimes ... \otimes \mathbb{I}$.

Let $\{e^i_j\}_{j\in\mathbb{N}}$ denote orthonormal basis for the spaces $L^2(\phi^i_\alpha(U_\alpha), dx_i)$ for $i = 2, ..., n$. Then, we can write

$$L^2(\phi_\alpha(U_\alpha), dx) = L^2(\phi^1_\alpha(U_\alpha), dx_1) \otimes \left\{\text{span}\{e^2_1\} \oplus \text{span}\{e^2_2\} \oplus ...\right\} \otimes ... \\ ... \otimes \{\text{span}\{e^n_1\} \oplus \text{span}\{e^n_2\} \oplus ...\}$$

By the distributive property between the tensor product and the direct sum, we have as a result a countable direct sum of tensor product spaces

$$L^2(\phi_\alpha(U_\alpha), dx) = \bigoplus_{a\in\mathbb{N}} \mathcal{L}_a, \tag{4.7}$$

where each element of the sum is of the form

$$\mathcal{L}_a = L^2(\phi^1_\alpha(U_\alpha), dx_1) \otimes \text{span}\{e^2_j\} \otimes ... \otimes \text{span}\{e^n_l\}$$

for some $j, l \in \mathbb{N}$.

Proposition 4.14. *Every $\mathcal{L}_a$, $a \in \mathbb{N}$, in equation* (4.7) *is invariant under $M_{1,\alpha}$ and has cyclic vector $f^a_c = \chi_{\phi^1_\alpha(U_\alpha)}(x_1) \otimes ... \otimes e^n_l(x_n)$.*

Proof. Call $\mathcal{L}_1$ the space

$$\mathcal{L}_1 = L^2(\phi^1_\alpha(U_\alpha), dx_1) \otimes \text{span}\{e^2_1\} \otimes ... \otimes \text{span}\{e^n_1\}$$

where the low index is 1 for all the spans. For simplicity and concreteness, we will prove the Proposition for $\mathcal{L}_1$, but the proof is essentially the same for every $a \in \mathbb{N}$. The operator $M_{1,\alpha}$ acts on $\mathcal{L}_1$ as

$$M_{1,\alpha}\left[f^1(x_1) \otimes ... \otimes e^n_1(x_n)\right] = x_1 f^1(x_1) \otimes ... \otimes e^n_1(x_n).$$

Because M_{x_1} is bounded on $L^2(\phi^1_\alpha(U_\alpha), dx_1)$, it is everywhere defined and $L^2(\phi^1_\alpha(U_\alpha), dx_1)$ is an invariant space for M_{x_1}. Since $M_{1,\alpha}$ acts as identity on the other components of the tensor

product, it follows that $\mathcal{L}_1$ is an invariant subspace under this position operator.

Let us check that f_c^1 is indeed a cyclic vector. We have that

$$(M_{1,\alpha})^m f_c^1 = (x_1)^m \otimes ... \otimes e_1^n(x_n),$$

for $m = 0, ..., \infty$. As a consequence of the Stone-Weierstrass Theorem, polynomials are dense in $L^2(\phi_\alpha^1(U_\alpha), dx_1)$ and the set $\{(M_{1,\alpha})^m f_c^1 | m = 0, ..., \infty\}$ is dense in $\mathcal{L}_1$. □

Therefore, we succeeded in our first purpose: we have found a countable direct sum decomposition of $L^2(\Sigma_t, \nu_t)$ in terms of invariant subspaces under $M_{1,\alpha}$ with cyclic vectors, which is given by

$$L^2(\Sigma_t, \nu_t) = \left(\bigoplus_{a \in \mathbb{N}} \mathcal{L}_a \right) \oplus \text{span}\{h_1\} \oplus \text{span}\{h_2\} \oplus ... \tag{4.8}$$

Let us now find the direct integral representation induced by the generalized position operator. The subspace $L^2(U_\alpha^c, \nu_t)$ is an eigenspace and, according to our discussion after Theorem A.25, it can be isometrically embedded into the direct integral representation. Since the restriction of $M_{1,\alpha}$ to each $\mathcal{L}_a$ has simple spectrum, we can apply Lemma 4.11 directly. Then, for each $\mathcal{L}_a$, there exists a unitary map

$$W_a : \mathcal{L}_a \to L^2(\overline{\phi_\alpha^1(U_\alpha)}, \mu_{f_c^a}),$$

where the measure is given by $\mu_{f_c^a}(B) = \langle f_c^a, P_{M_{1,\alpha}}(B) f_c^a \rangle_{\mathcal{L}_a}$, for $B \in \sigma(M_{1,\alpha}|_{\mathcal{L}_a}) = \sigma(M_{1,\alpha})$, and where $P_{M_{1,\alpha}}$ is the spectral measure of $M_{1,\alpha}$. For example, in $\mathcal{L}_1$, this measure is given by

$$\begin{aligned}
\mu_{f_c^a}(B) &= \langle f_c^a, P_{M_{1,\alpha}}(B) f_c^a \rangle_{\mathcal{L}_1} \\
&= \langle \chi_{\phi_\alpha^1(U_\alpha)}(x_1), \chi_B(x_1) \chi_{\phi_\alpha^1(U_\alpha)}(x_1) \rangle_{L^2(\phi_\alpha^1(U_\alpha), dx_1)} \langle e_1^2, e_1^2 \rangle_{\text{span}\{e_1^2\}} ... \langle e_1^n, e_1^n \rangle_{\text{span}\{e_1^n\}} \\
&= \text{volume}(B).
\end{aligned}$$

In fact, this measure will be the same for any $\mathcal{L}_a$, since the e_j^i's are orthonormal in their respective spaces. For this reason, we will drop the subscript a in the measure henceforth.

Recall from our Example A.24 that each space $L^2(\overline{\phi_\alpha^1(U_\alpha)}, \mu_{f_c^a})$ can be thought as a direct integral with respect to the measure $\mu_{f_c^a}$ where we associate to each $\lambda \in \sigma(M_{1,\alpha}|_{\mathcal{L}_a})$ the Hilbert

space $\mathcal{H}^a_\lambda = \mathbb{C}$. Next, define

$$\mathcal{H}_\lambda = \bigoplus_{a\in\mathbb{N}} \mathcal{H}^a_\lambda,$$

for $\lambda \in \sigma(M_{1,\alpha})$, where to the eigenvalue 0 we associate the space $\mathcal{H}_0 = L^2(U^c_\alpha, \nu_t)$. Since the measures $\mu_{f^a_c}$ are all equal, let us call it ξ. Let us perform the direct integral of this collection of Hilbert spaces with respect to the measure ξ. We have that

$$\begin{aligned}
\int^{\oplus}_{\sigma(M_{1,\alpha})} \mathcal{H}_\lambda d\xi(\lambda) &= \int^{\oplus}_{\sigma(M_{1,\alpha})} \bigoplus_{a\in\mathbb{N}} \mathcal{H}^a_\lambda d\xi(\lambda) \\
&= \bigoplus_{a\in\mathbb{N}} \int^{\oplus}_{\sigma(M_{1,\alpha})} \mathcal{H}^a_\lambda d\xi(\lambda) \\
&= \left(\bigoplus_{a\in\mathbb{N}} L^2(\phi^1_\alpha(U_\alpha), \xi)\right) \oplus L^2(U^c_\alpha, \nu_t) \\
&= \left(\bigoplus_{a\in\mathbb{N}} L^2(\phi^1_\alpha(U_\alpha), dx_1)\right) \oplus L^2(U^c_\alpha, \nu_t).
\end{aligned}$$

By the other hand, we have that $L^2(\Sigma_t, \nu_t) = \left(\bigoplus_{a\in\mathbb{N}} \mathcal{L}_a\right) \oplus L^2(U^c_\alpha, \nu_t)$. Then, finally, define the unitary map $U : L^2(\Sigma_t, \nu_t) \to \int^{\oplus}_{\sigma(M_{1,\alpha})} \mathcal{H}_\lambda d\xi(\lambda)$ given by

$$U : \begin{cases} L^2(U^c_\alpha, \nu_t) \xrightarrow{\mathbb{I}} L^2(U^c_\alpha, \nu_t) \\ \mathcal{L}_a \xrightarrow{W_a} L^2(\phi^1_\alpha(U_\alpha), dx_1) \end{cases}$$

That is, we have that

$$L^2(\Sigma_t, \nu_t) \xrightarrow{U} \left(\bigoplus_{a\in\mathbb{N}} L^2(\phi^1_\alpha(U_\alpha), dx_1)\right) \oplus L^2(U^c_\alpha, \nu_t) = \int^{\oplus}_{\sigma(M_{1,\alpha})} \mathcal{H}_\lambda d\xi(\lambda) \tag{4.9}$$

Therefore, we have found the direct integral representation of the position operator $M_{1,\alpha}$. This construction can be repeated for any $M_{i,\alpha}$.

4.3 Newton-Wigner operator on perturbed Minkowski spacetime

In this section, we aim to investigate the effects of perturbations in the spatial metric on the Newton-Wigner operators. This is motivated by the fact that our laboratories are situated in a

spacetime that is not perfectly flat, but rather slightly curved. We will do it in the framework of Minkowski in 1+1 dimensions for a massive, spinless, irreducible representation of $\mathcal{P}_+^\uparrow$. The Hilbert space of this system is $\mathcal{H} = L^2(\mathbb{R}, dp/\omega)$, and equation (3.4) gives a unitary map $W : L^2(\mathbb{R}, dp/\omega) \to L^2(\mathbb{R}, dx)$.

Our strategy is to add a perturbation on the Euclidean metric in the following way. The Euclidean metric is defined as

$$g(x)(v, w) = \langle v, w \rangle, \tag{4.10}$$

where $v, w \in T_x\mathbb{R}$ and $\langle , \rangle$ denotes the usual inner product. If we multiply this metric by a positive-definite real function $\phi(x)$, the resulting metric is still Riemannian. For example, for $\phi_\epsilon(x) = 1 - \epsilon e^{-x^2}$, where $0 < \epsilon < 1$, we have the perturbed metric

$$g_\epsilon(x)(v, w) = (1 - \epsilon e^{-x^2})\langle v, w \rangle. \tag{4.11}$$

The corresponding volume measure ν_ϵ will be

$$\nu_\epsilon(A) = \int_A \sqrt{\det g_\epsilon^{ij}}\, dx \tag{4.12}$$

$$= \int_A \sqrt{\phi_\epsilon(x)}\, dx, \tag{4.13}$$

where $A \in \mathcal{B}(\mathbb{R})$.

Our next steps are the following. We begin by finding a unitary operator between $L^2(\mathbb{R}, \nu_\epsilon)$ and $L^2(\mathbb{R}, dx)$ to map the position operator M_ϵ (multiplication operator by the coordinate x) on the first space into the second; then we use the unitary map W^{-1} to map it to $\mathcal{H}$. Let us define the map $T : L^2(\mathbb{R}, \nu_\epsilon) \to L^2(\mathbb{R}, dx)$ as:

$$(Tg)(x) = \phi_\epsilon^{1/4}(x) g(x), \tag{4.14}$$

where $g \in L^2(\mathbb{R}, \nu_\epsilon)$. It is straightforward to see that this is an unitary map

$$\langle Tg, Th \rangle_{L^2(\mathbb{R}, dx)} = \int_{\mathbb{R}} (Tg)(x)\overline{(Th)}(x)\, dx \tag{4.15}$$

$$= \int_{\mathbb{R}} g(x)\overline{h}(x)\sqrt{\phi_\epsilon(x)}\, dx \tag{4.16}$$

$$= \langle g, h \rangle_{L^2(\mathbb{R}, \nu_\epsilon)}. \tag{4.17}$$

It is also easy to check that the inverse map is given by $(T^{-1}g)(x) = \phi_\epsilon^{-1/4}(x) g(x)$. Next, we

examine how this unitary operator maps the position operator M_ϵ into $L^2(\mathbb{R}, dx)$. To this end, we Taylor expand the function $\phi_\epsilon^{1/4}(x)$ and its inverse around $\epsilon = 0$ at first order, finding that

$$\phi_\epsilon^{1/4}(x) \approx 1 - \frac{\epsilon}{4} e^{-x^2} \tag{4.18}$$

$$\phi_\epsilon^{-1/4}(x) \approx 1 + \frac{\epsilon}{4} e^{-x^2}. \tag{4.19}$$

Then, it follows that

$$(TM_\epsilon T^{-1} g)(x) = TM_\epsilon \left(g(x) + \frac{\epsilon}{4} e^{-x^2} g(x) \right) \tag{4.20}$$

$$= T \left(xg(x) + \frac{x\epsilon}{4} e^{-x^2} g(x) \right) \tag{4.21}$$

$$= \left(1 - \frac{\epsilon}{4} e^{-x^2} \right) \left(xg(x) + \frac{x\epsilon}{4} e^{-x^2} g(x) \right) \tag{4.22}$$

$$= xg(x) + \frac{x\epsilon}{4} e^{-x^2} g(x) - \frac{x\epsilon}{4} e^{-x^2} g(x) - \frac{x\epsilon^2}{16} e^{-2x^2} g(x) \tag{4.23}$$

$$= xg(x) - \frac{\epsilon^2 x}{16} e^{-2x^2} g(x). \tag{4.24}$$

Hence, the action of M_ϵ in $L^2(\mathbb{R}, dx)$ is the usual position operator plus a correction which depends quadratically on the perturbation parameter.

Let $\mathcal{F} : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dp)$ denote the Fourier transform into the momentum space and $\hat{g}(p) 0 \left(\mathcal{F}(g) \right)(p)$. We have that

$$\left[\mathcal{F} \left(TM_\epsilon T^{-1} \right) \mathcal{F}^{-1} \hat{g} \right](p) = \left[\mathcal{F} \left(xg - \frac{\epsilon^2 x}{16} e^{-x^2} g \right) \right](x) \tag{4.25}$$

$$= -i \frac{\partial \hat{g}(p)}{\partial p} - \frac{\epsilon^2}{16} \mathcal{F} \left(x e^{-2x^2} g(x) \right) \tag{4.26}$$

$$= -i \left[\frac{\partial \hat{g}(p)}{\partial p} + \frac{\epsilon^2}{128} p e^{-p^2/8} * \hat{g}(p) \right], \tag{4.27}$$

where in the last line we used the Convolution Theorem, and where $*$ denotes the convolution.

Finally, we can map this operator we obtained in the last equation to $\mathcal{H}$ using the unitary map $Z : L^2(\mathbb{R}, dp) \to L^2(\mathbb{R}, dp/\omega)$ given by $(Z\psi)(p) \equiv \sqrt{\omega}\psi(p)$. Since W^{-1} is the composition of $\mathcal{F}$ followed by Z, the resulting operator is the **perturbed Newton-Wigner operator**, Q_ϵ^{NW}.

We can compute directly to find:

$$(Q_\epsilon^{NW}\psi)(p) = \left[Z\mathcal{F}TM_\epsilon T^{-1}\mathcal{F}^{-1}Z^{-1}\psi\right](p) \tag{4.28}$$

$$= \left[Z\left(-i\frac{\partial}{\partial p}\right)Z^{-1}\psi\right](p) + \left[Z\left(-i\frac{\epsilon^2 p}{128}e^{-p^2/8}\right) * (Z^{-1}\psi)\right](p) \tag{4.29}$$

$$= i\left(\frac{\partial}{\partial p} - \frac{p}{2\omega^2}\right)\psi(p) - i\epsilon^2\left[\frac{\sqrt{\omega}}{128}\left(pe^{-p^2/8}\right) * \left(\frac{\psi}{\sqrt{\omega}}\right)\right](p). \tag{4.30}$$

Therefore, the perturbed Newton-Wigner operator corresponds to the usual operator plus a correction depending quadratically on the perturbation parameter. This construction was done for a simple gaussian perturbation, but the same steps could be followed for a more complicated perturbation.

Part II

Modular Localization and the Localizability Problem

Chapter 5

Logics and Measurements

> *I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our "creations," are simply the notes of our observations.*
>
> *G.H. Hardy*

When looking at the fundamental laws of Physics, it always astonished me that these laws are so general that they accommodate not only the reality of what is but also what could have been. Cosmologists say that observations of the matter distribution in the Universe indicate that the Universe is approximately flat, or at least with a mild curvature. However, General Relativity is so powerful, that even if the matter content were completely different from what it is, it would still be able to describe the Universe. In these alternative realities, we could have positively or negatively curved spacetime or even closed causal curves. General Relativity transcends what it is to whatever could have been. At least as far as our current understanding allows us to go. As if this was not far enough, Mathematics seems to go one step further and fluctuate above it all, governing and being obeyed. When General Relativity becomes old and obsolete, Mathematics will welcome the new theory with its timeless youthfulness.

Yet, even Mathematics itself has its own basic rules, and many would agree that *Logic* lies at the very bottom of its structure. Logical principles, such as implication, negation, and quantifiers, are at the axiomatic formulation of Set Theory, for instance, and appear explicitly or implicitly in any branch of Mathematics. On the other hand, some problems in Physics are so involved, and the most obvious attempts fail so drastically, that we are forced to give up on complex mathematical structures and demand only the bare logical minimum. This approach goes beyond specific models and relies only on fundamental logical principles. As we will see, this is the case for the Localizability Problem. However, before we focus on this specific problem, let us discuss a few more general examples. Given a physical system that we want to study

(and we don't specify at the moment whether it is classical or quantum), all that we can learn about this system comes from experiments. In the language introduced in the Introduction 1, these experiments are performed by instruments Q, whose equivalence classes form the (experimental) observables $[Q]$, as in Definition 1.1. However, the way in which these instruments are used *necessarily follows logical rules*. These rules can be translated into a precise mathematical language, which we will call a *logic* (Definition 5.5). With this, we are not providing a model for the physical system. We are simply adding constraints that necessarily will have to be incorporated into the model.

We illustrate with an example. Suppose we have an experiment that can determine whether a given system is inside a cylinder B_1 in $\mathbb{R}^3$ or not, as in the figure below (think about it as a *Geiger counter* represented by the set B_1).

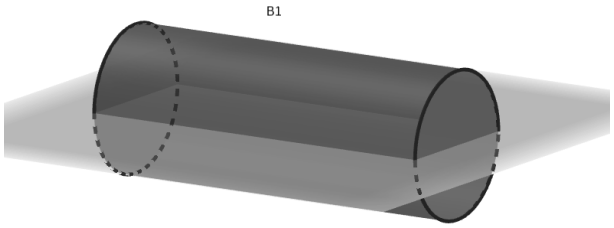


Figure 5.1: Cylinder B_1 in $\mathbb{R}^3$.

In this case, B_1 represents an instrument that measures the system's position. Consider now that we have another cylinder B_2 which represents a similar experiment that is equally capable of determining if the system is inside it or not. In addition, suppose that B_2 is contained in B_1, as in the figure below.

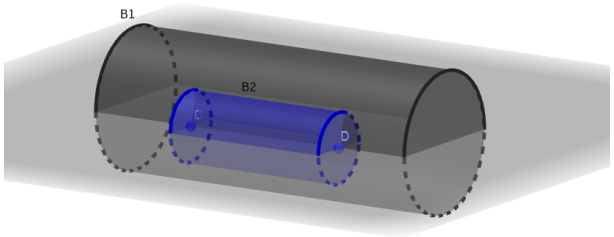


Figure 5.2: Cylinder $B_2 \subset B_1$ in $\mathbb{R}^3$.

Our first encounter with a logical principle is then very clear: if the system is inside B_2, this *implies* that it is also inside B_1. Suppose now (with a bit more abstraction) that these two cylinders intersect, as in the next figure.

If the system is detected in the region where these cylinders intersect, then it is inside both

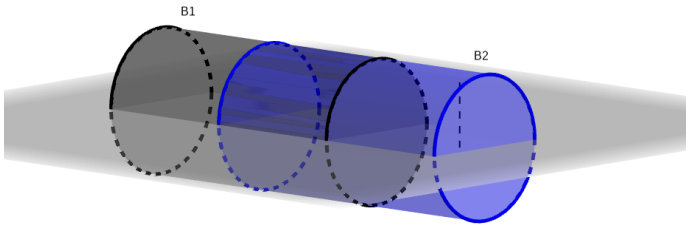


Figure 5.3: Intersecting cylinders in $\mathbb{R}^3$.

B_1 *and* B_2. Next, consider that we have a big cylinder B_3 that contains both B_1 and B_2 in such a way that B_3 is the union of B_1 and B_2 and such that these don't intersect.

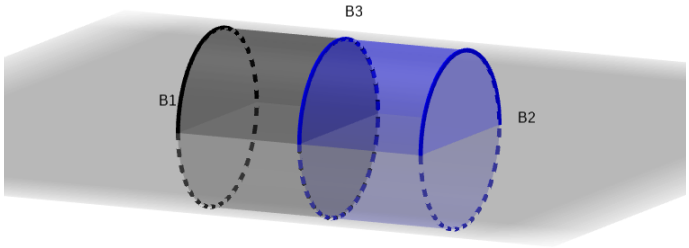


Figure 5.4: Union of cylinders in $\mathbb{R}^3$.

If B_3 detects the system, this means that it is inside B_1 *or* B_2. Finally, given a cylinder B_1, if the system is not detected in it, this means that it must be outside of B_1. Thus we say that the system is not in this cylinder. In other words, we *negate* that the system is in B_1.

Let us now translate all this into a precise mathematical language: we associate to each possible detector a subset of $\mathbb{R}^3$. We will choose these to be Borel subsets (that is, elements of the Borel σ-algebra $\mathcal{B}(\mathbb{R}^3)$) since these carry a sufficiently rich mathematical structure; the *implication* rule can be translated into a partial order "$\leq$" (that is, an antisymmetric, reflexive, transitive relation), in this case, given by the inclusion of sets; the logical connectives *and* and *or* can be translated into the intersection "$\cap$" and union "$\cup$" of sets, respectively; finally, the *negation* is translated into the set complement "^{c}". In summary, we have a mathematical structure $\mathcal{L}$ given by a quintuple $\mathcal{L} = (\mathcal{B}(\mathbb{R}^3), \leq, \cup, \cap, ^c)$. This structure is just a specific example of a general structure called *logic*, which we now define. The following definitions and concepts can be found in standard textbooks such as [60, 61].

5.1 Basic Concepts

Definition 5.1. Let $(X, \leq)$ be a partially ordered set. We call it a **lattice** if every pair of elements $a, b \in X$ has both a least upper bound (denoted by "$a \vee b$" and also called **join**) and a greater lower bound (denoted by "$a \wedge b$" and also called **meet**).

The operations join and meet are easily verified to be binary, associative, and commutative. Also, the **absorption laws** are satisfied, that is $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ for every $a, b \in X$. Based on these properties, we can give another definition of a lattice, based only on the algebraic relations just presented.

Definition 5.2 (Algebraic definition)**.** A **lattice** is an algebraic structure $\mathcal{L} = (X, \vee, \wedge)$ consisting of a set X, and two binary, commutative, and associative operations satisfying the absorption laws.

We already showed that a lattice according to the order-theoretic definition is a lattice with the algebraic definition. To go the other way round, we just notice that starting with an algebraic lattice we can define a partial order on it as: $a \leq b$ if $a = a \wedge b$. Hence, we will interchangeably refer to a lattice as an algebraic structure or a partially ordered set with the properties above. A lattice may have a least element and a greatest element, which we will denote by 0 and 1, respectively. In our example above, these are simply $0 = \emptyset$ and $1 = \mathbb{R}^3$. Another very important notion in a lattice is that of a complement.

Definition 5.3. An **orthocomplementation** in a lattice $\mathcal{L}$ is a mapping $\mathcal{L} \ni a \mapsto a^\perp \in \mathcal{L}$ such that:

1. $a^{\perp\perp} = a$.
2. $a \leq b$ implies $b^\perp \leq a^\perp$.
3. $a \wedge a^\perp = 0$ and $a \vee a^\perp = 1$.

A lattice with an orthocomplementation is called an **orthocomplemented lattice**. If only condition 3 is not satisfied we call it a **(pseudo-) orthocomplemented lattice**.

It follows from item 2 in this definition that the meet and join are not independent. In fact, the *De Morgan* laws hold:

$$\left(\bigwedge_{i\in I} a_i\right)^\perp = \bigvee_{i\in I} a_i^\perp, \quad \left(\bigvee_{i\in I} a_i\right)^\perp = \bigwedge_{i\in I} a_i^\perp, \tag{5.1}$$

where I is any finite set. In our example above, this orthocomplementation is given by the set complement. Two lattices can have similar structures, and this gives rise to our next definition.

Definition 5.4. Let $\mathcal{L}_1 = (X, \vee_X, \wedge_X, \perp_X, 0_X, 1_X)$ and $\mathcal{L}_2 = (Y, \vee_Y, \wedge_Y, \perp_Y, 0_Y, 1_Y)$ be two orthocomplemented lattices. A map $h : X \to Y$ is a **homomorphism** if

1. $h(0_X) = 0_Y$ and $h(1_X) = 1_Y$.
2. $h(a \vee_X b) = h(a) \vee_Y h(b)$ and $h(a \wedge_X b) = h(a) \wedge_Y h(b)$ for every $a, b \in \mathcal{L}_1$.
3. $h(a^{\perp_X}) = h(a)^{\perp_Y}$ for every $a \in \mathcal{L}_1$.

We call it an **isomorphism** if it has the further property that $h(a) = 0_Y$ if, and only if, $a = 0_X$.

As we see, the structure of an orthocomplemented lattice already captures (most of) the essential features of the logic behind the position measurements in the example above. However, as it stands, this structure is yet too general and hard to work with. We need some extra technical requirements, that are not physically motivated but are necessary if we want to go further without great technical difficulties.

Definition 5.5. Let $\mathcal{L}$ be an orthocomplemented lattice. We call it a **logic** if:

1. for any countably infinite sequence $\{a_i\}_{i\in\mathbb{N}}$ of elements in $\mathcal{L}$, $\bigwedge_{i\in\mathbb{N}} a_i$ and $\bigvee_{i\in\mathbb{N}} a_i$ exist in $\mathcal{L}$.
2. if $a_1, a_2 \in \mathcal{L}$ and $a_1 \leq a_2$, then there exists an element $b \in \mathcal{L}$ such that $b \leq a_1^{\perp}$ and $b \vee a_1 = a_2$.

If the element b in item 2 above exists, it is unique and equal to $a_1^{\perp} \wedge a_2$. In our example above (which is clearly a logic), this element is the relative complement of a_1 with respect to a_2. We have now come to a very important point. In the example given above, the logic $\mathcal{L} = (\mathcal{B}(\mathbb{R}^3), \cup, \cap, ^c)$ implements all the experimentally verifiable propositions about the localization of the system, that is, to every proposition of the form:

"the system is localized in the region $B \subseteq \mathbb{R}^3$",

we associate the set B in the logic $\mathcal{L}$. More generally, to any (experimental) observable Q (Definition 1.1) of the system, we can formulate a similar proposition:

"the measured value of the observable Q lies in the set $E \subseteq \mathbb{R}$".

Our basic assumption is that the set of all experimentally verifiable propositions about a physical system forms a logic. This condition puts constraints on the way the instruments Q themselves are used. We also assume, for technical reasons, that these sets E in $\mathbb{R}$ are elements of the Borel σ-algebra of the real line. The logic associated with a given physical system encodes its most profound characteristics. For instance, we will see that the logic of classical systems is intrinsically different from the logic of quantum systems. Let us proceed with some more basic definitions first.

From now on we will assume that a logic $\mathcal{L}$ associated with a physical system is given. On top of it, we will construct some important concepts.

Definition 5.6. Let $\mathcal{L}$ be a logic. An **observable associated with** $\mathcal{L}$ is a mapping:

$$x : E \mapsto x(E) \in \mathcal{L}$$

where $E \in \mathcal{B}(\mathbb{R})$, and such that:

1. $x(\emptyset) = 0$ and $x(\mathbb{R}) = 1$.
2. If $E, F \in \mathcal{B}(\mathbb{R})$ and $E \cap F = \emptyset$, then $x(E) \leq x(F)^{\perp}$.
3. If $E_1, E_2, ...$ is a sequence of mutually disjoint Borel sets in $\mathbb{R}$, then:

$$x\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \bigvee_{n\in\mathbb{N}} x(E_n).$$

We denote by $\mathcal{O}(\mathcal{L})$ the set of all observables associated with the logic $\mathcal{L}$.

The idea of this definition is the following: given that we used an instrument to measure the system, we associate the proposition that the measured value of this instrument lies in the Borel set E with the element $x(E) \in \mathcal{L}$. In other words, an observable in the logic-theoretic sense is a collection of elements of the logic formed by propositions of the type given above, and such that these three properties are satisfied. If $f : \mathbb{R} \to \mathbb{R}$ is measurable, we denote by:

$$f \circ x \tag{5.2}$$

the observable whose measured value is $f(r)$ whenever the measured value of x is r. This observable corresponds to the mapping $\mathbb{R} \ni E \mapsto x(f^{-1}(E)) \in \mathcal{L}$. For each observable, we can formulate the idea of a spectrum as follows.

Definition 5.7. Let x be an observable in a logic $\mathcal{L}$. We define the **spectrum** of x, denoted by $\sigma(x)$, as:

$$\sigma(x) = \bigcap_{C \text{ closed, } x(C)=1} C.$$

That is, $\sigma(x)$ is the intersection of all closed Borel sets C such that $x(C) = 1$.

The spectrum can be discrete, bounded, or unbounded (depending on whether this intersection has these properties). The experienced reader might note the resemblance with the notion of the spectrum of an observable usually given in Quantum Mechanics. These definitions will in fact coincide when we analyse the logic of Quantum Mechanics, which we will do later in this chapter. For now, we proceed with some more basic definitions.

Having defined a general notion of observables associated with a logic, we now need a corresponding notion of states. Loosely speaking, the idea of a state of a system is something that encodes all of its measurable characteristics, that is, knowing that the system is in a given state should allow us to know which would be the value of any observable to be read in an experiment, or at least it should give us the probability of reading such a value. The precise definition is as follows.

Definition 5.8. Let $\mathcal{L}$ be a logic and $\mathcal{O}(\mathcal{L})$ the set of all its observables. A **state of** $\mathcal{L}$ is a map

$$P : \mathcal{O}(\mathcal{L}) \ni x \mapsto P_x,$$

which assigns to each observable $x \in \mathcal{O}(\mathcal{L})$ a probability measure P_x on $\mathcal{B}(\mathbb{R})$ such that for any Borel function $f : \mathbb{R} \to \mathbb{R}$ and any observable x,

$$P_{f \circ x}(E) = P_x(f^{-1}(E)).$$

The interpretation is that, if the system is in the state P, then there is a probability $P_x(E)$ of reading the values $E \in \mathcal{B}(\mathbb{R})$ when measuring the observable x. It turns out that we can generalize the concept of probability measures (usually defined on a σ-algebras) to logics, as follows.

Definition 5.9. Let $\mathcal{L}$ be a logic. A **probability measure on** $\mathcal{L}$ is a function $p : \mathcal{L} \to \mathbb{R}$ such that:

1. $0 \leq p(a) \leq 1$ for all $a \in \mathcal{L}$.
2. $p(0) = 0$ and $p(1) = 1$.

3. If $a_1, a_2, ...$ is a sequence of mutually orthogonal elements of $\mathcal{L}$ and $a = \bigvee_n a_n$, then $p(a) = \sum_n p(a_n)$.

Since every σ-algebra is a logic, this definition coincides with the usual one when restricted to this case. Notice that, due to condition 2 in Definition 5.5, if $a_1, a_2 \in \mathcal{L}$ and $a_1 \leq a_2$, then $p(a_1) + p(b) = p(a_2)$, where b is the relative complement of a_1 with respect to a_2. This means that:

$$a_1 \leq a_2 \implies p(a_1) \leq p(a_2).$$

The following result shows that there is a one-to-one correspondence between states and probability measures.

Theorem 5.10. *Let $\mathcal{L}$ be a logic, $\mathcal{O}(\mathcal{L})$ the set of all its observables, and p a probability measure on $\mathcal{L}$. Then, for any observable $x \in \mathcal{O}(\mathcal{L})$, and any Borel set $E \in \mathcal{B}(\mathbb{R})$, the function defined as:*

$$P_x^p(E) \doteq p(x(E))$$

is a probability measure on $\mathcal{B}(\mathbb{R})$ and:

$$P^p : \mathcal{O}(\mathcal{L}) \ni x \mapsto P_x^p$$

is a state on $\mathcal{L}$. Conversely, if P is an arbitrary state on $\mathcal{L}$, there exists one, and only one, probability measure p such that:

$$P_x(E) = p(x(E))$$

for all $x \in \mathcal{O}(\mathcal{L})$ and all $E \in \mathcal{B}(\mathbb{R})$.

Proof. See Thereom 3.5 in [60]. □

5.2 The Logic of Classical Mechanics

Now that we have defined the necessary basic concepts, we deepen our study by investigating some important examples. We start by putting the Hamiltonian formulation of Classical Mechanics in this language. The first step is to define its logic. Consider a classical system with n

degrees of freedom, described in a phase space $\mathbb{F}$ of dimension $2n$ with the structure of a symplectic manifold. Since $\mathbb{F}$ is a topological space, we can consider the Borel σ-algebra generated from its topology, which we denote as $\mathcal{B}(\mathbb{F})$. We then define the logic of Classical Mechanics to be:

$$\mathcal{L}_{CM} \doteq (\mathcal{B}(\mathbb{F}), \cup, \cap, ^c).$$

It is easy to verify that this is indeed a logic according to Definition 5.5. Notice further that for any $a, b, c \in \mathcal{L}_{CM}$, the following identities, called the **distributive laws**, are satisfied:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{5.3}$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{5.4}$$

This gives rise to the following very important definition.

Definition 5.11. We call an orthocomplemented, distributive lattice a **Boolean algebra**. A **Boolean σ-algebra** is a Boolean algebra where $\bigvee_{i\in I} a_i$ and $\bigwedge_{i\in I} a_i$ exists for every countable subset I of the lattice.

Hence, the logic of Classical Mechanics is an example of a Boolean σ-algebra. The importance of Boolean algebras lies in the fact that they model the so-called "Propositional Calculus", that is, the formal language of the connectives of implication, negation, conjunction, and disjunction, and it finds applications in many areas, such as programming languages, statistics, and many others [62]. Notice that not every logic needs to be Boolean since the distributive law is not necessarily satisfied in every logic. Given an arbitrary set, the class of all its subsets is a Boolean algebra under set union, intersection, and complementation. Also, every σ-algebra is a Boolean σ-algebra. Does the opposite hold? That is, is it true that every Boolean algebra is isomorphic with some σ-algebra? The answer is no, but the *Loomis-Sikorski Theorem* (see Theorem 1.3 in [60]) states that any Boolean σ-algebra is isomorphic with a quotient $\Sigma/\mathcal{N}$, where Σ is a σ-algebra of measurable sets and $\mathcal{N} \subset \Sigma$ is a closed subset such that $0 \in \mathcal{N}$, $1 \notin \mathcal{N}$, and if $a \in \mathcal{N}$ and $b \leq a$, then $b \in \mathcal{N}$.

The next thing we want to investigate is the set $\mathcal{O}(\mathcal{L}_{CM})$ of observables associated with $\mathcal{L}_{CM}$. In the usual approach to Hamiltonian Classical Mechanics, the observables are taken to be the real-valued, measurable functions on the phase space. The next theorem shows that this notion of observable is equivalent to our logic-theoretic notion of observable defined in Definition 5.6 exactly because the logic of classical mechanics is a Boolean σ-algebra.

Theorem 5.12. *Let X be a set and $\mathcal{L}$ be a Boolean σ-algebra of subsets of X and let $x : \mathcal{B}(\mathbb{R}) \ni E \mapsto x(E) \in \mathcal{L}$ be an observable. Then there exists a unique measurable, real-valued function f on X such that*

$$x(E) = f^{-1}(E)$$

for all $E \in \mathcal{B}(\mathbb{R})$.

Proof. See Theorem 1.4 in [60]. □

Finally, it only remains to understand the states on $\mathcal{L}_{CM}$. As we saw in Theorem 5.10, these can be described by probability measures on $\mathcal{B}(\mathbb{F})$. Let $r(t) \in \mathbb{F}$ be an arbitrary point in the phase space for a given time t (we are assuming that the evolution equation is given by the integral curves of Hamilton's differential equations). We can define a *sharp state* as:

$$\nu_t(E) \doteq \delta_{r(t)}(E), \quad E \in \mathcal{B}(\mathbb{F}), \tag{5.5}$$

where δ_r is the Dirac measure. More generally, like in the case of Statical Mechanics or Thermodynamics, the state of the system is not known exactly and it can not be taken to be a sharp state. In these cases, we use statistical ensembles. These can be described as follows. Let $\rho(t, q, p) : \mathbb{F} \to \mathbb{R}$ be a measurable function such that:

$$\rho(t, q, p) \geq 0 \text{ and } \int_{\mathbb{F}} \rho(t, q, p) d\mu = 1,$$

where μ is the symplectic measure constructed from the symplectic form on $\mathbb{F}$. Then we can define a *probabilistic state* on $\mathcal{B}(\mathbb{F})$ as:

$$\nu_t(E) \doteq \int_E \rho(t, q, p) d\mu, \quad E \in \mathcal{B}(\mathbb{F}).$$

5.3 The Logic of Quantum Mechanics

As we have seen, the construction of the logic associated with a physical system is completely determined by the nature of the measurements we can do on this system. It is an experimental fact that measurements in Quantum Mechanics are fundamentally different from Classical Mechanics. This is due to the fact that in the former we have *incompatible observables*, and this changes completely the structure of the logic. We illustrate with the classical example of the position and momentum observables. Suppose we have a quantum system and we associate

with it a logic $\mathcal{L}$. Heisenberg's uncertainty principle states that:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2},$$

where σ_x and σ_p are the standard deviations of the position and momentum measurements, respectively. Let us translate this to the language of lattices and logics that we have been working with. The standard deviation gives the precision of the measurement, and we can associate with it an interval in the real line, meaning that the result of the experiment will be inside this interval. Hence, we identify σ_x and σ_p with the intervals I and J in $\mathbb{R}$. We can formulate a proposition about the system in the following way:

"The measured value of the position of the system lies in the interval $I \subset \mathbb{R}$".

And similarly for the momentum. As we did in the last sections, we associate to this statement an element of the logic $\mathcal{L}$. However, if the product $|I| \cdot |J|$ is smaller than $\hbar/2$, *there is no experimental setup that can verify both of these propositions at the same time*. As we will see, the consequence of this fact is that a logic that admits incompatible elements cannot be a Boolean algebra and therefore is fundamentally different from the logic of Classical Mechanics. The precise definition of compatibility of elements in a logic is the following.

Definition 5.13. Let $\mathcal{L}$ be a logic. We say that two elements $a, b \in \mathcal{L}$ are **compatible** (or simultaneously verifiable) if there are elements $a_1, b_1, c \in \mathcal{L}$ such that

1. a_1, b_1, c are mutually orthogonal.
2. $a = a_1 \vee c$ and $b = b_1 \vee c$.

If two elements are compatible we write it as $a \leftrightarrow b$.

This definition doesn't look like the usual definition of incompatibility in Quantum Mechanics formulated in terms of commutation rules. However, we will see below that these definitions coincide. Let $x, y \in \mathcal{O}(\mathcal{L})$ be two observables. We say that they are **compatible** (or simultaneously observable) if for any two Borel subsets $E, F \in \mathcal{B}(\mathbb{R})$, $x(E) \leftrightarrow x(F)$. We denote compatible observables by $x \leftrightarrow y$. We have the following fundamental result.

Theorem 5.14. *Let $\mathcal{L}$ be a logic and $a, b \in \mathcal{L}$. Then, the following statements are equivalent.*

1. $a \leftrightarrow b$.
2. *There exists an observable x and two Borel sets $A, B \in \mathcal{B}(\mathbb{R})$ such that $x(A) = a$ and $x(B) = b$.*

3. There exists a sublogic of $\mathcal{L}$ which is Boolean and contains a and b.

Proof. See Lemma 3.7 in [60]. □

This theorem shows that if a logic has incompatible elements, then it can not be Boolean, and hence it is structurally different from Classical Mechanics. The following theorem characterizes the compatible elements.

Theorem 5.15. *Let $\mathcal{L}$ be a logic and $a, b \in \mathcal{L}$. If $a \leftrightarrow b$, then the elements a_1, b_1 and c in Definition 5.13 are uniquely determined by:*

$$a_1 = a \wedge b^{\perp}$$
$$b_1 = b \wedge a^{\perp}$$
$$c = a \wedge b.$$

Proof. See Theorem 12.1.2 in [61]. □

Our next goal is to extend our understanding of compatible elements in a logic to compatible observables. We have the following fundamental theorem.

Theorem 5.16. *Let $\mathcal{L}$ be a logic and $\{x_\lambda\}_{\lambda \in D}$ a family of observables. Suppose that $x_\lambda \leftrightarrow x_{\tilde{\lambda}}$ for all $\lambda, \tilde{\lambda} \in D$. Then, there exists a set X, a σ-algebra $\mathcal{B}$ of subsets of X, real-valued $\mathcal{B}$-measurable functions g_λ on X, and a homomorphism $\tau : \mathcal{B} \to \mathcal{L}$ such that*

$$\tau(g_\lambda^{-1}(E)) = x_\lambda(E)$$

for all $\lambda \in D$ and $E \in \mathcal{B}(\mathbb{R})$. Suppose further that D is countable. Then there exists an observable x and Borel functions $f_\lambda : \mathbb{R} \to \mathbb{R}$ such that (remember equation (5.2)*):*

$$x_\lambda = f_\lambda \circ x$$

for all $\lambda \in D$.

Proof. See Theorem 3.9 in [60]. □

The first part of this theorem shows that compatible observables can always be described by functions on a σ-algebra of sets (which is always Boolean), and the second part shows that in a countable set of compatible observables, all observables are functions of a single one. It also proves that logics in which all observables are mutually compatible are necessarily Boolean σ-algebras, which resemble (and are usually understood to be associated with) classical systems.

Furthermore, it indicates that the structure of observables in non-Boolean logics is potentially more complicated since we cannot say that they are all functions of a single observable. This is indeed the case for Quantum Mechanics, as we now investigate.

Following the same steps as we did in the Classical Mechanics case, we first try to define the logic of Quantum Mechanics. Since we don't have a phase space at our disposal anymore, the answer to this question is not so obvious. Following the axioms of Quantum Mechanics, we assume that there is a separable, complex Hilbert space $\mathcal{H}$ with inner product $\langle , \rangle$ associated with our quantum system. The first to propose a logic for quantum systems was von Neumann and Birkhoff in their seminal paper [63]. The proposed idea is to *define the logic of quantum mechanical systems using the set of all closed subspaces of* $\mathcal{H}$, which we denote by $\mathcal{M}(\mathcal{H})$. Let us define some operations on this set. We define the *join* of two closed subspaces $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{M}(\mathcal{H})$ as:

$$\mathcal{M}_1 \vee \mathcal{M}_2 \doteq \overline{\mathcal{M}_1 + \mathcal{M}_2},$$

and the *meet* as:

$$\mathcal{M}_1 \wedge \mathcal{M}_2 \doteq \mathcal{M}_1 \cap \mathcal{M}_2.$$

The orthocomplementation is naturally given by the orthogonal complement:

$$\mathcal{M}_1^{\perp} \doteq \{\psi \in \mathcal{H} | \langle \psi, \phi \rangle = 0 \text{ for all } \phi \in \mathcal{M}_1\}.$$

With these definitions, it is easy to see that:

$$\mathcal{L}_{QM} \doteq (\mathcal{M}(\mathcal{H}), \vee, \wedge, \perp)$$

is a logic (where $0 = \emptyset$, $1 = \mathcal{H}$), which we call the *logic of Quantum Mechanics*. Notice that, since there is a one-to-one relation between closed subspaces in a Hilbert space and orthogonal projections, we could equivalently write this logic as:

$$\mathcal{L}_{QM} \doteq (\mathcal{P}(\mathcal{H}), \vee, \wedge, \perp), \tag{5.6}$$

where $\mathcal{P}(\mathcal{H})$ is the set of all orthogonal projections, and the operations are defined as:

$$P_{\mathcal{M}_1} \vee P_{\mathcal{M}_2} \doteq P_{\overline{\mathcal{M}_1+\mathcal{M}_2}}$$
$$P_{\mathcal{M}_1} \wedge P_{\mathcal{M}_2} \doteq P_{\mathcal{M}_1 \cap \mathcal{M}_2}$$
$$P^{\perp}_{\mathcal{M}_1} \doteq P_{\mathcal{M}_1^{\perp}}$$

where $P_{\mathcal{M}_i}$ denotes the orthogonal projection into $\mathcal{M}_i$. With these definitions, there is an obvious isomorphism (see Definition 5.4) between these two logics, and we shall work mostly with the second.

To convince ourselves that this logic can indeed implement the experimentally verifiable propositions about a quantum system, we need to study what makes Quantum Mechanics intrinsically different from Classical Mechanics: the existence of non-compatible propositions. Let us investigate what incompatibility in the sense of Definition 5.13 means for orthogonal projections. Let $P, Q \in \mathcal{P}(\mathcal{H})$ and suppose they are compatible, that is, there exists $P_1, Q_1, R \in \mathcal{P}(\mathcal{H})$ pairwise orthogonal and such that $P = P_1 + R$ and $Q = Q_1 + R$ (check Theorem 5.15). Then:

$$\begin{aligned} PQ &= (P_1 + R)(Q_1 + R) \\ &= R^2 \\ &= (Q_1 + R)(P_1 + R) \\ &= QP. \end{aligned}$$

That is, the *compatibility of P and Q implies that they commute*. Let us check the other direction. Suppose P and Q commute. Then define:

$$P_1 \doteq P - PQ$$
$$Q_1 \doteq Q - PQ$$
$$R \doteq PQ.$$

It is easy to verify that these operators are indeed orthogonal projections and that they are pairwise orthogonal. Hence, P an Q are compatible. Therefore, for this logic, *compatibility is equivalent to commutativity*. We can also conclude, due to Theorem 5.14, that $\mathcal{L}_{QM}$ is not a Boolean algebra.

Next, we want to study the set $\mathcal{O}(\mathcal{L}_{QM})$ of observables associated with the logic of quantum mechanics. It is straightforward to see that, for this logic, the Definition 5.6 of an observable coincides with the definition of a spectral measure (Definition A.10) on the real line. Hence, due

to the Spectral Theorem (Theorem A.19), there is a unique self-adjoint operator associated with it. Therefore, $\mathcal{O}(\mathcal{L}_{QM})$ coincides with the set of all (possibly unbounded) self-adjoint operators acting on $\mathcal{H}$. Also, the notion of the spectrum of an observable of a logic given by Definition 5.7 coincides with the usual definition for linear operators on a Hilbert space (Definition A.6).

Finally, it only remains to study the set of states of this logic. For that, we have the following remarkable theorem.

Theorem 5.17 (**Gleason's Theorem**). *Let $\mathcal{H}$ be a finite-dimensional Hilbert space with dimension equal to or bigger than 3, or infinite-dimensional and separable. Let μ be a probability measure on $\mathcal{L}_{QM}$ (and hence a state, due to Theorem 5.10). Then, there exists a trace-class, positive operator $T \in B(\mathcal{H})$ such that*

$$\mu(P) = \mathrm{Tr}(TP),$$

for all $P \in \mathcal{P}(\mathcal{H})$.

Proof. See [60, 64]. □

This is a very powerful result. In Quantum Mechanics textbooks, states defined in terms of traces over density matrices are usually presented with a motivation coming from Physics, namely, to accommodate the possibility of having a statistical ensemble. That is a good motivation since in real-life experiments we can rarely know with certainty the state of the probed system. However, this theorem shows that this is an *emerging* fact, which depends solely on the structure of the logic behind Quantum Mechanics, and no other type of states exist. We can go even further. In our construction, we assumed the existence of a Hilbert space, and we constructed the logic from there. However, there are reconstruction theorems stating that a logic that satisfies some specific properties (all of them satisfied by $\mathcal{L}_{QM}$) is isomorphic with the logic of closed subspaces of a Hilbert space. Hence, also the Hilbert space itself emerges from the structure of the logic. These reconstruction theorems are very complicated, and we refrain to get into the details. We refer to [60] and [61] for a complete exposition.

To conclude this section, we would like to mention that we could form logics with a subset of closed subspaces of $\mathcal{H}$, instead of taking all of them, as we did with $\mathcal{L}_{QM}$. Suppose that $\mathcal{A} \subseteq B(\mathcal{H})$ is a von Neumann algebra (see Definition 6.7), and let us denote by $\mathcal{P}(\mathcal{H})_{\mathcal{A}}$ the set of all projections in $\mathcal{A}$. As is well known, this set determines the algebra completely by the relation:

$$\mathcal{A} = \mathcal{P}(\mathcal{H})^{cc}_{\mathcal{A}},$$

where ^{c} denotes the commutant. The set $\mathcal{P}(\mathcal{H})_{\mathcal{A}}$ is also a logic, under the same operations (see Chapter 4, Section 4, in [60], and also [65]). However, this logic is in general not isomorphic with $\mathcal{P}(\mathcal{H})$, and its properties can vary greatly. These logics are, therefore, harder to study, since the general results showed in this section do not apply. However, these logics could be of great interest to Physics, since von Neumann algebras are often associated with local algebras of observables in Algebraic Quantum Field Theory (see Chapter 6). We refer to [65] for more discussion.

5.4 The Logic of Spacetime

In the example given at the beginning of this chapter, we intentionally omitted one important ingredient: time. It was enough for our purposes to consider only spatial regions, but in this section, we want to examine what happens when we include time. As we will see, also in this case there is a logic constructed with subsets of the spacetime. Moreover, this logic encompasses its most important ingredient, namely, the causal structure. We will focus our efforts on understanding the relativistic spacetimes, even though the logic of the Galilean spacetime is also very interesting, and finds many applications in the structural construction of non-relativistic quantum mechanics. However, since our goal is to apply these methods to relativistic scenarios, we will not discuss the Galilean logic. We refer the interested reader to [66].

The single most important structure in a relativistic spacetime is its causal structure. This limitation on which points of spacetime can have any influence, or be influenced by, other points constitutes the background structure of any fundamental theory in Physics. We can translate this structure to the mathematical language of lattices and logics in the following way. Let $M = (\mathbb{R}^4, g)$ denote the four-dimensional Minkowski spacetime, where g is the Minkowski metric with signature $(+,-,-,-)$. This metric gives rise to the following quadratic form, known as the *spacetime interval*:

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2,$$

where $x = (x_0, x_1, x_2, x_3) \in M$. We can define a partial order on M with this form: $x \leq y$ if $y - x$ is a time-like, future-oriented vector, that is, $Q(y-x) > 0$ and $x_0 < y_0$. This partial order structure already encodes a lot of information about the spacetime. To show this, we enunciate the following theorem.

Theorem 5.18. *Consider the partially-ordered set $(M, \leq)$, as defined above. Let G denote the set of all functions $f : M \to M$ that are one-to-one (not necessarily linear or continuous), and*

that both f and f^{-1} preserve the partial order, that is:

$$x \leq y \iff f(x) \leq f(y)$$

for all $x, y \in M$. Then, G is a group under composition and is equal to the orthochronus Poincaré group plus dilations.

Proof. See [67] □

This theorem is quite surprising since it does not even require that the functions be linear or continuous: the preservation of the partial order is enough to determine the group of isometries. This suggests that we should put some effort into understanding the mathematical structure of the spacetime from this point of view. Even though our results in Part II of this thesis are restricted to flat spacetimes, we decided to introduce the lattice/logic-theoretic structure for more general spacetimes in this section. We do this because, first, the necessary effort to include these more general spacetimes is not much bigger than to study the flat case alone; and second, we open the possibility to generalize our results to curved spacetimes. Our first goal is to construct a logic associated with the spacetime.

From now on, we consider a Lorentzian spacetime of dimension $d \geq 2$, equipped with a pseudo-riemannian metric g with signature $(+, -, ..., -)$. We say that two points $p, q \in M$ are **time-connected** if there exists a time-like curve that passes through p and q. We start by defining a "complementation" relation in M. Let S be any subset of M. Then, we define the **causal complement** of S as:

$$S^{\perp} \doteq \{p \in M | p \text{ is not time-connected to any point of } S\}.$$

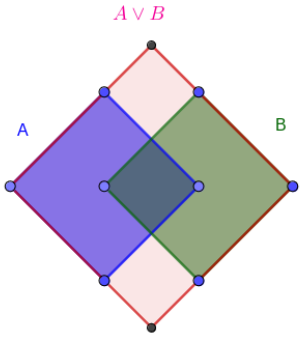
Note that the set of all subsets of M is a lattice under set union and intersection, but this complementation does not define an orthocomplementation relation on it (in the sense of Definition 5.3): that is because it is not necessarily true that $S \cup S^{\perp} = M$. Hence, since we want to construct an orthocomplemented lattice, we need to select subsets of M. We denote by $\mathcal{C}(M)$ the set of all **causally-closed** (or causally complete) subsets of M, that is:

$$\mathcal{C}(M) \doteq \{S \subseteq M | S = S^{\perp\perp}\}.$$

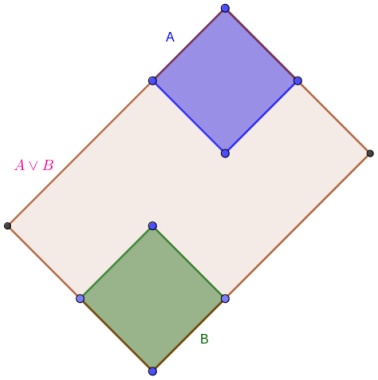
We equip this set with the following join and meet operations:

$$A \vee B \doteq (A \cup B)^{\perp\perp}$$
$$A \wedge B \doteq A \cap B,$$

where $A, B \in \mathcal{C}(M)$. In the following figure, we have some examples for two-dimensional Minkowski spacetime.



(a) Join of intersecting regions.



(b) Join of causally connected regions.

Figure 5.5: Join of causally closed regions in Minkowski spacetime

Theorem 5.19. *The quadruple* $\mathcal{L}_M \doteq (\mathcal{C}(M), \vee, \wedge, \perp)$ *is a logic.*

Proof. See [68]. □

The prototypical elements of this logic in Minkowski spacetime are diamonds, wedges, and sets constructed from them by the join and meet. Note, however, that single points are also in this logic. In the lattice-theoretic language, these are called *atoms*. More precisely, an **atom** is a non-zero element b of a lattice such that the only other element it majorizes is the zero element 0. In symbols: $0 \leq a \leq b$ implies either $a = 0$ or $b = 0$. The lattice is called **atomic** if every

non-zero element contains an atom, and **atomistic** if every non-zero element is the join of its atoms. The lattice $\mathcal{L}_M$ is both atomic and atomistic (check Theorem 2 in [68]). As another example, we notice that $\mathcal{L}_{QM}$ is also atomic and atomistic, where the atoms are given by the orthogonal projections projecting on one-dimensional subspaces.

The interesting resemblance between $\mathcal{L}_M$ and $\mathcal{L}_{QM}$ goes even further. As we saw, the distinguishing characteristic between $\mathcal{L}_{CM}$ and $\mathcal{L}_{QM}$ is that the former contains incompatible elements, which implies that it is not a Boolean algebra, and hence the distributive law (remember equations (5.3)) is not satisfied. Surprisingly, *this is also the case for* $\mathcal{L}_M$.

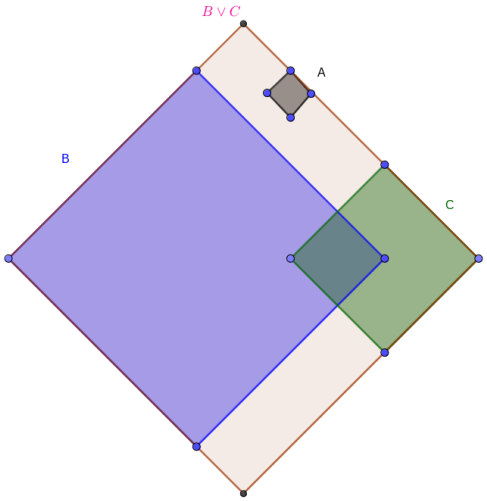


Figure 5.6: Lack of distributivity in $\mathcal{L}_M$.

In the figure above we can see that the distributive law does not hold: $A \wedge B = A \wedge C = 0$, but $A \wedge (B \vee C) = A$. This is an interesting fact: Classical Physics (meaning non-quantum) is described by a Boolean algebra, but the structure of the relativistic spacetime is non-Boolean. In this example, it is easy to check that the elements A and C are incompatible (according to Definition 5.13). Let us try to understand more deeply this common property between $\mathcal{L}_M$ and $\mathcal{L}_{QM}$. The distributive law is the crucial property of Boolean algebras, but since it is not satisfied by many important logics, mathematicians have been studying weakened versions of this law.

Definition 5.20. Let $\mathcal{L}$ be an orthocomplemented lattice. We say that the lattice is **modular** if for all $a, b, c \in \mathcal{L}$:

$$b \leq a \implies a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

The lattice is called **orthomodular** if:

$$b \leq a, c \leq a^{\perp} \implies a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Thus, it is clear that these are weaker conditions of the distributive law and that:

$$\text{distributivity} \implies \text{modularity} \implies \text{orthomodularity}.$$

Theorem 5.21. *$\mathcal{L}_{QM}$ is modular if, and only if, $\mathcal{H}$ is finite dimensional, and it is always orthomodular. $\mathcal{L}_M$ is always non-modular and always orthomodular.*

Proof. For the proof that $\mathcal{L}_{QM}$ is always orthomodular check [61], and for the proof that $\mathcal{L}_M$ is always non-modular and orthomodular see [68]. □

Therefore, these two lattices share a weakened version of the distributive law. We mention briefly that the reconstruction theorems remarked at the end of the previous sections do not apply to $\mathcal{L}_M$, since it does not satisfy all the requirements: hence, we cannot naturally reconstruct a Hilbert space from the spacetime logic such that $\mathcal{L}_M$ is isomorphic with the lattice of orthogonal projections in this space. We refer to [68] and [69] for more on this relation.

As we saw, the logic $\mathcal{L}_M$ includes many elements since all single points are in there. For our future purposes (see Chapter 7), this will be, in a sense that we explain later, "too big". Hence, we study some sublogics. From now on we will impose one more assumption on the structure of the spacetime: we will restrict our attention to globally hyperbolic spacetimes, which means that $M \simeq \mathbb{R} \times \Sigma$, where Σ is a Cauchy surface. Let $\mathcal{B}(\Sigma)$ denote the Borel σ-algebra on Σ. It is clear that for any $b \in \mathcal{B}(\Sigma)$, the element $D \doteq b^{\perp\perp}$ belongs to the logic $\mathcal{L}_M$ (since $D^{\perp\perp} = D$). Let us denote by $\mathcal{L}_\Sigma$ the set of all elements formed in this way, that is:

$$\mathcal{L}_\Sigma \doteq \{D \in \mathcal{L}_M | D = b^{\perp\perp}, \text{ where } b \in \mathcal{B}(\Sigma)\}. \tag{5.7}$$

Proposition 5.22. *$\mathcal{L}_\Sigma$ is a Boolean sublogic of $\mathcal{L}_M$.*

Proof. This proposition is discussed in [69], but we give our own proof here. Let us start by showing that it is indeed a sublogic. For that, we need to show that $\mathcal{L}_\Sigma$ is closed under the logic operations. In fact, for any two arbitrary elements $D_1 = b_1^{\perp\perp}$ and $D_2 = b_2^{\perp\perp}$ in $\mathcal{L}_\Sigma$ we have:

$$D_1 \vee D_2 = b_1^{\perp\perp} \vee b_2^{\perp\perp} = (b_1^{\perp} \cap b_2^{\perp})^{\perp} = (b_1 \cup b_2)^{\perp\perp},$$

and

$$D_1 \wedge D_2 = b_1 \cap b_2 = (b_1^\perp \vee b_2^\perp)^\perp = (b_1 \cap b_2)^{\perp\perp},$$

where we used the De Morgans laws given in equation (5.1). For the orthocomplementation, it is clear that:

$$D_1^\perp = b_1^\perp, \tag{5.8}$$

and hence $D_1^\perp \in \mathcal{L}_\Sigma$ since it is also determined by an element in $\mathcal{B}(\Sigma)$. From these relations, it is straightforward to show that $\mathcal{L}_\Sigma$ is an orthocomplemented lattice and that the join and meet of any countably infinite sequence of elements in $\mathcal{L}_\Sigma$ exist in $\mathcal{L}_\Sigma$. The relative orthocomplement given in item 2 of the Definition 5.5 is just the double orthocomplementation of the relative set complement of elements in $\mathcal{B}(\Sigma)$. Hence, $\mathcal{L}_\Sigma$ is a logic.

It only remains to show that this logic is Boolean. We do that by showing that the distributive law (5.3) is satisfied. Let $D_1 = b_1^{\perp\perp}, D_2 = b_2^{\perp\perp}, D_3 = b_3^{\perp\perp}$ be arbitrary elements in $\mathcal{L}_\Sigma$. Then:

$$\begin{aligned} D_1 \wedge (D_2 \vee D_3) &= b_1^{\perp\perp} \wedge (b_2^{\perp\perp} \vee b_3^{\perp\perp}) \\ &= (b_1 \cap (b_2 \cup b_3))^{\perp\perp} \\ &= (b_1 \cap b_2 \cup b_1 \cap b_3)^{\perp\perp} \\ &= (D_1 \wedge D_2) \vee (D_1 \wedge D_3). \end{aligned}$$

The second relation in (5.3) can be similarly verified. □

The logic $\mathcal{L}_\Sigma$ is simpler than $\mathcal{L}_M$ since, as we saw in the above proof, much of its structure comes from $\mathcal{B}(\Sigma)$, which is a Boolean σ-algebra. The prototypical elements of this logic are diamonds and wedges whose base lies on Σ. This sublogic also has single points as its atoms, but only those which belong to the Cauchy surface. As we will see, these will be much easier to handle.

Chapter 6

Quantum Fields and Modular Theory

> *All beginnings are obscure. Inasmuch as the mathematician operates with his conceptions along strict and formal lines, he, above all, must be reminded from time to time that the origins of things lie in greater depths than those to which his methods enable him to descend.*
>
> *H. Weyl*

When it comes to the greater depths of reality, the work of a mathematical physicist is to translate this structure into a precise mathematical language and develop techniques to deal with the broader scope. Quantum Field Theory (QFT) is, since a few decades, the most fundamental theory of matter ever created. Its mathematical complexity goes far beyond what is currently understood, and its full comprehension lies in the distant dreams of optimistic physicists and mathematicians.

Nonetheless, a good part of the mathematical structure of QFT has been developed, especially for non-interacting theories. We will follow the approach of Algebraic Quantum Field Theory (AQFT). See [70–72] for introductory material. The philosophy behind this framework is to assume that the set of all possible observables of a given relativistic quantum system that can be observed in a given region of spacetime $B \subseteq M$ forms a local algebra $\mathcal{A}(B)$. These algebras and their relations with the regions of spacetime and the causal structure of Minkowski spacetime follow some axioms, which we describe in the first section of this chapter. In Part II of this thesis, we are mainly concerned with flat spacetime, and hence we restrict our efforts to this case. However, these axioms can also be generalized to curved spacetimes (see [73]).

Many techniques of Functional Analysis are required to give a solid mathematical basis to AQFT. Among them, the Modular Theory of Tomita-Takesaki [74, 75] plays a distinguished role, with profound physical and mathematical implications. In Section 6.2 we show the relation between AQFT and this subarea of Functional Analysis, paying special attention to an important

application, the so-called Modular Localization, which is the main technique we use in Chapter 7 to propose a new approach to the Localizability Problem.

6.1 Algebraic Quantum Field Theory

We start with some basic definitions.

Definition 6.1. Let V be a vector space over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$, with binary operations $\cdot : \mathbb{K} \times V \to V$, and $+ : V \times V \to V$. Let us equip this vector space further with an associative product $\circ : V \to V$; an involution $* : V \to V$; and a norm $\|\cdot\| : V \to \mathbb{R}_+$ such that this vector space is a Banach space and an algebra satisfying:

$$\|A \circ B\| \leq \|A\|\|B\|,$$

for all $A, B \in V$. We call the structure $(V, \mathbb{K}, \cdot, +, \circ, *, \|\cdot\|)$ a **Banach *-algebra**.

Definition 6.2. A **abstract C^*-algebra** $\mathcal{A}$ is a Banach $*$-algebra satisfying the condition (called the C^* condition) that:

$$\|A^* \circ A\| = \|A^*\|\|A\|, \tag{6.1}$$

for all $A \in \mathcal{A}$. We say that $\mathcal{A}$ is **unital** if it has an identity, that is, an element $\mathbb{I}$ such that $\mathbb{I}A = A\mathbb{I}$ for all $A \in \mathcal{A}$.

One of the most important examples of a C^*-algebra is the set $B(\mathcal{H})$ of all bounded operators acting on a complex Hilbert space $\mathcal{H}$, equipped with the usual vector space structure and where $\circ$ is given by composition and the involution is the adjoint. It is easy to check that the C^* condition is satisfied. In addition, it is clear that any norm closed $*$-subalgebra of $B(\mathcal{H})$ is again a C^*-algebra. This motivates the following definition.

Definition 6.3. A **concrete C^*-algebra** is a norm closed subalgebra of $B(\mathcal{H})$ which is invariant under involution.

The following theorem shows that every abstract C^*-algebra is actually concrete for some Hilbert space $\mathcal{H}$.

Theorem 6.4. *Let $\mathcal{A}$ be an abstract C^*-algebra. Then, there exists an isometric $*$-isomorphism (that is, an isomorphism that preserves the norm, and maps the involution in $\mathcal{A}$ to the involution in $B(\mathcal{H})$) between $\mathcal{A}$ and a concrete C^*-subalgebra of $B(\mathcal{H})$ for some $\mathcal{H}$.*

Proof. See Theorem 2.1.10 in [75]. □

The theory of C^*-algebras is very rich and finds many applications in Physics. These are often associated with the observables of a quantum system. For example, in non-relativistic Quantum Mechanics, the Weyl algebra, which incorporates the Canonical Commutation Relations, is a C^*-algebra. Our next goal is to introduce an specific type of C^*-algebra, namely, von Neumann algebras.

Let V be an arbitrary vector space and $p : V \to \mathbb{R}$ be a seminorm, that is, a norm without the property that it is positive definite. Given a family of seminorms $\mathcal{S}$ on V, we can always generate a topology on this set in the following way. Let $x_0 \in V$, $\epsilon > 0$, $n \in \mathbb{N}$, and $p_1, ..., p_n \in \mathcal{S}$. Then, we define the following subset on V:

$$V(x_0, p_1, ..., p_n; \epsilon) \doteq \{x \in V | p_i(x - x_o) < \epsilon, i = 1, ..., n\}.$$

For each $x \in V$, we denote by $\mathcal{V}_x$ the collection of all subsets $V(x, p_1, ..., p_n; \epsilon)$ as given above.

Theorem 6.5. *Let $\mathcal{S}$ be a family of seminorms on a vector space V. Then:*

1. *There exists a topology $\tau_{\mathcal{S}}$ in V such that, for each $x \in V$, it admits $\mathcal{V}_x$ as a neighborhoud basis, that is:*

$$\tau_{\mathcal{S}} = \{G \subseteq V | \textit{for each } x \in G \textit{ there exists } U \in \mathcal{V}_x \textit{ such that } U \subseteq G\}.$$

2. *$(V, \tau_{\mathcal{S}})$ is a locally convex topological vector space.*

3. *Each seminorm $p \in \mathcal{S}$ is continuous with respect to $\tau_{\mathcal{S}}$.*

Proof. See Chapter 1 in [76]. □

We call $\tau_{\mathcal{S}}$ the **topology generated by the family $\mathcal{S}$ of seminorms**. We can apply these results to generate some topologies on $B(\mathcal{H})$.

Definition 6.6. Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ denote the set of all bounded operators acting on it. We define the following topologies which are generated by the respective family of seminorms.

1. **norm topology**: $p(A) = \|A\|$.

2. **strong operator topology**: $p_v(A) = \|Av\|, v \in \mathcal{H}$.

3. **weak operator topology**: $p_{u,v}(A) = |\langle u, Av\rangle|, u, v \in \mathcal{H}$.

We can now give the important definition of a von Neumann algebra.

Definition 6.7. A **von Neumann algebra** is a weak-operator closed $*$-invariant subspace of $B(\mathcal{H})$.

One of the fundamental results about von Neumann algebras is that they have interesting algebraic properties. Let $S \subseteq B(\mathcal{H})$ be any subset of $B(\mathcal{H})$. We define the **commutant** of S to be:

$$S^c \doteq \{A \in B(\mathcal{H}) | BA = AB, \text{ for all } B \in S\}. \tag{6.2}$$

The following theorem, due to von Neumann, shows the intrinsic relations between the topological and algebraic properties of von Neumann algebras discussed above.

Theorem 6.8 (**Double Commutant Theorem**). *Let $\mathcal{A}$ be a unital $*$-invariant subalgebra of $B(\mathcal{H})$. The following are equivalent.*

1. $\mathcal{A} = \mathcal{A}^{cc}$.
2. *$\mathcal{A}$ is weak-operator closed.*
3. *$\mathcal{A}$ is strong-operator closed.*

Proof. See Theorem 2.4.11 in [75]. □

Hence, this theorem gives us a few alternative definitions of von Neumann algebras, one of them being purely algebraic. One particularly important subset of any von Neumann algebra is its subset of orthogonal projections.

Theorem 6.9. *Let $\mathcal{A}$ be a von Neumann algebra. Denote by $\mathcal{A}_P$ its subset of orthogonal projections, that is:*

$$\mathcal{A}_P \doteq \{P \in \mathcal{A} | P = P^* = P^2\}.$$

Then $\mathcal{A} = \mathcal{A}_P^{cc}$.

Proof. See [75]. □

We are now in a position to formulate the axioms of AQFT.

Axioms of Algebraic Quantum Field Theory

1. **Local algebras:** There is a unital C^*-algebra $\mathcal{A}(\mathbb{M})$ and, to each causally convex bounded region $O \subset \mathbb{M}$ of the Minkowski spacetime, there is a unital C^*-subalgebra $\mathcal{A}(O)$ containing the unit of $\mathcal{A}(\mathbb{M})$, such that the $\mathcal{A}(O)$ collectively generate $\mathcal{A}(\mathbb{M})$.

2. **Isotony:** If $O_1 \subset O_2$, then:

$$\mathcal{A}(O_1) \subset \mathcal{A}(O_2).$$

3. **Causality:** If O_1 and O_2 are causally disjoint, then:

$$[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0.$$

4. **Poincaré covariance:** Let $\mathcal{P}_+^\uparrow$ denote the identity connected component of the Poincaré group. Then, to every $g \in \mathcal{P}_+^\uparrow$, there exists an $*$-automorphism $\alpha(g)$ of $\mathcal{A}(\mathbb{M})$ such that:

$$\begin{aligned} \alpha(g) : \mathcal{A}(O) &\to \mathcal{A}(gO), \\ \alpha(e) &= \mathbb{I}, \\ \alpha(g)\alpha(h) &= \alpha(gh), \text{ for every } g, h \in \mathcal{P}_+^\uparrow. \end{aligned}$$

5. **Existence of dynamics:** If $O_1 \subset O_2$ and O_1 contains a Cauchy surface of O_2, then:

$$\mathcal{A}(O_2) = \mathcal{A}(O_1).$$

We see that this list of axioms contains only the very basic requirements a relativistic quantum theory should satisfy. They were inspired by the success of the mathematical structure of Quantum Mechanics and Special Relativity, and it's hard to imagine any physical theory that does not satisfy these requirements. Due to the great physical and mathematical generality of these axioms, it is often useful to impose further conditions. Due to the vast technical literature on von Neumann algebras, and due to the fact that for free theories this is the case, we will assume that the local algebras are von Neumann algebras. One of the great advantages of this further assumption is that it allows us to use the techniques coming from Modular Theory, which we develop in the next section. Furthermore, we saw that von Neumann algebras are weak-operator closed algebras. These are defined by a family seminorms that are interpreted in Physics as expectation values of observables. Hence, von Neumann algebras have properties that closely resemble a physical interpretation.

There is still one ingredient missing in our abstract formulation of Quantum Fields: the states. Notice that so far we've been talking exclusively about algebras of observables, and we didn't specify a Hilbert space. That is, in fact, one of the advantages of AQFT. Hence, we cannot define states as elements in a Hilbert space, as is done in the traditional approach to Quantum Mechanics. We need a more abstract definition.

Definition 6.10. Let $\mathcal{A}$ be a unital C^*-algebra. A (algebraic) **state on** $\mathcal{A}$ is a positive, normalized linear functional $\omega : \mathcal{A} \to \mathbb{C}$, that is:

$$\omega(A^*A) \geq 0 \quad \text{and} \quad \omega(\mathbb{I}) = 1,$$

for all $A \in B(\mathcal{H})$. The state is **mixed** if its is a convex combination of distinct states, and its **pure** otherwise.

Consider $\mathcal{A} = B(\mathcal{H})$. Then it is clear that, if $\psi \in \mathcal{H}$, $\|\psi\| = 1$, the map:

$$\omega_\psi(A) \doteq \langle \psi, A\psi \rangle$$

defines a (pure) state on $B(\mathcal{H})$ for every $\psi \in \mathcal{H}$. Consider now that $D \in B(\mathcal{H})$ is any trace class, positive operator such that $\mathrm{Tr}\{D\} = 1$. Then, it is also straightforward to check that the map:

$$\omega_D(A) \doteq \mathrm{Tr}\{AD\}$$

defines a state on $B(\mathcal{H})$, that can be mixed, depending on D. Hence, the abstract definition of states on C^*-algebras includes the traditional definition of Quantum Mechanics and it implements the expected values on observables. Note, however, that not every state is necessarily of one of the two types presented above (see [77] for more on this). Going back to our notation given in the Introduction 1 in equation (1.1), in the present context, $\mathfrak{T}_{RQT}$ is relativistic quantum theory, The$(\mathfrak{T}_{RQT})$ is the AQFT formulation, $\mathcal{O}_{[Q]}$ the self-adjoint elements of the (local) C^*-algebras, $\mathcal{O}_{[\alpha]}$ the algebraic states on these algebras, and $\mu^{\mathcal{O}_{[Q]}}_{\mathcal{O}_{[\alpha]}}$ the expectation values of the orthogonal projections in the spectral decomposition of the self-adjoint operators.

Even though the abstract algebraic construction of field theories has proven to be the correct approach to deal with the foundations of QFT, it can also be quite useful to work on concrete Hilbert spaces. In this case, we will be dealing with "representations" of the algebraic relations on the Hilbert space. More precisely:

Definition 6.11. A **representation** of a unital C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a

Hilbert space, and π a map from $\mathcal{A}$ to linear operators on $\mathcal{H}$, such that:

1. $\pi(\mathbb{I}) = \mathbb{I}$.

2. π is linear and respect products (that is, $\pi(AB) = \pi(A)\pi(B)$).

3. Each $\pi(A)$ has an adjoint such that $\pi(A)^* = \pi(A^*)$.

A representation is said to be **faithful** if $\ker \pi = \{0\}$. It is called **irreducible** if there are no subspaces of $\mathcal{H}$ invariant under $\pi(A)$, for every $A \in \mathcal{A}$, that are not either trivial or dense in $\mathcal{H}$.

Definition 6.12. Let $\mathcal{A}$ be a unital C^*-algebra and consider two representations, $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$. These representations are called **unitarily equivalent** if there is a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(A) = \pi_2(A)U$ holds for all $A \in \mathcal{A}$. They are called **unitarily inequivalent** otherwise.

The existence of inequivalent representations of the algebra of observables was one of the reasons why the algebraic approach was proposed. This fact is particularly important when considering curved backgrouds and in the study of superselection sectors (see [70, 71, 78, 79]). There is a nice way to construct representations starting from states on the algebra. This result is known as the GNS (Gel'fand, Naimark, Segal) construction.

Theorem 6.13 (GNS construction). *Let ω be a state on a unital C^*-algebra $\mathcal{A}$. Then there is a representation $(\mathcal{H}_\omega, \pi_\omega)$ of $\mathcal{A}$ and a unit vector $\Omega_\omega \in \mathcal{D}_\omega$ such that:*

1. *Ω_ω is cyclic, that is, $\mathcal{H}_\omega = \pi_\omega(\mathcal{A})\Omega_\omega$.*

2. *$\omega(A) = \langle \Omega_\omega, \pi(A)\Omega_\omega \rangle$ for all $A \in \mathcal{A}$.*

3. *The triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is unique up to unitary equivalence.*

4. *ω is pure if and only if the representation is irreducible, and if π_ω is faithful, then $\|\pi_\omega(A)\| = \|A\|_{\mathcal{A}}$.*

Proof. See Theorem 10 in [71]. □

6.1.1 The Free Scalar Field

Let us finish this section with an example. We will give an overview of the construction of the algebra of observables for a free, bosonic field on Minkowski spacetime. Let us summarize our procedure: we start by assuming a (classical) dynamical equation (the Klein-Gordon equation); using techniques coming from the study of Green hyperbolic operators, we construct the (real)

vector space of solutions of this equation; this vector space can be endowed with the structure of a symplectic vector space; to every symplectic vector space there is a unique C^*-algebra associated with it: this is the Weyl C^*-algebra, which satisfies all of the axioms for a field theory described above, and hence is the algebra of observables of a free, bosonic field. To begin, we will explain the relation between symplectic vector spaces and Weyl algebras.

Definition 6.14. Let X be a (possibly infinite-dimensional) real vector space. A pair (X, σ) is called a **symplectic vector space** if $\sigma : X \times X \to \mathbb{R}$ is bilinear, skew-symmetric, and weakly non-degenerate, that is: $\sigma(u, v) = 0 \, \forall u \in X \implies v = 0$. Such a map is called a **symplectic form**.

Definition 6.15. Let (X, σ) be a symplectic vector space. A $*$-algebra $\mathcal{W}(X, \sigma)$ is said to be the **Weyl $*$-algebra of** (X, σ) if there exists a family $\{W(u)\}_{u \in X}$ of non-zero elements, called the **generators**, such that:

1. The Weyl commutation relations hold:

$$W(u)W(v) = e^{-\frac{i}{2}\sigma(u,v)}W(u+v)$$
$$W(u)^* = W(-u)$$

   for every $u, v \in X$.

2. The $*$-algebra $\mathcal{W}(X, \sigma)$ is generated by the family $\{W(u)\}_{u \in X}$, that is, the algebra coincides with the linear span of finite combinations of finite products of $\{W(u)\}_{u \in X}$.

We call it the **Weyl C^*-algebra of** (X, σ), and denote it by $C\mathcal{W}(X, \sigma)$, if the norm on $\mathcal{W}(X, \sigma)$ satisfies the C^* condition (6.1).

The next theorem shows that, given a symplectic vector space, there is always a Weyl $*$-algebra (and a Weyl C^*-algebra) associated with it.

Theorem 6.16. *Let (X, σ) be a symplectic vector space of arbitrary dimension. Then:*

1. *There exists a Weyl $*$-algebra $\mathcal{W}(X, \sigma)$ associated with (X, σ).*

2. *Any Weyl $*$-algebra has a unit, and it holds that:*

$$W(0) = \mathbb{I}, \quad W(u)^* = W(-u) = W(u)^{-1}, u \in X.$$

   In addition, the generators are linearly independent.

3. *There exists a unique norm on $\mathcal{W}(X,\sigma)$ satisfying the C^* condition.*

4. *The algebras $\mathcal{W}(X,\sigma)$ and $C\mathcal{W}(X,\sigma)$ are unique up to $*$-isomorphism.*

Proof. See Theorem 11.26 in [80]. □

The first and easiest example is given by the finite-dimensional symplectic vector space $(\mathbb{R}^{2n},\sigma)$, where $\sigma = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$ is the canonical symplectic form. This symplectic space is associated with the traditional Weyl algebra of non-relativistic Quantum Mechanics, represented in the Hilbert space of a quantum system moving in $\mathbb{R}^n$, and where the Weyl generators are given by the exponential of the sums of position and momentum operators. Furthermore, the *Stone-von Neumann Theorem* guarantees that all representations of Weyl $*$-algebras of finite-dimensional symplectic spaces are unique up to isomorphism (see Theorem 11.22 in [80]). As we will see, this result doesn't apply to field theories.

We are now ready to construct the algebra of observables of a free, bosonic theory. Consider the four-dimensional Minkowski spacetime with coordinates $(t,x) \in \mathbb{R}^4$, consider the ***Klein-Gordon equation:***

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi = 0,$$

where ϕ is a smooth, real-valued function on the spacetime. We rewrite this equation as $P\phi = 0$, where $P \doteq \partial^2/\partial t^2 - \nabla^2 + m^2$ is a (Green Hyperbolic) partial differential operator. It is well known that the initial value problem of this equation admits unique solutions (see Proposition 3.2.9 [72] and for a more complete treatment [81]). Let us define the solution space as:

$$\text{Sol} = \left\{\phi \in C^\infty(\mathbb{R}^4) | P\phi = 0\right\} \subset C^\infty(\mathbb{R}^4).$$

We can find these solutions as follows. By definition, a Green hyperbolic operator admits the existence of retarded E^+ and advanced E^- Green operators. Defining the *advanced-minus-retarded* Green operator as $E = E^- - E^+$, it can be proved that for any $f \in C_0^\infty(\mathbb{R}^4)$, the function $\phi = Ef$ defines a solution of the Klein-Gordon equation. The space of solutions is a real vector space, and it has a natural symplectic form $\sigma : \text{Sol} \times \text{Sol} \to \mathbb{R}$ given by:

$$\sigma(\phi,\phi') \doteq \int_{\mathbb{R}^3} \left(\phi\frac{\partial\phi'}{\partial t} - \phi'\frac{\partial\phi}{\partial t}\right) dx.$$

Hence, the pair (Sol,σ) is a symplectic vector space. This is an infinite-dimensional vector space, and hence the Stone-von Neumann Theorem doesn't apply. Using Theorem 6.16, we

have a unique C^*-algebra associated with this space, which we denote by $\mathcal{CW}(\mathrm{Sol}, \sigma)$. Furthermore, we can define *local algebras* by defining $\mathcal{CW}(\mathrm{Sol}, \sigma)(\mathcal{O})$ as the C^*-algebra generated by functions $\phi \in \mathrm{Sol}$ with support in $\mathcal{O}$, where $\mathcal{O} \subseteq M$ is any causally complete, bounded, open subset of M.

Theorem 6.17. *The net $\mathcal{O} \mapsto \mathcal{CW}(Sol, \sigma)(\mathcal{O})$ satisfies all axioms of Algebraic Quantum Field Theory.*

Proof. See Theorem 3.3.1 in [72], and [71]. □

This procedure can be extended to great generality: any wave function defined by a Green hyperbolic operator on a globally hyperbolic manifold has unique solutions for the initial value problem (see [81]). This includes theories with spin different than zero, as well as massless theories, and therefore the construction of free field theories is completely understood. However, *it doesn't include interacting theories* since the differential operators defining the dynamical equations are not Green hyperbolic in this case. Hence, the enduring challenge of constructing interacting theories might require completely different methodologies. Notice that what we have done was to start with a classical theory and (second) quantize it. Alternatively, initiating with a quantum theory from the outset would be more advantageous. We will see in the next section that the *Modular Localization* method does this job. This method also allows us to construct a net of C^*-algebras for free theories, but still based on second quantization. Extensions to interacting theories are a current topic of research (see Chapter 10 in [72], for example).

6.2 The Modular Theory of Tomita-Takesaki

The Modular Theory, also referred to as the Theory of Tomita-Takesaki, was initiated with Tomita's presentation of an unpublished pre-print in 1967 at the Baton Rouge conference. However, it became more popular with Takesaki's paper in 1970 [74]. As it was first presented, this theory provided pure mathematical tools to better understand von Neumann algebras acting on a Hilbert space with a cyclic and separating vector. Nevertheless, it soon became clear that it was also very useful for Physics. The first to realize an application of these techniques were Haag, Hugenholtz, and Winnik in their description of equilibrium states using the KMS condition [82]. Later on, Bisognano and Wichmann [83] made a significant discovery linking the modular group (as described below) with the one-parameter boosts on Minkowski spacetime, giving origin to a (KMS) thermodynamical equilibrium state. A phenomenon that would be called, in a later published paper [84], the Unruh effect. The Modular Theory was also very useful for pure

mathematical results, playing an essential role in the classification of von Neumann algebras [85].

The Modular Theory can be presented in two ways: the first, through the study of von Neumann algebras acting on a Hilbert space with a cyclic and separating vector[75, 86, 87]; and the second, through the introduction of standard subspaces [88–90]. We will give an overview of the first, and focus on the second, as it will be more useful for the presentation of Modular Localization.

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a vector Ω that is *cyclic* ($\mathcal{M}\Omega \subset \mathcal{H}$ is dense in $\mathcal{H}$) and *separating* ($A\Omega = 0, A \in \mathcal{M} \implies A = 0$). Let us define the operator S_0 as:

$$S_0 A\Omega \doteq A^*\Omega,$$

where $A \in \mathcal{M}$. This operator is clearly anti-linear and due to the cyclicity of Ω, also densely defined. In addition, by showing that the adjoint of S_0 is densely defined, we can conclude that it is closable. We will denote its closure by S. We can then perform its polar decomposition:

$$S = J\Delta^{1/2} = \Delta^{-1/2}J, \tag{6.3}$$

where Δ is the unique, positive, self-adjoint operator (called the *modular operator*) and J is the unique anti-unitary operator (called the *modular conjugation*) associated with the pair $(\mathcal{M}, \Omega)$. Note that $J^2 = \mathbb{I}$ and $J = J^*$. The theorem that follows stands as the fundamental result of Tomita-Takesaki's theory.

Theorem 6.18 (**Tomita-Takesaki**). *Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω. Then $J\Omega = \Omega = \Delta\Omega$ and it holds that:*

$$J\mathcal{M}J = \mathcal{M}^c \quad \textit{and} \quad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}, \quad \forall t \in \mathbb{R}.$$

Proof. See Theorem 2.5.14 in [75]. □

Note that the unitaries $\Delta^{it}, t \in \mathbb{R}$, induce a one-parameter group of automorphism of $\mathcal{M}$ through the map:

$$\sigma_t : \mathcal{M} \ni A \mapsto \Delta^{it}A\Delta^{-it} \in \mathcal{M}.$$

This group is referred to as the *modular automorphism group* associated with the pair $(\mathcal{M}, \Omega)$.

If $\mathcal{M}$ is abelian, this group is trivial.

We now present a second, analogous formulation of Tomita-Takesaki's Theorem, this time in terms of the so-called *stardard subspaces*.

Definition 6.19. Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle , \rangle = \langle , \rangle_{\mathbb{R}} + i\langle , \rangle_I$, and $H \subset \mathcal{H}$ a real linear subspace. We define the **symplectic complement** H' of H as the real Hilbert subspace:

$$H' \doteq \{\xi \in \mathcal{H} | \langle \xi, \eta \rangle_I = 0 \, \forall \eta \in H\} . \tag{6.4}$$

Note that *the symplectic complement is not an orthocomplementation* in the sense of Definition 5.3 since condition 3 is not necessarily satisfied. In fact, real closed subspaces that satisfy this condition are called *factors*. We gather in the following proposition some properties of the symplectic complement that will be useful later.

Proposition 6.20. *Let M be a subset of a complex Hilbert space $\mathcal{H}$ and $'$ the symplectic complement as above. Then:*

1. *M' is a closed, real subspace of $\mathcal{H}$.*
2. *If $M \subset N$, then $N' \subset M'$.*
3. *M'' is the closed, real subspace of $\mathcal{H}$ generated by M.*
4. *$(M + iM)' = M' \cap iM'$.*
5. *$M' = \{0\}$ if M is a dense subspace of $\mathcal{H}$.*

Proof. See Proposition 1.2.1 in [91]. □

Definition 6.21. A closed real subspace H is called **cyclic** if $H + iH$ is dense in $\mathcal{H}$, and **separating** if $H \cap iH = \{0\}$. If it is both cyclic and separating, we call it **standard**.

Let $H \subset \mathcal{H}$ be a standard subspace. Let us define an anti-linear operator $S_H : D(S) \to \mathcal{H}$ as:

$$S_H(\xi + i\eta) \doteq \xi - i\eta, \qquad \xi, \eta \in H,$$

where $D(S_H) = H + iH$. Since H is standard, S_H is densely defined. It is also clear that $S_H^2 = \mathbb{I}_{D(S_H)}$. Hence, S_H is an involution. We will see below that it is also a closed operator. The following lemma shows that any operator with these characteristics can produce a standard subspace.

Lemma 6.22. *Let S be a closed, densely defined, anti-linear involution on $\mathcal{H}$. Then $H \doteq ker(1-S)$ is a standard subspace of $\mathcal{H}$.*

Proof. See Lemma 2.2.1 in [89]. □

Proposition 6.23. *The map:*

$$H \mapsto S_H \tag{6.5}$$

is a bijection between the set of standard subspaces of $\mathcal{H}$ and the set of closed, densely defined, anti-linear involutions on $\mathcal{H}$, where:

$$S \mapsto ker(1-S)$$

is the inverse map of (6.5)*. In addition, the map is order-preserving:*

$$H_1 \subset H_2 \iff S_{H_1} \subset S_{H_2},$$

and

$$S_H^* = S_{H'}.$$

Proof. See Proposition 2.1.2 in [89]. □

We see that the operator S_H has similar properties to the operator (6.3). Thus, it is worth analyzing its polar decomposition.

Proposition 6.24. *Let*

$$S_H = J_H \Delta_H^{1/2}$$

be the polar decomposition of S_H. Then

1. *J_H is an anti-unitary involution with*

$$J = J^* = J^{-1}.$$

2. *$J_{H'} = J_H$ and $\Delta_{H'} = \Delta_H^{-1}$.*

Proof. See Proposition 2.1.3 in [89]. □

Finally, the following theorem gives the real Hilbert subspace version of the Tomita-Takesaki theorem.

Theorem 6.25 (**Tomita-Takesaki- Real subspace version**). *Let Δ_H and J_H be as above. Then, for all $t \in \mathbb{R}$:*

$$\Delta_H^{it} H = H, \quad J_H H = H'.$$

Proof. See Theorem 2.1.4 in [89]. □

One might wonder about the connection between the two approaches to the Tomita-Takesaki Theory. Let us first show how to go from von Neumann algebras with a cyclic and separating vector Ω to standard subspaces. Suppose $\mathcal{M}$ is a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Denote by $\mathcal{M}_{sa}$ the subset of $\mathcal{M}$ consisting of self-adjoint operators. Then, due to the assumption that Ω is cyclic and separating, the real vector subspace:

$$K \doteq \mathcal{M}_{sa}\Omega = \{\psi \in \mathcal{H} | \psi = A\Omega, A \in \mathcal{M}_{sa}\} \subset \mathcal{H}$$

is a standard subspace. Furthermore, if Δ and J are the modular operators associated with the pair $(\mathcal{M}, \Omega)$, then $\Delta = \Delta_K$ and $J = J_K$. In fact, if $K + iK \ni \psi = A\Omega + iB\Omega$, where $A, B \in \mathcal{M}_{sa}$, then:

$$S(A\Omega + iB\Omega) = A\Omega - iB\Omega,$$

which coincides with S_K.

In the other direction, a von Neumann algebra with a cyclic and separating vector may be obtained from a standard subspace via a technique known as *second quantization*. We will follow the notation defined in [91]. We start by defining the *Bosonic Fock space* over a Hilbert space $\mathcal{H}$ with an inner product $\langle , \rangle$, which we denote by $e^{\mathcal{H}}$, and we define as:

$$e^{\mathcal{H}} \doteq \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_{Sym} n},$$

where $\mathcal{H}^{\otimes_{Sym} n} = \text{Sym}(\mathcal{H}^{\otimes n})$, and Sym is the orthogonal projection defined as:

$$\text{Sym}(h_1 \otimes ... \otimes h_n) \doteq \frac{1}{n!} \sum_{\sigma \in P(n)} h_{\sigma(1)} \otimes ... \otimes h_{\sigma(n)},$$

where $P(n)$ is the permutation group of n elements. The vectors of the form:

$$e^h = \bigoplus_{n=0}^{\infty} \frac{h^{\otimes n}}{\sqrt{n!}}$$

form a total set in $e^{\mathcal{H}}$ (see Lemma 2.8 in [86]), and are called *coherent vectors*. The exponential notation is adequate since the inner product in this space is clearly:

$$\langle e^h, e^k \rangle = e^{\langle h,k \rangle}.$$

If A is an operator acting on $\mathcal{H}$, then its second-quantized version, which acts on $e^{\mathcal{H}}$, is defined as:

$$e^A \doteq \bigoplus_{n=0}^{\infty} A^{\otimes n}.$$

In particular, if U is unitary, then e^U is also unitary.

An important class of operators acting on $e^{\mathcal{H}}$ are the *Weyl unitaries*. These are defined as:

$$W(h)e^0 = e^{\left(-\frac{1}{4}\|h\|^2\right)} e^{\frac{i}{\sqrt{2}}h}, \quad h \in \mathcal{H}$$
$$W(h)W(k) = e^{-\frac{i}{2}\langle h,k \rangle_I} W(h+k), \quad h,k \in \mathcal{H}.$$

The vector e^0 is called the *vacuum state*, and the second equality above implements the Canonical Commutation Relations in the Weyl form. These unitaries are well-defined on a dense set of vectors spanned by the coherent vectors and therefore can be extended to unitaries on $e^{\mathcal{H}}$. Finally, given a closed real subspace $H \subset \mathcal{H}$, we can construct an associated von Neumann algebra $\mathcal{R}(H)$ as:

$$\mathcal{R}(H) \doteq \{W(h) | h \in H\}^{cc}. \tag{6.6}$$

The following theorems give some fundamental results concerning these algebras.

Theorem 6.26. *Consider the algebra $\mathcal{R}(H)$ as above. Then, the vacuum e^0 is cyclic and separating for this algebra if, and only if, H is standard. In this case, the modular operators*

associated with the pair $(\mathcal{R}(H), e^0)$ *are:*

$$S = e^{S_H}$$
$$\Delta = e^{\Delta_H}$$
$$J = e^{J_H}.$$

Proof. See Theorem 2.6 in [86] and references therein. □

Theorem 6.27. *For closed real subspaces* H, K *in* $\mathcal{H}$*, the map* $H \mapsto \mathcal{R}(H)$ *has the following properties:*

1. e^0 *is cyclic for* $\mathcal{R}(H)$ *if, and only if,* $H + iH$ *is dense in* $\mathcal{H}$*.*
2. e^0 *is separating for* $\mathcal{R}(H)$ *if, and only if,* $H \cap iH = \{0\}$*.*
3. $\mathcal{R}(H)^c = \mathcal{R}(H')$*.*
4. $(\mathcal{R}(H) \cup \mathcal{R}(K))^{cc} = \mathcal{R}(H + K)$*.*
5. $\mathcal{R}(H) \cap \mathcal{R}(K) = \mathcal{R}(H \cap K)$*, and hence* $\mathcal{R}(H)$ *is a factor (in the von Neumann algebra sense) if, and only if,* H *is a factor (in the real subspace sense).*

Proof. See Theorem 1.3.2 in [91]. □

To close this section, we give some applications of this theory to Physics. Historically, the first application, as already mentioned, was to thermal equilibrium states. Since this topic is not connected with the main topic of this thesis, we only comment briefly on this result. Defining a state ω on $\mathcal{M}$ as:

$$\omega(A) \doteq \frac{1}{\|\Omega\|^2}\langle \Omega, A\Omega \rangle,$$

it can be proved that this state is KMS with respect to the modular group (see Theorem 2.5.14 in [75]). This fact has profound consequences for Quantum Statistical Mechanics and Thermal Field Theory [70, 75, 82].

The starting point of the study of Modular Theory, as we saw, is a von Neumann algebra with a cyclic and separating state. In QFT, one could wonder when (or if) these conditions are satisfied. In AQFT, the local algebras are described by von Neumann algebras, as discussed in Section 6.1. Do we have a state with the necessary properties? The following groundbreaking theorem answers this question.

Theorem 6.28 (**Reeh-Schlieder**). *Let $\mathbb{R}^4 \ni O \mapsto \mathcal{M}(O)$ be the net of C^*-algebras describing a free scalar field (see Subsection 6.1.1), where O is any open set with interior points and such that the causal complement also has interior points. Let Ω denote the vacuum state (that is, the state with the least energy). Then, Ω is cyclic and separating for any $\mathcal{M}(O)$.*

Proof. See [71, 92, 93]. □

This finding is quite robust and unintuitive. For example, consider that we have a laboratory contained in a given region $O \in \mathbb{R}^4$ of spacetime. The theorem says that the action of the local algebra $\mathcal{M}(O)$ on the vacuum can not only create states that are "localized" in the laboratory, but it can create almost any state (that is, a dense subset), even states that are "localized" in faraway regions. Of course, the precise definition of localization we are using needs to be clarified, and we will do it in Chapter 7.

This theorem also allows us to use all the techniques of Modular Theory on QFT. Hence, a natural question that arises is: what is the modular group in this context? To answer this question, we need to specialize our investigation to a particular class of regions in spacetime, namely, the wedges.

Definition 6.29. The **right wedge** (also known as the right Rindler wedge), denoted by W_1, is defined as the following subset of Minkowski spacetime:

$$W_1 \doteq \{x \in \mathbb{R}^4 | x_1 > |x_0|\}. \tag{6.7}$$

All other wedges are defined as Poincaré transforms of this wedge. That is, denoting by $\mathcal{W}$ the set of all wedges, we have that:

$$\mathcal{W} = \left\{W \subset \mathbb{R}^4 | W = gW_1, g \in \mathcal{P}\right\}.$$

These are causally closed regions and hence belong to the logic of spacetime (check Section 5.4). It is possible to assign to each wedge a one-parameter group of transformations, denoted by $\Lambda_W(t)$, $t \in \mathbb{R}$, such that it is covariant with respect to $\Lambda_W(t)$, that is:

$$\Lambda_W(t)(W) = W \tag{6.8}$$

$$\Lambda_{gW}(t) = g\Lambda_W(t)g^{-1}, g \in \mathcal{P}_+^{\uparrow} \tag{6.9}$$

$$\Lambda_{gW}(t) = g\Lambda_W(-t)g^{-1}, g \in \mathcal{P}_+^{\downarrow}. \tag{6.10}$$

It is also possible to assign to each wedge a time-reversing reflection $R_W \in \mathcal{P}_+$ such that W is

covariant with respect to R_W, that is:

$$R_W(W) = W^{\perp} \tag{6.11}$$

$$R_{gW} = gR_W g^{-1}, g \in \mathcal{P}_+, \tag{6.12}$$

where $\perp$ is the causal complement defined in Section 5.4 for $M = \mathbb{R}^4$, the Minkowski spacetime.

For example, for the right wedge, the one-parameter group can be chosen as the (rescaled) boosts preserving W_1, that is:

$$\mathbb{R} \ni t \mapsto \Lambda_{W_1}(t) = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t & 0 & 0 \\ -\sinh 2\pi t & \cosh 2\pi t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.13}$$

The element R_{W_1} is the reflection with respect to the edge of the wedge, that is:

$$R_{W_1}(x_0, x_1, x_2, x_3) = (-x_0, -x_1, x_2, x_3). \tag{6.14}$$

For the regions in the set $\mathcal{W}$, we have the following striking result, which shows that the modular group has a geometric action.

Theorem 6.30 (**Bisognano-Wichmann**). *Let U denote a positive-energy representation of the Poincaré group with mass m and spin 0, and consider the pair $(\mathcal{M}(W), \Omega)$, where $\mathcal{M}(W)$ is the local von Neumann algebra of a massive, scalar field associated with the wedge $W \in \mathcal{W}$, and Ω is the vacuum state. Then, the modular group associated with the pair $(\mathcal{M}(W), \Omega)$ is given by:*

$$\Delta^{it} = U(\Lambda_W(t), 0), t \in \mathbb{R}.$$

Proof. See [83, 87]. □

That is, the modular group implements the boosts under the representation U. Moreover, according to our previous discussion, Ω is a KMS state (that is, a thermal state) under the dynamics given by the modular group, a phenomenon nowadays known as the *Unruh effect*. The original paper of Bisognano-Wichmann (1975, [83]) predates Unruh's paper (1976, [84]). Furthermore, Bisognano and Wichmann's derivation of this effect is non-perturbative, while Unruh's utilizes perturbation techniques.

6.3 Modular Localization

In the last section, we showed that if $\mathcal{R}(O)$, $O \subseteq \mathbb{R}^4$, is a local von Neumann algebra with a cyclic and separating vector Ω (Theorem 6.28), we can construct a (local) standard subspace from the vacuum as $H(O) = \overline{\mathcal{R}(O)_{sa}\Omega}$. The *Modular Localization* framework, proposed by Brunetti, Guido and Longo [1], is a method to reverse this process: we start by constructing local standard subspaces and then, by second quantization, we obtain a net of von Neumann algebras satisfying the axioms of AQFT.

The idea, inspired by the Bisognano-Wichmann Theorem (Theorem 6.30), is the following: the conclusion of this theorem is that the modular group of the von Neumann algebra associated with a wedge is equal to the representation of the one-parameter group of boosts preserving this wedge. In the Modular Localization framework, we start with a representation of the Poincaré group and build (local) standard subspaces with the representation of the boosts preserving a given wedge. The algebra of observables constructed by second quantization of these standard subspaces is only dependent on the choice of representation of the Poincaré group and is, therefore, *independent of classical models*. Remember that this was not the case when we constructed the algebra of observables of a bosonic field with the techniques used in Section 6.1.1.

Let $\mathcal{P}_+ = \mathcal{P}_+^\uparrow \cup \mathcal{P}_+^\downarrow$ denote the proper part of the Poincaré group, and $\mathcal{P}_+ \ni g \mapsto U(g)$ be a strongly continuous (anti-) unitary representation on the Hilbert space $\mathcal{H}$, meaning that $U(g)$ is unitary if $g \in \mathcal{P}_+^\uparrow$, and is antiunitary if $g \in \mathcal{P}_+^\downarrow$. Consider a wedge $W \in \mathcal{W}$ as defined in Definition 6.29, and let $\Lambda_W(t)$, $t \in \mathbb{R}$, and R_W be the one-parameter group of boosts preserving W, and its time-reversing reflection, respectively, defined by equations (6.8) and (6.11). Let us define the following operators associated with W:

$$\begin{aligned} \Delta_W &\doteq e^{H_W} \\ J_W &\doteq U(R_W) \\ S_W &\doteq J_W \Delta_W^{1/2}, \end{aligned}$$

where H_W is the self-adjoint generator of $U(\Lambda_W(t))$.

Proposition 6.31. *Let Δ_W and J_W be as above. Then:*

1. *Δ_W is a densely defined, closed, positive non-singular operator on $\mathcal{H}$.*
2. *J_W is an antiunitary operator on $\mathcal{H}$ and $J_W^2 = \mathbb{I}$.*
3. *$J_W \Delta_W J_W^{-1} = \Delta_W^{-1}$.*

4. *S_W is a densely defined, antilinear, closed operator on $\mathcal{H}$ with Range$(S_W) = D(S_W)$ and $S_W^2 \subset \mathbb{I}$.*

Proof. See Propositions 2.1 and 2.2 in [1]. □

Thus, the operator S_W just defined has all the properties of the operator S defined in the last section in Tomita-Takesaki's Theory. Remember that, according to Proposition 6.23, there is a standard subspace associated with S_W, which we denote by $H(W)$, given by:

$$H(W) = \ker(1 - S_W) = \{h \in D(S_W) | S_W h = h\}. \tag{6.15}$$

It follows that $D(S_W) = H(W) + iH(W)$ and $S_W(h + ik) = h - ik$, where $h, k \in H(W)$. Moreover, from Tomita-Takesaki's Theorem 6.25 in the real subspace version, it is true that:

$$\Delta_W^{it} H(W) = H(W)$$
$$J_W H(W) = H(W)',$$

where $'$ is the symplectic complement (6.4).

Hence, we have a net of local standard subspaces given by the map $\mathcal{W} \ni W \mapsto H(W) \subset \mathcal{H}$. Let us explore some of the properties of this family.

Theorem 6.32. *Let U be a (anti-) unitary representation of $\mathcal{P}_+$ and $W \mapsto H(W)$ as above. Then:*

1. *Wedge duality holds, namely: $H(W^\perp) = H(W)'$.*
2. *The representation U acts covariantly: $U(g)H(W) = H(gW)$, $g \in \mathcal{P}_+$.*
3. *The following are equivalent:*
   (a) *The spaces $H(W)$ are factors, that is: $H(W) \cap H(W)' = \{0\}$.*
   (b) *The representation U does not contain the trivial representation.*
   (c) *The net is irreducible, namely: $\bigcap_{W \in \mathcal{W}} H(W) = \{0\}$.*

Proof. See Proposition 2.4 and Theorem 2.5 in [1]. □

Following the process of second quantization described in the last section (remember equation (6.6)), we can obtain a net of local von Neumann algebras $\mathcal{W} \ni W \mapsto H(W) \mapsto \mathcal{R}(W)$. Does this net satisfy the axioms of AQFT in Section 6.1 (restricted to wedge regions)? Let us focus on the first 4 axioms, since wedges do not contain Cauchy surfaces (we will address this

problem later): the first axiom, *Local Algebras*, is obviously satisfied; the third, the *Causality* axiom, is also satisfied due to wedge duality in Theorem 6.32 and the relation between the symplectic complement and the commutant of the respective von Neumann algebra given in Theorem 6.27; the fourth axiom, *Poincaré covariance*, is also clearly satisfied due to the covariance of the net $W \to H(W)$ shown in Theorem 6.32. It only remains to check the conditions under which the second axiom, *Isotony*, is satisfied. As we will see, this axioms holds if, and only if, U is a positive energy representation. To prove this, we must grasp a better understanding of inclusions of real subspaces and wedges.

The following theorem is a one-particle analog of Borchers' Theorem for von Neumann algebras [94] and will serve as a very important technical tool.

Theorem 6.33. *Let $H \subset \mathcal{H}$ be a standard subspace and $U(a) = e^{iaH}$ a one-paramenter group of unitaries on $\mathcal{H}$ satisfying:*

$$\Delta_H^{it} U(a) \Delta_H^{-it} = U(e^{\mp 2\pi t} a)$$
$$J_H U(a) J_H = U(-a).$$

Then, the following are equivalent:

1. *$U(a)H \subset H$ for $a \geq 0$.*
2. *The generator $\pm H$ of the representation is positive.*

Proof. See [90]. □

Next, let us look at some geometric properties of inclusions of wedges. Let C denote the cone in the Lie algebra of $\mathcal{P}_+^\uparrow$ consisting of the generators of future-pointing light-like or time-like translations.

Definition 6.34. Let $W_0, W \in \mathcal{W}$ be any two wedges such that $W_0 \subset W$. We say that W_0 is **positively included** in W if

1. W_0 can be obtained by W via a suitable translation $\exp(a_0 h)$, $a_0 \geq 0$, such that $\pm h \in C$, and where $\exp$ denotes the exponential map from the Lie algebra to the Lie group.
2. We have the following relations:

$$\Lambda_W \exp(ah) \Lambda_W(-t) = \exp\left(e^{\mp 2\pi t} ah\right)$$
$$R_W \exp(ah) R_W = \exp(-ah),$$

for $a, t \in \mathbb{R}$.

Now we have the tools to prove the following fundamental result on inclusions of standard subspaces in the net $\mathcal{W} \ni \mapsto H(W)$.

Theorem 6.35. *Let U be a (anti-) unitary representation of $\mathcal{P}_+$, and let $W, \tilde{W} \in \mathcal{W}$ be any two wedges such that $W \subset \tilde{W}$. Then $H(W) \subset H(\tilde{W})$ if, and only if, U is a positive energy representation.*

Proof. It follows straightforwardly from Theorem 6.33 and the fact that any inclusion of wedges is the composition of finitely many positive inclusions. □

So far we constructed the method of Modular Localization for special regions of Minkowski spacetime, namely, the wedges. Can we extend this construction to other types of regions? Since causally closed, convex regions are given by intersections of wedges (check [95]), for any region $\mathcal{C}$ of this type, we can define:

$$H(\mathcal{C}) \doteq \bigcap_{W \supset \mathcal{C}} H(W),$$

that is, $H(\mathcal{C})$ is the intersection of all wedges containing $\mathcal{C}$. For general causally-closed regions $\mathcal{O}$, we define:

$$H(\mathcal{O}) \doteq \bigvee_{\mathcal{C} \subset \mathcal{O}} H(\mathcal{C}), \tag{6.16}$$

that is, $H(\mathcal{O})$ is the real subspace generated (meaning the closed real span) by all $H(\mathcal{C})$ such that $\mathcal{C}$ is contained in $\mathcal{O}$. We will refer to the map $\mathcal{O} \to H(\mathcal{O})$ as the **modular localization map**.

Proposition 6.36. *Let U be a (anti-) unitary positive energy representation of $\mathcal{P}_+$. Then:*

1. *Isotony holds for the net $\mathcal{O} \to H(\mathcal{O})$, where $\mathcal{O}$ is any causally-closed region.*
2. *If $\mathcal{O}_1 \subset \mathcal{O}_2^{\perp}$, then $H(\mathcal{O}_1) \subset H(\mathcal{O}_2)'$.*
3. *If $\mathcal{O}$ is, in addition, convex, then $H(\mathcal{O}^{\perp}) = H'(\mathcal{O})$ (Haag duality).*

Proof. See Corollary 3.5 in [1]. □

We now have a net of closed real subspaces such that the corresponding net of second quantized algebras satisfies all the axioms of AQFT (except axiom number 5 in Section 6.1). Note,

however, that the subspaces $H(\mathcal{O})$ defined above for causally-closed regions do not necessarily satisfy the standard property. As a consequence, the vacuum will not necessarily be cyclic for the algebra. More precisely:

Theorem 6.37. *Let $H(\mathcal{O})$ be a closed, real subspace as above, where $\mathcal{O}$ is a causally-closed region, and let $\mathcal{R}(\mathcal{O})$ denote the corresponding von Neumann algebra obtained via second quantization. Then, the vacuum state in the second quantization is cyclic and separating if, and only if, $H(\mathcal{O})$ is standard.*

Proof. Follows immediately from items 1 and 2 in Theorem 6.27. □

The above property of the vacuum is known as the *Reeh-Schilider property*. This property is very important in QFT and it has a simple formulation in terms of standard subspaces. If the representation U in Proposition 6.36 is the irreducible representation of a massive particle without spin, then the net of local algebras $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ coincides with the algebra derived in Section 6.1.1. The success of this method of producing local algebras from standard subspaces led physicists to reformulate the axioms of AQFT in terms of standard subspaces, which are simpler in nature (see Section 2.3 in [96], for example). Note that Proposition 6.36 is valid for all positive-energy representations, which includes the infinite spin representation. This is a surprising result since it was shown in [97] that the construction of Wightman fields for this representation is not possible. However, it has been shown that the real subspace of a double cone in this representation is not only non-standard, but it is trivial [98]. This leads to the localization of infinite spin particles in *infinite strings*, where the standard property is recovered [99].

Finally, we point out that the Modular Localization formalism can also be extended to other spacetimes. The somehow tricky part is to find analogs of wedges in other geometries. For de Sitter spacetime, this can be done (check Section 5.2 in [1]). For other spacetimes, the construction is a current topic of research (see [100], for instance).

Chapter 7

Modular Localization and the Localizability Problem

Leben—das heisst für uns Alles, was wir sind, beständig in Licht und Flamme verwandeln, auch Alles, was uns trifft, wir können gar nicht anders.

Nietzsche

The Modular Localization formalism, as presented in the last chapter, is a method to produce a covariant net of closed, real subspaces associated with regions of spacetime and the corresponding net of von Neumann algebras. Even though it carries the word "localization" in its name, it has, in principle, no direct connection with the Localizability Problem. The goal of Part II of this work (and of this chapter) is to show that Modular Localization can be used to provide a new approach to this problem.

As already discussed, the Newton-Wigner approach is based on orthogonal projections defined on the Borel sets of a spatial section Σ_t in spacetime, giving rise to a spectral measure, and a probability measure on the logic $\mathcal{L} = (\mathcal{B}(\Sigma_t), \cup, \cap, ^c)$ for each quantum state. The new approaches that followed Newton-Wigner also constructed a probability measure on this same logic, even though with a different mathematical object, namely, Positive-Operator-Valued-Measures (POVM's). Our approach is different: we want to construct a probability measure on the spacetime logic $\mathcal{L}_\Sigma$. Note, however, that the general picture is just the same when we look from the perspective of the mathematical structure we constructed in Chapter 5: we are fabricating a probability measure on a logic (which contains regions of space/spacetime) for each quantum state. A necessary intermediate step is, of course, to connect the classical features of spacetime with the quantum world. As we will show, this can be done with the aid of Modular Localization, solving all the causality problems in the Newton-Wigner approach. Our program has the distinguishing feature of allowing localization in spacetime rather than only in

space for a fixed time, as is done in the approaches using PVM's and POVM's. Furthermore, our method is valid for any positive energy, positive mass representation of the Poincaré group. Finally, we analyze what are the relations between our method and Newton-Wigner's approach, showing that these can be approximated in some specific sense.

7.1 Implementing the Spacetime Logic

As we saw in Section 6.3, the Modular Localization map is a function from spacetime regions to real closed subspaces of a Hilbert space describing our quantum system. The goal of this section is to understand which logical structures of the spacetime logic are preserved under this map. Is it a logic homomorphism in the sense of Definition 5.4? As we will see, this is not exactly true, but the important properties for our goals are preserved. This idea is inspired by the fact that the Newton-Wigner Localization map $\mathcal{B}(\mathbb{R}^3) \ni O \to E(O)$ (remember equation (3.4)) is a homomorphism of the logic $\mathcal{L} = (\mathcal{B}(\mathbb{R}^3), \cup, \cap, ^c)$ into the Hilbert space. Let us start by proving this.

Theorem 7.1. *Let $\mathfrak{A}$ be a σ-algebra on the set Ω, and $\mathfrak{A} \ni A \mapsto E(A) \in \mathcal{P}(\mathcal{H})$ a spectral measure (Definition A.10), where $\mathcal{H}$ is a Hilbert space. Then, $A \to E(A)$ is a logic homomorphism from $\mathcal{L} \doteq (\mathfrak{A}, \cup, \cap, ^c)$ to $\mathcal{L}_{QM} = (\mathcal{P}(\mathcal{H}), \vee, \wedge, \perp)$ (remember equation (5.6)). Furthermore, $E(\mathfrak{A}) \subset \mathcal{P}(\mathcal{H})$ is a boolean σ-algebra.*

Proof. We want to prove properties 1 to 3 in Definition 5.4 of a lattice homomorphism. It is obvious that $E(\emptyset) = 0$ and $E(\Omega) = \mathbb{I}$. By additivity of spectral measures, we have for any $A, B \in \mathfrak{A}$:

$$E(A \cup B) = E(A \setminus B) + E(B \setminus A) + E(A \cap B).$$

We want to to show that the right-hand side is the least upper bound of $\{E(A), E(B)\}$ (and hence equal to $E(A) \vee E(B)$, according to our discussion in Section 5.1). Suppose that there is another projection $\tilde{E} \in \mathcal{P}(\mathcal{H})$ that is bigger than both $E(A)$and $E(B)$ but is smaller then $E(A \cup B)$. This implies:

$$\begin{gathered}\tilde{E}E(A) = E(A), \quad \tilde{E}E(A \setminus B) = E(A \setminus B) \\ \tilde{E}E(B) = E(B), \quad \tilde{E}E(B \setminus A) = E(B \setminus A) \\ \tilde{E}E(A \cup B) = \tilde{E}.\end{gathered}$$

We can then compute:

$$\begin{aligned}\tilde{E} = \tilde{E}E(A \cup B) &= \tilde{E}E(A \setminus B) + \tilde{E}E(B \setminus A) + \tilde{E}E(A \cap B) \\ &= E(A \setminus B) + E(B \setminus A) + E(A \cap B) \\ &= E(A \cup B),\end{aligned}$$

and hence $E(A \cup B) = \tilde{E} = E(A) \vee E(B)$. Proposition A.11 shows that $E(A \cap B) = E(A)E(B) = E(A) \wedge E(B)$, and hence property 2 is proved.

To prove property 3, note that for any $A \in \mathfrak{A}$:

$$\begin{aligned}E(A^c \cup A) &= E(A^c) + E(A) \\ E(\Omega) &= E(A^c) + E(A) \\ \mathbb{I} &= E(A^c) \vee E(A),\end{aligned}$$

which implies that $E(A^c) = E(A)^\perp$ and property 3 is proved. To prove that it is a logic homomorphism, we note that $E\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \bigvee_{i\in\mathbb{N}} E(A_i)$ and $E\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \bigwedge_{i\in\mathbb{N}} E(A_i)$ for any countable sequence $\{A_i\}_{i\in\mathbb{N}}$. Also, given any $A, B \in \mathfrak{A}$ such that $A \subset B$, it is true that $E(A^c \cap B) = E(A)^\perp \cap E(B)$, and hence $E(\mathfrak{A})$ is a sublogic of $\mathcal{L}_{QM}$. Finally, it is clear that for any $E(A), E(B), E(C) \in E(\mathfrak{A})$ the distributive law 5.3 holds, due to the distributivity of the elements in $\mathfrak{A}$. □

Corollary 7.2. *The Newton-Wigner map $\mathcal{B}(\mathbb{R}^3) \ni O \to P^{NW}(O)$ is a logic homomorphism from $\mathcal{L} = (\mathcal{B}(\mathbb{R}^3), \cup, \cap, ^c)$ to $\mathcal{L}_{QM} = (\mathcal{P}(\mathcal{H}), \vee, \wedge, \perp)$, and $P^{NW}(\mathcal{B}(\mathbb{R}^3)) \subset \mathcal{P}(\mathcal{H})$ is a boolean sub-logic.*

As discussed in Chapter 5, the logic of a physical system encodes the nature of the measurements that can be performed on it. The Newton-Wigner formalism is, therefore, encoding particular measurements performed on spatial sections that know nothing about time, and is thus not a surprise that problems with the causality structure appear in this approach. If the time scale in which the measurement is performed is very short, the Newton-Wigner localization can be seen as a good approximation (check [4]). Nonetheless, in Theoretical Physics we seek to understand not only approximations of reality but reality itself to the maximum extent our intellectual capacities allow us. It is then desired to have a logic of measurements that takes into consideration the causal structure of spacetime. As a consequence of the previous corollary, the most natural approach would be to find a homomorphism from $\mathcal{L}$ into $\mathcal{H}$.

Following the example of the Newton-Wigner localization, it is worth investigating if it is

possible to implement this logic in terms of orthogonal projections, that is, we want to find a homomorphism $P : \mathcal{L}_M \ni O \mapsto P(O) \in \mathcal{P}(\mathcal{H})$ such that $\mu_\psi(O) \doteq \langle \psi, P(O)\psi \rangle$ is, for every $\psi \in \mathcal{H}$, a probability measure on $\mathcal{L}_M$ (as in Definition 5.9), and such that it represents the probability of finding the system in O if it is in the state $\psi \in \mathcal{H}$. Note that the map $O \to P(O)$ *is not a spectral measure* since $\mathcal{L}_M$ is not a σ-algebra: for instance, a double-cone is in this logic but its set complement is not. Therefore, the No-go Theorems presented in Chapter 4 do not apply and it is not obvious that this strategy doesn't work. For our analysis, we will need the following technical lemma due to Borchers.

Lemma 7.3. *Let $V(t) = e^{itH}$ be a strongly continuous, one-parameter group of unitary operators acting on a Hilbert space whose generator H has a spectrum bounded from below. Let P_1 and P_2 be two orthogonal projections such that:*

1. $P_1 P_2 = 0$.
2. *There is an $\epsilon > 0$ such that for all t with $|t| < \epsilon$:*

$$[P_1, V(t)P_2V(-t)] = 0.$$

Then, $P_1V(t)P_2V(-t) = 0$ for all $t \in \mathbb{R}$.

Proof. Check Theorem 3.1 in [101]. □

We can now prove the following No-go theorem, which expresses the impossibility of implementing the spacetime logic in terms of orthogonal projections.

Theorem 7.4 (No-Go Theorem). *Consider the Minkowski spacetime logic $\mathcal{L}_M$, and let $E : \mathcal{L}_M \ni O \mapsto E(O) \in \mathcal{P}(\mathcal{H})$ be a logic homomorphism into the orthogonal projections acting on the Hilbert space $\mathcal{H}$. Let $\mathbb{R}^4 \ni v \mapsto U(v)$ be a strongly continuous representation of the translation group on $\mathcal{H}$, and consider the sublogic $\mathcal{L}_\Sigma$ as defined in equation* (5.7)*. Suppose that the following conditions are satisfied:*

1. *Translation covariance:*

$$E(O + v) = U(v)E(O)U(-v),$$

where $v \in \mathbb{R}^4$.

2. *If v is time-like and future-directed and*

$$U(tv) = e^{iH(v)t}, \quad t \in \mathbb{R},$$

where $H(v)$ is the generator of the translation in the v direction, then $\sigma(H(v))$ is bounded below.

Then, $E(b^{\perp\perp}) = 0$ for any bounded $b \in \mathcal{B}(\Sigma)$.

Proof. Because E is assumed to be a homomorphism, we have that for any $O_1, O_2 \in \mathcal{L}_M$, $O_2 \subseteq O_1^{\perp}$ and:

$$E(O_1)E(O_2) = E(O_2)E(O_1) = 0, \tag{7.1}$$

since $E(O_2)\mathcal{H} \subset E(O_1^{\perp})\mathcal{H} = (E(O_1)\mathcal{H})^{\perp}$. Let us choose a bounded $b_1 \in \mathcal{B}(\Sigma)$. Define $b_2 \doteq b_1 + w$ for some $w \in \mathbb{R}^3$ (spatial translation) such that b_1 and b_2 are disjoint. From equation (7.1), we have that $\left[E(b_1^{\perp\perp}), E(b_2^{\perp\perp})\right] = \left[E(b_1^{\perp\perp}), E(b_1^{\perp\perp} + w)\right] = 0$. Let $w_1 \in \mathbb{R}^4$ be a future-directed, time-like vector. Then, for a small enough $\epsilon > 0$, and for any t such that $|t| < \epsilon$, it follows from the translation covariance condition that:

$$\left[E(b_1^{\perp\perp}), U(tw_1)E(b_1^{\perp\perp} + w)U(-tw_1)\right] = \left[E(b_1^{\perp\perp}), E(b_1^{\perp\perp} + w + tw_1)\right] = 0.$$

The Figure 7.1 shows the setup for an example in 1+1 dimensions.

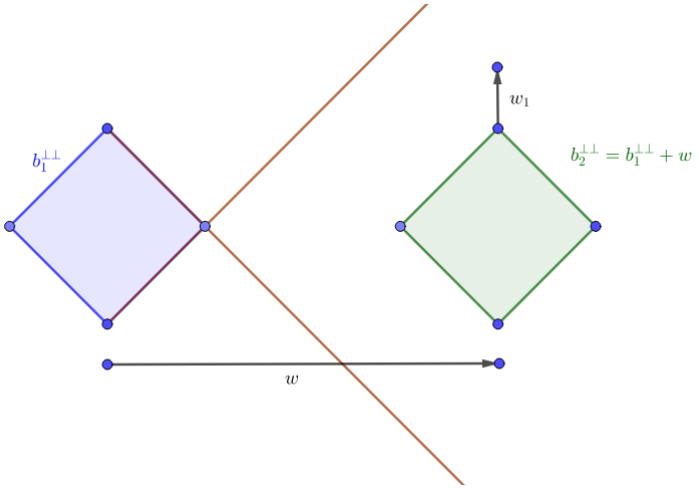


Figure 7.1: Relation between the regions $b_1^{\perp\perp}$ and $b_2^{\perp\perp}$.

We can now apply Lemma 7.3 and conclude that:

$$E(b_1^{\perp\perp})U(tw_1)E(b_2^{\perp\perp})U(-tw_1) = E(b_1^{\perp\perp})E(b_1^{\perp\perp} + w + tw_1) = 0 \tag{7.2}$$

for all $t \in \mathbb{R}$.

Consider now a future-directed, time-like vector w_2 and a big enough $t_2 \in \mathbb{R}$ such that $b_1^{\perp\perp} + t_2 w_2$ is inside the causal future of $b_2^{\perp\perp}$ and, in addition, for small enough $\epsilon > 0$, $b_1^{\perp\perp} + (t + t_2)w_2$ is still inside the causal future of $b_2^{\perp\perp}$ for all $t \in \mathbb{R}$ such that $|t| < \epsilon$. In particular, there exists a time-like vector $z \in \mathbb{R}^4$ and $s \in \mathbb{R}$ such that $b_1^{\perp\perp} + t_2 w_2 = b_2^{\perp\perp} + sz$. See Figure 7.2.

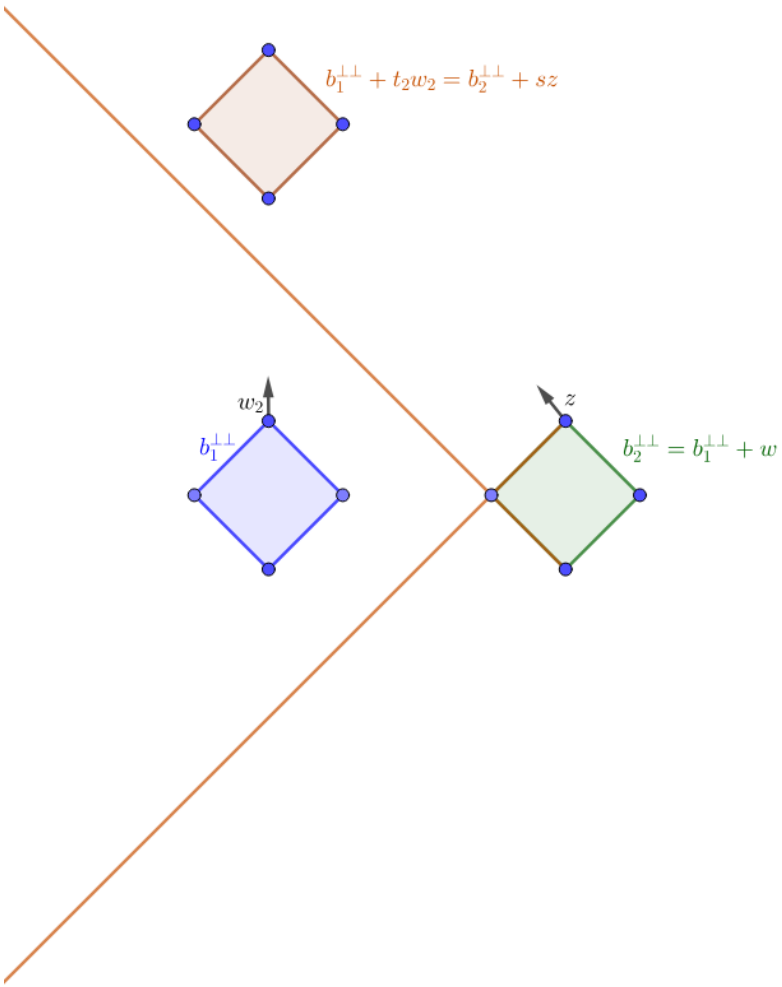


Figure 7.2: Translations of regions $b_1^{\perp\perp}$ and $b_2^{\perp\perp}$.

Therefore:

$$E(b_1^{\perp\perp})E(b_1^{\perp\perp} + t_2 w_2) = E(b_1^{\perp\perp})E(b_1^{\perp\perp} + w + sz) = 0, \tag{7.3}$$

where we used equation (7.2) (with $t = s$ and $w_1 = z$ in this case). Besides, for each t with

$|t| < \epsilon$, there exists a time-like vector $\hat{z} \in \mathbb{R}^4$ and $\hat{s} \in \mathbb{R}$ such that:

$$E(b_1^{\perp\perp})E(b_1^{\perp\perp} + (t_2 + t)w_2) = E(b_1^{\perp\perp})E(b_1^{\perp\perp} + w + \hat{s}\hat{z}) = 0,$$

that is:

$$\left[E(b_1^{\perp\perp}), U(tw_2)E(b_1^{\perp\perp} + t_2w_2)U(-tw_2)\right] = 0 \tag{7.4}$$

for all $|t| < \epsilon$. Hence, we can use equations (7.3) and (7.4) to apply Lemma 7.3 once more and conclude that:

$$E(b_1^{\perp\perp})U(tw_2)E(b_1^{\perp\perp} + t_2w_2)U(-tw_2) = E(b_1^{\perp\perp})E(b_1^{\perp\perp} + (t_2 + t)w_2) = 0$$

for all $t \in \mathbb{R}$. Finally, choose $t = -t_2$ and we have that

$$E(b_1^{\perp\perp}) = 0$$

for all $b_1 \in \mathcal{B}(\Sigma)$. □

Therefore, it is clear that it is impossible to implement the logic $\mathcal{L}_M$ in terms of orthogonal projections, even if it is not a spectral measure, meaning that there is no position operator attached to it (it would have, however, a position observable in the sense of Definition 5.6).

We see from the above theorem that it is not at all obvious how to incorporate the space-time logic into the Hilbert space structure. *It is at this point that Modular Localization plays its role*. We will show that the map (6.16) defining modular localization preserves the necessary logic structure, for our purposes, of the sublogic $\mathcal{L}_\Sigma$. Before that, we need to study the lattice of closed, real subspaces in a complex Hilbert space $\mathcal{H}$. This lattice has a pseudo-orthocomplementation, namely, the symplectic complement. Remember that it is not exactly an orthocomplementation because condition 3 in Definition 5.3 is not always satisfied.

Definition 7.5. Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle , \rangle$, and $\mathcal{C}(\mathcal{H})_\mathbb{R}$ be the set of all closed, real subspaces of $\mathcal{H}$. We define the **(pseudo-)orthocomplemented lattice of closed, real subspaces** as the quadruple $\mathcal{L}(\mathcal{H})_\mathbb{R} \doteq (\mathcal{C}(\mathcal{H})_\mathbb{R}, \vee_\mathbb{R}, \wedge_\mathbb{R}, \prime)$ where:

$$\begin{aligned} H_1 \vee_\mathbb{R} H_2 &\doteq \overline{H_1 + H_2} \\ H_1 \wedge_\mathbb{R} H_2 &\doteq H_1 \cap H_2 \\ H_1' &\doteq \{\xi \in \mathcal{H} | \langle \xi, \eta \rangle_I = 0 \forall \eta \in H_1\}, \end{aligned}$$

for $H_1, H_2 \in \mathcal{C}(\mathcal{H})_{\mathbb{R}}$, and where the overline denotes the closure with respect to the topology induced by $\langle , \rangle_{\mathbb{R}}$, the real part of the inner product $\langle , \rangle = \langle , \rangle_{\mathbb{R}} + i\langle , \rangle_I$.

That the above operations are indeed well-defined can be seen from Proposition 6.20. Similarly to what we did with the logic $\mathcal{L}_{QM}$ in (5.6), we can rewrite the lattice $\mathcal{L}(\mathcal{H})_{\mathbb{R}}$ in terms of real projections: considering $(\mathcal{H}, \langle , \rangle_{\mathbb{R}})$ as a real Hilbert space, all the closed, real subspaces are in one-to-one correspondence with orthogonal projections (with respect to $\langle , \rangle_{\mathbb{R}}$). Note, however, that these projections are not necessarily linear operators in the complex Hilbert space $\mathcal{H}$. Hence, we can equivalently write:

$$\mathcal{L}(\mathcal{H})_{\mathbb{R}} = (\mathcal{P}(\mathcal{H})_{\mathbb{R}}, \vee_{\mathbb{R}}, \wedge_{\mathbb{R}}, ') , \tag{7.5}$$

where $\mathcal{P}(\mathcal{H})_{\mathbb{R}}$ denotes the set of real linear projections in $\mathcal{H}$. We will interchangeably use the notation $\mathcal{L}(\mathcal{H})_{\mathbb{R}}$ to refer to both the lattice of closed real subspaces and the lattice of corresponding real linear projections, depending on the convenience.

Proposition 7.6. *Let U be an (anti-) unitary positive energy representation of $\mathcal{P}_+$. Let $\mathbb{R}^4 \supseteq \mathcal{O} \mapsto H(\mathcal{O})$ denote the modular localization map* (6.16)*. Then:*

1. *The map $\mathcal{L}_\Sigma \ni b^{\perp\perp} \mapsto H(b^{\perp\perp}) \in \mathcal{L}(\mathcal{H})_{\mathbb{R}}$, $b \in \mathcal{B}(\Sigma)$, preserves joins:*

$$H(b_1^{\perp\perp} \vee_M b_2^{\perp\perp}) = H(b_1^{\perp\perp}) \vee_{\mathbb{R}} H(b_2^{\perp\perp}),$$

   if:

   (a) $b_1^{\perp\perp} \cap b_2^{\perp\perp} = \emptyset$.

   (b) $b_1^{\perp\perp} \subseteq b_2^{\perp\perp}$.

2. *The map $\mathcal{L}_\Sigma \ni b^{\perp\perp} \mapsto H(b^{\perp\perp}) \in \mathcal{L}(\mathcal{H})_{\mathbb{R}}$, $b \in \mathcal{B}(\Sigma)$, preserves meets:*

$$H(b_1^{\perp\perp} \wedge_M b_2^{\perp\perp}) = H(b_1^{\perp\perp}) \wedge_{\mathbb{R}} H(b_2^{\perp\perp}),$$

   if:

   (a) *$b_1^{\perp\perp}$ and $b_2^{\perp\perp}$ are convex, disjoint, and U doesn't contain the trivial representation. In this case:*

$$H(b_1^{\perp\perp} \wedge_M b_2^{\perp\perp}) = H(b_1^{\perp\perp}) \wedge_{\mathbb{R}} H(b_2^{\perp\perp}) = \{0\}.$$

(b) $b_1^{\perp\perp} \subseteq b_2^{\perp\perp}$. *In this case:*

$$H(b_1^{\perp\perp} \wedge_M b_2^{\perp\perp}) = H(b_1^{\perp\perp}) = H(b_1^{\perp\perp}) \vee_{\mathbb{R}} H(b_2^{\perp\perp}).$$

3. *Haag duality (in other words, preservation of the orthocomplementation) holds:*

$$H(b^{\perp\perp})' = H(b^{\perp\perp\perp}), \quad b \in \mathcal{B}(\Sigma),$$

if $b^{\perp\perp}$ *is convex.*

Proof. 1. Let b_1 and b_2 be disjoint. Then $b_1^{\perp\perp} \vee_M b_2^{\perp\perp} = b_1^{\perp\perp} \cup b_2^{\perp\perp}$. Remember that the modular localization map is given by:

$$H(\mathcal{O}) \doteq \bigvee_{\mathcal{C}\subset\mathcal{O}} H(\mathcal{C}), \tag{7.6}$$

and hence:

$$H(b_1^{\perp\perp} \vee_M b_2^{\perp\perp}) = \bigvee_{\mathcal{C}\subset b_1^{\perp\perp}\cup b_2^{\perp\perp}} H(\mathcal{C}),$$

where the join is taken over all convex regions C that are contained in $b_1^{\perp\perp} \cup b_2^{\perp\perp}$. Because $b_1^{\perp\perp}$ and $b_2^{\perp\perp}$ are disjoint, the convex region $\mathcal{C}$ is either in $b_1^{\perp\perp}$ or $b_2^{\perp\perp}$, and we can write:

$$\begin{aligned} H(b_1^{\perp\perp} \vee_M b_2^{\perp\perp}) &= \bigvee_{\mathcal{C}\subset b_1^{\perp\perp}\cup b_2^{\perp\perp}} H(\mathcal{C}) \\ &= \left(\bigvee_{\mathcal{C}_1\subset b_1^{\perp\perp}} H(\mathcal{C}_1)\right) \bigvee_{\mathbb{R}} \left(\bigvee_{\mathcal{C}_2\subset b_2^{\perp\perp}} H(\mathcal{C}_2)\right) \\ &= H(b_1^{\perp\perp}) \vee_{\mathbb{R}} H(b_2^{\perp\perp}), \end{aligned}$$

where C_1 and C_2 are the convex sets contained in in $b_1^{\perp\perp}$ and $b_2^{\perp\perp}$, respectively. If $b_1^{\perp\perp} \subseteq b_2^{\perp\perp}$ the joint is also preserved as a consequence of the isotony of the net, as shown in Proposition 6.36. The claim follows immediately.

2. Let us now prove the preservation of the meet. If $b_1^{\perp\perp}$ and $b_2^{\perp\perp}$ are convex and disjoint, then there exists a wedge W that contains $b_1^{\perp\perp} \subset W$ and such that $b_2^{\perp\perp} \subset W^{\perp}$. If U does not contain the trivial representation, the claim follows from item 3 in Theorem 6.32. If $b_1^{\perp\perp} \subseteq b_2^{\perp\perp}$, the claim follows again from the isotony of the net.

3. Follows directly from item 3 in Proposition 6.36.

□

Hence, we see that the modular localization map partially implements the logic $\mathcal{L}_\Sigma$ into the space of real closed subspaces for any positive energy representation. Note that in this proposition we do not provide an *if, and only if* condition: it is not clear if the joins and meets are preserved when the regions in $\mathcal{L}_\Sigma$ are neither disjoint nor such that one is a subset of the other. If we consider the big logic $\mathcal{L}_M$, this is certainly not the case. A simple counter-example is to take single points (which are causally closed), as in the figure below. It is clear that $H(p_1 \vee p_2) \neq \{0\}$, but $H(p_1) \vee_{\mathbb{R}} H(p_2) = \{0\} \vee_{\mathbb{R}} \{0\} = \{0\}$. This is not the case if the two points belong to the same Cauchy surface, which is another evidence that the logic $\mathcal{L}_\Sigma$ is much easier to handle.

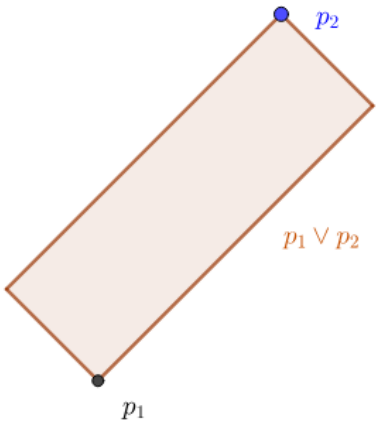


Figure 7.3: Join of single points.

To close this section, we point out that the logic $\mathcal{L}_\Sigma$ can also be partially implemented in von Neumann algebras through second quantization. Let us define the logic of von Neumann algebras.

Definition 7.7. Let $\mathcal{H}$ be a complex Hilbert space, $B(\mathcal{H})$ the set of bounded operators acting on it, and $\mathcal{A}(\mathcal{H})$ the set of all von Neumann algebras in $B(\mathcal{H})$. We define the **(pseudo-) orthocomplemented lattice of von Neuman algebras** as the quadruple $\mathcal{L}_{vN} \doteq (\mathcal{A}(\mathcal{H}), \vee_{vN}, \wedge_{vN}, c)$, where c is the commutant defined in equation (6.2) and:

$$\mathcal{A}_1 \vee_{vN} \mathcal{A}_2 \doteq (\mathcal{A}_1 \cup \mathcal{A}_2)^{cc}$$
$$\mathcal{A}_1 \wedge_{vN} \mathcal{A}_2 \doteq \mathcal{A}_1 \cap \mathcal{A}_2.$$

Similar to what we had in the real subspace case, the commutant is not a true orthocomplementation because condition 3 in Definition 5.3 is not necessarily satisfied for every element (it holds only for factors).

Proposition 7.8. *Consider the second quantization map $\mathcal{H} \supset H \mapsto \mathcal{R}(H) \in \mathcal{A}(\mathcal{H})$ defined in equation* (6.6)*. Then:*

1. *The map $\mathcal{L}_\Sigma \ni b^{\perp\perp} \mapsto \mathcal{R}(H(b^{\perp\perp}))$, $b \in \mathcal{B}(\Sigma)$, preserves joins:*

$$\mathcal{R}(H(b_1^{\perp\perp} \vee_M b_2^{\perp\perp})) = \mathcal{R}(H(b_1^{\perp\perp})) \vee_{vN} \mathcal{R}(H(b_2^{\perp\perp})),$$

   if:

   (a) $b_1^{\perp\perp} \cap b_2^{\perp\perp} = \emptyset$.

   (b) $b_1^{\perp\perp} \subseteq b_2^{\perp\perp}$.

2. *The map $\mathcal{L}_\Sigma \ni b^{\perp\perp} \mapsto \mathcal{R}(H(b^{\perp\perp}))$, $b \in \mathcal{B}(\Sigma)$, preserves meets:*

$$\mathcal{R}(H(b_1^{\perp\perp} \wedge_M b_2^{\perp\perp})) = \mathcal{R}(H(b_1^{\perp\perp})) \wedge_{vN} \mathcal{R}(H(b_2^{\perp\perp})),$$

   if:

   (a) *$b_1^{\perp\perp}$ and $b_2^{\perp\perp}$ are convex, disjoint, and U doesn't contain the trivial representation. In this case:*

$$\mathcal{R}(H(b_1^{\perp\perp} \wedge_M b_2^{\perp\perp})) = \mathcal{R}(H(b_1^{\perp\perp})) \wedge_{vN} \mathcal{R}(H(b_2^{\perp\perp})) = \{0\}.$$

   (b) *$b_1^{\perp\perp} \subseteq b_2^{\perp\perp}$. In this case:*

$$\mathcal{R}(H(b_1^{\perp\perp} \wedge_M b_2^{\perp\perp})) = \mathcal{R}(H(b_1^{\perp\perp})) = \mathcal{R}(H(b_1^{\perp\perp})) \wedge_{vN} \mathcal{R}(H(b_2^{\perp\perp})).$$

3. *Haag duality holds:*

$$\mathcal{R}(H(b^{\perp\perp})') = \mathcal{R}(H(b^{\perp\perp}))^c, \quad b \in \mathcal{B}(\Sigma),$$

   if $b^{\perp\perp}$ is convex.

Proof. The claims follow directly from Proposition 7.6 and Theorem 6.27. □

7.2 The (quasi-) Probability Measure

So far, we have found a way to partially implement the logic $\mathcal{L}_\Sigma$ into the Hilbert space structure describing a relativistic quantum system. This brought us halfway to our goal: we want to attribute probability distributions to each state in $\mathcal{H}$ and to each region of spacetime in $\mathcal{L}_\Sigma$ such that we can interpret it as the probability of detection of the system in the corresponding region. This function describing the probability distribution cannot be the traditional probability measure defined on a σ-algebra because $\mathcal{L}_\Sigma$ *is not a* σ*-algebra*. Fortunately, we have already studied a method to extend this definition to logics (Definition 5.9). Our goal in this section is to find a probability measure on the logic $\mathcal{L}_\Sigma$ with the above-mentioned interpretation.

As usual, let us seek inspiration in the Newton-Wigner formalism. As shown in Corollary 7.2, the Newton-Wigner map $\mathcal{B}(\Sigma) \ni O \mapsto P^{NW}(O) \in \mathcal{P}(\mathcal{H})$ implements the logic $\mathcal{B}(\Sigma)$ in terms of orthogonal projections. Then, a probability measure (in the traditional and in the logic sense) is defined as:

$$\mu^{NW}_\psi(O) \doteq \langle \psi, P^{NW}(O)\psi \rangle,$$

for each $\psi \in \mathcal{H}$, $\|\psi\| = 1$. Following this path, it would be natural to attempt to define a probability measure on $\mathcal{L}_\Sigma$ as:

$$\nu_\psi(b^{\perp\perp}) \doteq \langle \psi, E(b^{\perp\perp})\psi \rangle,$$

where $E(b^{\perp\perp})$ is the real linear projection into the real closed Hilbert space $H(b^{\perp\perp})$. However, *this strategy doesn't work* simply due to the fact that $E(b^{\perp\perp})$ is not necessarily a positive operator which implies that this function is not necessarily positive. Nonetheless, this problem can easily be solved if we *take the expectation value with respect to the real part of the inner product*, that is:

$$\mu_\psi(b^{\perp\perp}) \doteq \langle \psi, E(b^{\perp\perp})\psi \rangle_{\mathbb{R}}. \tag{7.7}$$

In this case, this function is positive, and conditions 1 and 2 in Definition 5.9 are satisfied. It only remains to check condition 3, namely, that if $b^{\perp\perp} = \bigvee_M b_i^{\perp\perp}$ where all b_i's are pairwise disjoint, then $\mu_\psi(b^{\perp\perp}) = \sum_i \mu(b_i^{\perp\perp})$. However, this is true if, and only if,

$$E(b^{\perp\perp}) = \sum_n E(b_n^{\perp\perp}),$$

which can not be true unless the real subspaces $H(b_n^{\perp\perp})$ are pairwise orthogonal (for every such $b^{\perp\perp}$) with respect to the real inner product $\langle , \rangle_{\mathbb{R}}$. These subspaces are already orthogonal with respect to the imaginary part of the inner product and being real orthogonal would imply they are orthogonal with respect to the full inner product $\langle , \rangle$, which is not true in general. Furthermore, we saw in our No-go Theorem 7.4 that it is not a good idea to try to implement the spacetime logic in terms of orthogonal projections.

Therefore, our attempt to use (7.7) as a probability measure seems to be doomed to failure. Nevertheless, let us persist with this function for the time being. The pertinent question to consider is: How nearly orthogonal are they to each other? This is an intricate question and we will dedicate the rest of this subsection to answer it. One possible strategy to answer it is to obtain bounds on the inner products between elements in real subspaces corresponding to disjoint elements in $\mathcal{L}_\Sigma$. Equivalently, we look at operator norm bounds on the product of real linear projections.

Theorem 7.9. *Let U be a (anti-) unitary positive energy, positive mass (where the smallest mass is $m > 0$) representation of $\mathcal{P}_+$. Let $b_1^{\perp\perp}, b_2^{\perp\perp} \in \mathcal{L}_\Sigma$, $b_1 \cap b_2 = \emptyset$, be such that the distance δ between b_1 and b_2 is bigger than zero. Consider the corresponding real linear projections $E(b_1^{\perp\perp})$ and $E(b_2^{\perp\perp})$. Then,*

$$\left\| E(b_1^{\perp\perp}) E(b_2^{\perp\perp}) \right\| \leq e^{-m\delta}. \tag{7.8}$$

Proof. The proof of this theorem follows closely the strategy and techniques used in [102] where the clustering property for massive quantum field theories was proved. Let $\mathcal{H}$ denote the Hilbert space where U is acting, and let us define the functions:

$$\begin{aligned} h(t) &\doteq \langle E(b_1^{\perp\perp})\psi, e^{itH} E(b_2^{\perp\perp})\phi \rangle \\ l(t) &\doteq \langle E(b_2^{\perp\perp})\phi, e^{-itH} E(b_1^{\perp\perp})\psi \rangle, \end{aligned}$$

where $t \in \mathbb{R}$, $\psi, \phi \in \mathcal{H}$ are arbitrary elements, and H denotes the self-adjoint generator of time translation coming from U. The spectrum of H has a mass gap due to the choice of U. Notice that the functions h and l are analytic on the upper and lower half-planes, respectively. Due to the existence of a positive distance δ between b_1 and b_2, there exists some $\tau > 0$ such that we can translate $H(b_2^{\perp\perp})$ in such a way that it remains in the symplectic complement of $H(b_1^{\perp\perp})$, that is:

$$e^{itH} H(b_2^{\perp\perp}) \subset H(b_1^{\perp\perp})', \tag{7.9}$$

for every $|t| < \tau$. This is illustrated in the figure below for an example in $1+1$ dimensions.

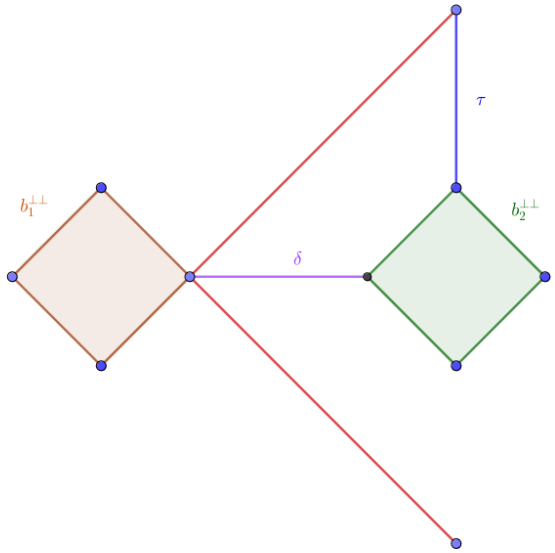


Figure 7.4: Distance δ and the symplectic complement of $b_1^{\perp\perp}$.

Notice that, for t in the real line, we have:

$$h(t) = \overline{l(t)}, \quad \forall t \in \mathbb{R}.$$

In addition, for every $|t| < \tau$, the imaginary part of h and l are zero because of equation (7.9), which means that:

$$h(t) = l(t), \quad \text{for } |t| < \tau.$$

It follows that we can apply the Edge of the Wedge Theorem to guarantee that there is an analytic function k on the twofold cut plane $\mathcal{G}_\tau = \{t \in \mathbb{C} | \operatorname{Im} t \neq 0 \text{ or } |\operatorname{Re} t| < \tau\}$ such that:

$$k(t) = \begin{cases} \langle E(b_1^{\perp\perp})\psi, e^{itH} E(b_2^{\perp\perp})\phi \rangle, & \operatorname{Im} t > 0 \\ \langle E(b_2^{\perp\perp})\phi, e^{-itH} E(b_1^{\perp\perp})\psi \rangle, & \operatorname{Im} t < 0 \\ h(t) = l(t), & \operatorname{Im} t = 0 \text{ and } |\operatorname{Re} t| < \tau. \end{cases}$$

Consider the unit disc $D = \{y \in \mathbb{C} | |y| < 1\}$. Then, the map

$$D \ni y \mapsto \frac{2y\tau}{1+y^2} \in \mathcal{G}_\tau$$

defines a conformal mapping between D and $\mathcal{G}_\tau$. Hence, the function

$$g(y) \doteq k\left(\frac{2y\tau}{1+y^2}\right)$$

is analytic on the unit disc. Our goal is to find an upper bound for $|\langle E(b_1^{\perp\perp})\psi, E(b_2^{\perp\perp})\phi\rangle| = |k(0)|$. For that, we use Jensen's formula (check Chapter 6 in [103]), which gives for $r < 1$:

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} d\nu \log|g(re^{i\nu})| + \sum_a \log\frac{|a|}{r},$$

where the sum on the right-hand side is over all zeros a of g with $|a| < r$. Hence, since this term is non-positive, we have the bound:

$$|g(0)| \leq \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} d\nu \log|g(re^{i\nu})|\right\}. \tag{7.10}$$

Defining $\lambda \doteq 2r/(1-r^2)$, we write the formula:

$$\mathrm{Im}\left\{\frac{2re^{i\nu}}{1+r^2e^{2i\nu}}\right\} = \frac{\lambda\sin(\nu)}{1+\lambda^2\cos^2(\nu)}.$$

Next, we want to find a bound for the function $|g(re^{i\nu})|$. Note that the relevant part of the argument is the imaginary part, as can be seen from the definition of the function k. Using the formula for the imaginary part and the assumption that the spectrum of H is bounded below by m, we can straightforwardly compute from the definition of k that:

$$|g(re^{i\nu})| \leq \exp\left\{\frac{-m\tau\lambda|\sin(\nu)|}{1+\lambda^2\cos^2(\nu)}\right\} \times \begin{cases} \left\|E(b_1^{\perp\perp})\psi\right\|\left\|E(b_2^{\perp\perp})\phi\right\|, & 0 \leq \nu \leq \pi \\ \left\|E(b_2^{\perp\perp})\phi\right\|\left\|E(b_1^{\perp\perp})\psi\right\|, & \pi \leq \nu \leq 2\pi. \end{cases}$$

Substituting this equation in equation (7.10), and defining

$$c(r) \doteq (1/2\pi)\int_0^{2\pi} d\nu|\sin(\nu)|\lambda/(1+\lambda^2\cos^2(\nu)),$$

we obtain:

$$\begin{aligned} |g(0)| &= |\langle E(b_1^{\perp\perp})\psi, E(b_2^{\perp\perp})\phi\rangle| \\ &\leq e^{-m\tau c(r)}\left\|E(b_1^{\perp\perp})\psi\right\|\left\|E(b_2^{\perp\perp})\phi\right\|, \end{aligned}$$

where the integration was carried by substituting $u = \lambda\cos(\nu)$, obtaining

$$c(r) = (2/\pi)\arctan\big(2r/(1-r^2)\big).$$

Since this relation is valid for any $r < 1$, we take the limit $\lim_{r\to 1} c(r) = 1$ to obtain:

$$|g(0)| \leq e^{-m\tau}\big\|E(b_1^{\perp\perp})\psi\big\|\big\|E(b_2^{\perp\perp})\phi\big\|.$$

Furthermore, since the elements $E(b_1^{\perp\perp})\psi$ and $E(b_2^{\perp\perp})\phi$ are in the symplectic complement of each other, it is true that:

$$|\langle E(b_1^{\perp\perp})\psi, E(b_2^{\perp\perp})\phi\rangle| = |\langle E(b_1^{\perp\perp})\psi, E(b_2^{\perp\perp})\phi\rangle_{\mathbb{R}}|.$$

The last step is to obtain a bound on the operator norm $\big\|E(b_1^{\perp\perp})E(b_2^{\perp\perp})\big\|$. This can be computed straightforwardly:

$$\begin{aligned}
\big\|E(b_1^{\perp\perp})E(b_2^{\perp\perp})\big\| &= \sup_{\|\xi\|=1}\big\|E(b_1^{\perp\perp})E(b_2^{\perp\perp})\xi\big\| \\
&= \sup_{\|\xi\|=1}|\langle E(b_1^{\perp\perp})E(b_2^{\perp\perp})\xi, E(b_1^{\perp\perp})E(b_2^{\perp\perp})\xi\rangle_{\mathbb{R}}| \\
&= \sup_{\|\xi\|=1}|\langle E(b_2^{\perp\perp})\xi, E(b_1^{\perp\perp})E(b_2^{\perp\perp})\xi\rangle_{\mathbb{R}}| \\
&= \sup_{\|\xi\|=1}|\langle E(b_2^{\perp\perp})\phi, E(b_1^{\perp\perp})\phi\rangle_{\mathbb{R}}| \qquad (\phi \doteq E(b_2^{\perp\perp})\xi) \\
&\leq \sup_{\|\xi\|=1} e^{-m\tau}\big\|E(b_1^{\perp\perp})\phi\big\|\big\|E(b_2^{\perp\perp})\phi\big\| \\
&= \sup_{\|\xi\|=1} e^{-m\tau}\big\|E(b_1^{\perp\perp})E(b_2^{\perp\perp})\xi\big\|\big\|E(b_2^{\perp\perp})\xi\big\| \\
&\leq e^{-m\tau}.
\end{aligned}$$

□

Therefore, we see that real subspaces associated with disjoint elements in $\mathcal{L}_\Sigma$ which are separated by a positive distance become very close to being orthogonal when the distance between them is "big enough". At this point, one might ask: what distance is "big enough" to consider these spaces as "very close" to being orthogonal? This is a question related to the *scale* of our system. Massless systems might have one further symmetry beyond the Poincaré group transformations: they also have scale invariance, that is, they are invariant under dilations. In this case, our question about the distance being big enough would be pointless. However, mas-

sive systems do have a fixed scale which is determined by its *Compton wavelength*, given by $\lambda_{compton} = 1/m$. We see this fact playing a role since the bigger the mass, the faster the decay. See Section 7.4 below for more discussion on this point.

Let us go back to the main question in this section, namely, we want to understand the additivity property for orthogonal (with respect to the symplectic complement) real projections. In other words, if $E(b_1^{\perp\perp})$ and $E(b_2^{\perp\perp})$ are as in the above theorem, what is the relation between $E(b_1^{\perp\perp}) \bigvee_{\mathbb{R}} E(b_2^{\perp\perp})$ and $E(b_1^{\perp\perp}) + E(b_2^{\perp\perp})$? The following lemma answers this question.

Lemma 7.10. *Let U be a (anti-) unitary positive energy, positive mass (where the smallest mass is $m > 0$) representation of $\mathcal{P}_+$. Let $b_1^{\perp\perp}, ..., b_N^{\perp\perp} \in \mathcal{L}_\Sigma$, $b_i \cap b_j = \emptyset$ for $i \neq j$, be such that the distance $\delta_{i,j}$ between b_i and b_j is bigger than zero. Let $\delta \doteq \min\{\delta_{i,j} | i, j \in \{1, ..., N\},\ i \neq j\}$. Then:*

$$\left\| \bigvee_{i=1}^{N} E(b_i^{\perp\perp}) - \sum_{i=1}^{N} E(b_i^{\perp\perp}) \right\| \leq N(N-1)e^{-m\delta}. \tag{7.11}$$

Proof. Let $\mathcal{H}$ denote the Hilbert space where U acts. Let $\psi \in \mathcal{H}$ be an arbitrary element with $\|\psi\| = 1$. For simplicity of notation, let us denote $H \doteq \overline{H(b_1^{\perp\perp}) + ... + H(b_N^{\perp\perp})}$. Define:

$$\xi \doteq \left(\bigvee_{i=1}^{N} E(b_i^{\perp\perp}) \right) \psi \in H.$$

Since every element in H is the limit of elements in the sum of the Hilbert spaces, we can write:

$$\xi = \lim_{k \to \infty} (\phi_k^1 + ... + \phi_k^N),$$

where $\phi_k^i \in H(b_i^{\perp\perp})$ for every $k \in \mathbb{N}$. Note that:

$$\left(\bigvee_{i=1}^{N} E(b_i^{\perp\perp}) \right) E(b_i^{\perp\perp}) = E(b_i^{\perp\perp}) = E(b_i^{\perp\perp}) \left(\bigvee_{i=1}^{N} E(b_i^{\perp\perp}) \right) \tag{7.12}$$

for every $1 \leq i \leq N$, since every $H(b_i^{\perp\perp})$ is a subspace of H. Next, we want to find an upper bound for the quantity:

$$Q \doteq \left\| \left(\bigvee_{i=1}^{N} E(b_i^{\perp\perp}) - \sum_{i=1}^{N} E(b_i^{\perp\perp}) \right) \psi \right\|.$$

Hence, using equations (7.8) and (7.12), we can calculate:

$$\begin{aligned}
Q &= \left\| \left(\left(\mathbb{I} - \sum_{i=1}^{N} E(b_i^{\perp\perp}) \right) \bigvee_{i=1}^{N} E(b_i^{\perp\perp}) \right) \psi \right\| \\
&= \left\| \left(\mathbb{I} - \sum_{i=1}^{N} E(b_i^{\perp\perp}) \right) \xi \right\| \\
&= \lim_{k\to\infty} \left\| \left(\mathbb{I} - \sum_{i=1}^{N} E(b_i^{\perp\perp}) \right) (\phi_k^1 + ... + \phi_k^N) \right\| \\
&= \lim_{k\to\infty} \left\| \left(E(b_1^{\perp\perp})\phi_k^2 + ... + E(b_1^{\perp\perp})\phi_k^N \right) + ... + \left(E(b_N^{\perp\perp})\phi_k^1 + ... + E(b_N^{\perp\perp})\phi_k^{N-1} \right) \right\| \\
&= \lim_{k\to\infty} \| \left(E(b_1^{\perp\perp})E(b_2^{\perp\perp})\phi_k^2 + \cdots + E(b_1^{\perp\perp})E(b_N^{\perp\perp})\phi_k^N \right) + \cdots + \\
&\qquad \left(E(b_N^{\perp\perp})E(b_1^{\perp\perp})\phi_k^1 + \cdots + E(b_N^{\perp\perp})E(b_{N-1}^{\perp\perp})\phi_k^{N-1} \right) \| \\
&\leq \sum_{i\neq 1}^{N} \left\| E(b_1^{\perp\perp})E(b_i^{\perp\perp}) \right\| + \sum_{i\neq 2}^{N} \left\| E(b_2^{\perp\perp})E(b_i^{\perp\perp}) \right\| + ... + \sum_{i\neq N}^{N} \left\| E(b_N^{\perp\perp})E(b_i^{\perp\perp}) \right\| \\
&\leq \sum_{i\neq 1}^{N} e^{-m\delta_{1,i}} + ... + \sum_{i\neq N}^{N} e^{-m\delta_{N,i}} \\
&\leq N(N-1)e^{-m\delta},
\end{aligned}$$

where the symbol $\sum_{i\neq j}^{N}$ denotes the sum over all $i \in \{1, .., N\}$ except j. The operator norm follows straightforwardly, and the lemma is proved. □

Corollary 7.11. *Let U and the regions $b_i^{\perp\perp} \in \mathcal{L}_\Sigma$, $i \in \{1, .., N\}$, be as in the above Lemma, and let $\omega : B(\mathcal{H}) \to \mathbb{C}$ be a state (in the sense of Definition 6.10) on $B(\mathcal{H})$. Then:*

$$\left| \omega \left(\bigvee_{i=1}^{N} E(b_i^{\perp\perp}) \right) - \sum_{i=1}^{N} \omega(E(b_i^{\perp\perp})) \right| \leq N(N-1)e^{-m\delta}.$$

Proof. The bound is obtained by direct computation:

$$\begin{aligned}
\left| \omega \left(\bigvee_{i=1}^{N} E(b_i^{\perp\perp}) - \sum_{i=1}^{N} E(b_i^{\perp\perp}) \right) \right| &\leq \|\omega\| \left\| \bigvee_{i=1}^{N} E(b_i^{\perp\perp}) - \sum_{i=1}^{N} \omega(E(b_i^{\perp\perp}) \right\| \\
&\leq N(N-1)e^{-m\delta}
\end{aligned}$$

□

Therefore, we see that even though the additivity property does not hold exactly, it holds approximately in the sense of the above lemma. Finally, we are able to define an approximate probability measure for a general state on $B(\mathcal{H})$.

Definition 7.12. Let U be a (anti-) unitary positive energy, positive mass representation of $\mathcal{P}_+$. Then, for each state $\omega : B(\mathcal{H}) \to \mathbb{C}$, we associate a **quasi-probability measure** on $\mathcal{L}_\Sigma$ given by:

$$\mu_\omega(b^{\perp\perp}) \doteq \omega(E(b^{\perp\perp}))_\mathbb{R},$$

where $b^{\perp\perp} \in \mathcal{L}_\Sigma$ and $\omega(\,\cdot\,)_\mathbb{R}$ denotes the real part of the state.

The reason we call it a quasi-probability measure is because the additivity property is not exactly satisfied. Nonetheless, the exponential decay allows us to see that for large separation of the disjoint regions, the above function behaves as a probability measure with finite additivity. See Section 7.4 for more discussion and physical interpretation. Observe that since the trivial representation of $\mathcal{P}_+$ is not included in our results above, we cannot attribute a quasi-probability measure to the vacuum state.

7.3 An Example: Irreducible massive representations in $1+1$D

Note that, up to this point, all of our results are valid for general positive energy, positive mass, (anti-) unitary representations of the proper Poincaré group, for any spacetime dimension. In this section, we restrict our attention to the $1+1$ dimensional Minkowski spacetime and we choose a (anti-) unitary, irreducible, massive representation U. The goal is to show explicit calculations that will agree with our general treatment done in the last section. In particular, as we will show, the almost orthogonality of real subspaces associated with disjoint spacetime regions will manifest itself in the decay properties of the Laplace-Beltrami operators. Finally, we do a comparison of our quasi-probability measure with the Newton-Wigner probability measures.

We start by choosing a positive energy, irreducible representation of $\mathcal{P}_+^\uparrow$. Remember that in Example 2.48 we discussed how to obtain all the irreducible representations by the induction method. We choose the following positive mass, $m > 0$, representation, often referred to as the *rapidity representation*:

$$\begin{aligned}(T(a)\psi)(\theta) &= e^{im(a_0\cosh\theta - a_1\sinh\theta)}\psi(\theta), \quad a = (a_0, a_1) \in \mathbb{R}^2,\\ (\Delta^{it}\psi)(\theta) &= \psi(\theta - 2\pi t), \quad t \in \mathbb{R},\end{aligned}$$

where $T(a)$ is implementing the translations, Δ^{it} the (rescaled) boosts and $\psi \in \mathcal{H} = L^2(\mathbb{R}, d\theta)$.

To extend this representation to the proper Poincaré group, we define the following operators:

$$\begin{aligned}(Z\psi)(\theta) &\doteq \psi(-\theta),\\ (J\psi)(\theta) &\doteq \overline{\psi(\theta)},\\ (\Gamma\psi)(\theta) &\doteq (ZJ\psi)(\theta) = \overline{\psi(-\theta)}.\end{aligned}$$

Note that:

$$\begin{aligned}ZT(a_0,a_1)Z &= T(a_0,-a_1),\\ \Gamma T(a_0,a_1)\Gamma &= T(-a_0,a_1),\end{aligned}$$

which means that Z and Γ implement spatial reflection (parity) $\mathcal{P}$, and time reflection $\mathcal{T}$, respectively. Any element in $\mathcal{P}_+^\downarrow$ can be written as $\mathcal{P}\Lambda\mathcal{T}$. It is easy to check that $Z\Delta^{it}\Gamma$ is an anti-unitary operator and thus we have a positive energy (anti-) unitary representation of $\mathcal{P}_+$ given by:

$$U(g) = \begin{cases} T(a)\Delta^{it}, \text{ if } g=(a,\Lambda)\in\mathcal{P}_+^\uparrow,\\ T(a)Z\Delta^{it}\Gamma, \text{ if } g=(a,\mathcal{P}\Lambda\mathcal{T})\in\mathcal{P}_+^\downarrow.\end{cases}$$

We can change variables and write this representation in momentum space. We refer to the representation in this space as the *momentum representation*. To go from the rapidity to the momentum representation, we simply perform a unitary map $M : L^2(\mathbb{R}, d\theta) \to L^2(\mathbb{R}, dp/\omega)$ given by:

$$(M\psi)(p) = \psi\left(\operatorname{arcsinh}\frac{p}{m}\right), \quad \psi\in L^2(\mathbb{R},d\theta), \tag{7.13}$$

and

$$(M^{-1}\phi)(\theta) = \phi(m\sinh\theta), \quad \phi\in L^2(\mathbb{R},dp/\omega).$$

We then have:

$$\begin{aligned}(T(a)\psi)(p) &= e^{i(\omega a_0 + p a_1)}\psi(p), \quad a = (a_0, a_1) \in \mathbb{R}^2,\\ (\Delta^{it}\psi)(p) &= \psi(\cosh(2\pi t)p - \sinh(2\pi t)\omega), \quad t \in \mathbb{R}\\ (Z\psi)(p) &= \psi(-p)\\ (J\psi)(p) &= \overline{\psi(p)}.\end{aligned}$$

We will use the representation in both of these variables depending on the convenience.

Having chosen a representation, the next step is to understand the construction of the modular localization map $H : \mathcal{L}_\Sigma \to \mathcal{L}(\mathcal{H})_\mathbb{R}$, where we are fixing a given Cauchy surface $\Sigma = \Sigma_t$ (which in this case is just $\mathbb{R}$ for all $t \in \mathbb{R}$). As we saw, this map is first constructed for wedges, which is then extended to the other elements in the logic $\mathcal{L}_\Sigma$. Let $W_1 \in \mathcal{W}$ be the right wedge with base in Σ, that is, $W_1 = \mathbb{R}_+^{\perp\perp}$. The real closed subspace $H(W_1)$ is constructed as in equation (6.15), that is, we want to understand which are the states $\psi \in D(S_{W_1}) \subset L^2(\mathbb{R}, d\theta)$ such that $S_{W_1}\psi = J_{W_1}\Delta_{W_1}^{1/2} = \psi$. Note that, for $\psi \in D(\Delta_{W_1}^{1/2})$:

$$\begin{aligned}(\Delta_{W_1}^{1/2}\psi)(\theta) &= (\Delta_{W_1}^{i(-i/2)}\psi)(\theta)\\ &= \psi(\theta + i\pi),\end{aligned}$$

which means that these states must be analytic on the strip S_π. Likewise, the action of the Tomita operator is:

$$(S_{W_1}\psi)(\theta) = \overline{\psi(\theta + i\pi)}, \quad \psi \in D(S_{W_1}).$$

Hence, we are looking for analytical functions on the strip such that its boundary values (on $\mathbb{R}$) lie in $L^2(\mathbb{R}, d\theta)$. This property resembles the definition of the *Hardy space of the strip*, namely:

$$\mathbb{H}^2(S_\pi) \doteq \left\{\psi : S_\pi \to \mathbb{C} \text{ analytic} \middle| \sup_{0<\lambda<\pi} \int_\mathbb{R} |\psi(\theta + i\lambda)|^2 d\theta < \infty\right\}.$$

We will consider this Hardy space as the subspace of all functions in $L^2(\mathbb{R}, d\theta)$ which are boundary values of functions in $\mathbb{H}^2(S_\pi)$. In fact, we have that:

Lemma 7.13. *In the rapidity representation:*

$$H(W_1) = \{\psi \in \mathbb{H}^2(S_\pi) | \overline{\psi(\theta + i\pi)} = \psi(\theta) \textit{ almost-everywhere}\}.$$

Proof. See Lemma 4.1 in [104]. □

This construction extends to arbitrary wedges by remembering that all wedges can be obtained by the action of the proper Poincaré group on W_1, and the subspaces $H(W)$, $W \in \mathcal{W}$, can likewise be obtained by the action of the representation U on $H(W_1)$. To extend to all elements in $\mathcal{L}_\Sigma$ we use equation (6.16).

The next thing we want to do is to characterize the real subspaces $H(I^{\perp\perp})$ where $I = [a, b]$, $a, b \in [-\infty, \infty]$, is any interval in $\mathbb{R}$. For convenience, we change to the momentum representation in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dp/\omega)$. For any $\psi \in \mathcal{H}$, there is a unique pair (ψ_+, ψ_-) related to the Cauchy data formulation of the solutions of the Klein-Gordon equation that determines it. We can write ψ in terms of these functions as:

$$\psi = \psi_+ + i\omega\psi_-, \quad \psi_+ \doteq \frac{1}{2}(1+\Gamma)\psi, \quad \psi_- \doteq \frac{1}{2i\omega}(1-\Gamma)\psi. \tag{7.14}$$

Theorem 7.14. *Let $I = [a, b]$, $a, b \in [-\infty, \infty]$, be any interval in $\mathbb{R}$. Considering the elements $\psi \in \mathcal{H} = L^2(\mathbb{R}, dp/\omega)$ as tempered distributions, let us define the real subspaces:*

$$K(I^{\perp\perp}) \doteq \{\psi \in \mathcal{H} | \operatorname{supp} \check{\psi}_\pm \subset I\}$$
$$L(I^{\perp\perp}) \doteq \{\psi \in \mathcal{H} | \check{\psi}_\pm \in C^\infty_{c,\mathbb{R}}(I)\},$$

where $\check{}$ denotes the inverse of the time-independent Fourier transform. Then $H(I^{\perp\perp}) = K(I^{\perp\perp})$ and $L(I^{\perp\perp}) \subset H(I^{\perp\perp})$ is a cyclic subspace of $H(I^{\perp\perp})$.

Proof. Check Lemma A.5 and A.8 in [104]. □

Our next goal is to understand "how much orthogonal" two real subspaces are if they are determined by disjoint intervals. The crucial step in this step, as will soon become clear, is to understand the decay properties of the modified Laplace-Beltrami operators. In the Minkowski spacetime $M = \mathbb{R} \times \mathbb{R}$, the Klein-Gordon equation can be written in the form:

$$(\partial_t^2 + A)\psi = 0,$$

where $A \doteq (-\Delta + m^2)$, $m > 0$, is the modified Laplace-Beltrami operator, understood as acting on $L^2(\mathbb{R}, dx)$, the space of square-integrable functions on the Cauchy surface. It is defined as acting on the dense domain $C_c^\infty(\mathbb{R})$. For any $\alpha \in \mathbb{R}$ and $f \in C_c^\infty(\mathbb{R})$, the vector $A^\alpha f$ is in the domain of all powers of A, so it is a smooth function (see Corollary 6.4.9 in [105]) and, when $\alpha \leq 0$, A^α is bounded. We have the following decay properties for this operator:

Proposition 7.15. *Let $\alpha \in \mathbb{R}$ and let $f, g \in C^\infty(\mathbb{R})$, where f is bounded and g has compact support, and the supports of f and g are separated by a distance $\delta > 0$. Then $fA^\alpha g$ (seen as a multiplication operator) is a well-defined operator on $L^2(\mathbb{R})$ and it has the bound:*

$$\|fA^\alpha g\| < C(\alpha)\|f\|_\infty \|g\|_\infty \frac{1}{m^{3/2}\delta^{\alpha+1}} \left(1 + \frac{|\alpha(\alpha+1)|}{2m\delta}\right) e^{-m\delta}, \tag{7.15}$$

for some $C(\alpha) > 0$ depending only on α.

Proof. See Proposition 4.3 in [106]. □

The following theorem presents an analogous result to Theorem 7.9, but obtained with techniques based on the decay properties of the Laplace-Beltrami operators just described.

Theorem 7.16. *Consider the massive momentum representation of $\mathcal{P}_+$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dp/\omega)$. Let I, J be any two intervals in $\mathbb{R}$ such that $I \cap J = \emptyset$, and the distance between I and J is $\delta > 0$. Then, for any $\psi \in H(I^{\perp\perp})$ and any $\phi \in H(J^{\perp\perp})$, we have that:*

$$|\langle \psi, \phi \rangle| < \frac{e^{-m\delta}}{m^{3/2}\delta^{1/2}} \Big[C(-1/2) \big\|\check{\psi}_+\big\|_{L^2(\mathbb{R},dx)} \big\|\check{\phi}_+\big\|_{L^2(\mathbb{R},dx)} \tag{7.16}$$

$$+ C(1) \frac{1}{\delta^{3/2}} \big\|\check{\psi}_-\big\|_{L^2(\mathbb{R},dx)} \big\|\check{\phi}_-\big\|_{L^2(\mathbb{R},dx)} \Big], \tag{7.17}$$

where $C(\,\cdot\,)$ is the constant appearing in 7.15.

Proof. We use equation (7.14) to write:

$$\psi = \psi_+ + i\omega\psi_-, \quad \phi = \phi_+ + i\omega\phi_-.$$

Then:

$$\begin{aligned} |\langle \psi, \phi \rangle| &= |\langle \psi_+, \phi_+ \rangle + \langle i\omega\psi_-, \phi_+ \rangle + \langle \psi_+, i\omega\phi_- \rangle + \langle i\omega\psi_-, i\omega\phi_- \rangle| \\ &\leq |\langle \psi_+, \phi_+ \rangle| + |\langle i\omega\psi_-, \phi_+ \rangle| + |\langle \psi_+, i\omega\phi_- \rangle| + |\langle i\omega\psi_-, i\omega\phi_- \rangle|. \end{aligned}$$

We now compute bounds for each of the above terms:

- For the first term we have:

$$|\langle \psi_+, \phi_+ \rangle| = \Big| \int_{\mathbb{R}} \overline{\psi}_+ \phi_+ \frac{dp}{\omega} \Big| = |\langle \psi_+, \omega^{-1}\phi_+ \rangle_{L^2(\mathbb{R},dp)}| = |\langle \check{\psi}_+, A^{-1/2}\check{\phi}_+ \rangle_{L^2(\mathbb{R},dx)}|,$$

  where in the last step we performed the inverse Fourier transform from $L^2(\mathbb{R}, dp)$ to $L^2(\mathbb{R}, dx)$. From Theorem 7.14, we know that $\mathrm{supp}(\check{\psi}_+) \subseteq I$ and $\mathrm{supp}(\check{\phi}_+) \subseteq J$.

Let $\chi_{\check{\psi}_+}$ and $\chi_{\check{\phi}_+}$ denote the characteristic functions of the supports of $\check{\psi}_+$ and $\check{\phi}_+$, respectively. Furthermore, let us denote by $\chi^{\infty}_{\check{\psi}_+}$ and $\chi^{\infty}_{\check{\phi}_+}$ the smoothed versions of these functions. Then:

$$\begin{aligned}
|\langle \check{\psi}_+, A^{-1/2}\check{\phi}_+\rangle_{L^2(\mathbb{R},dx)}| &= |\langle \check{\psi}_+\chi^{\infty}_{\check{\psi}_+}, A^{-1/2}\chi^{\infty}_{\check{\phi}_+}\check{\phi}_+\rangle_{L^2(\mathbb{R},dx)}| \\
&= |\langle \check{\psi}_+, \chi^{\infty}_{\check{\psi}_+}A^{-1/2}\chi^{\infty}_{\check{\phi}_+}\check{\phi}_+\rangle_{L^2(\mathbb{R},dx)}| \\
&\leq \big\|\check{\psi}_+\big\|_{L^2(\mathbb{R},dx)}\big\|\check{\phi}_+\big\|_{L^2(\mathbb{R},dx)}\Big\|\chi^{\infty}_{\check{\psi}_+}A^{-1/2}\chi^{\infty}_{\check{\phi}_+}\Big\| \\
&< C(-1/2)\big\|\check{\psi}_+\big\|_{L^2(\mathbb{R},dx)}\big\|\check{\phi}_+\big\|_{L^2(\mathbb{R},dx)}\frac{e^{-m\delta}}{m^{3/2}\delta^{1/2}}\left(1+\frac{1}{8m\delta}\right),
\end{aligned}$$

where in the last step we applied Proposition 7.15 with $\alpha = -1/2$.

- The second and third terms are equal to zero. Let us show for the second term:

$$|\langle i\omega\psi_-, \phi_+\rangle| = \Big|\int_{\mathbb{R}} \omega\overline{\psi}_-\phi_+\frac{dp}{\omega}\Big| = |\langle \psi_-, \phi_+\rangle_{L^2(\mathbb{R},dp)}| = |\langle \check{\psi}_-, \check{\phi}_+\rangle_{L^2(\mathbb{R},dx)}| = 0$$

due to the disjointness of the support of the functions in the last equation. An analogous computation shows that the third term is also equal to zero.

- The fourth term can be computed following the exact same steps as used in the first term. The only difference is that the power of A is 1 instead of $-1/2$. The result is:

$$|\langle i\omega\psi_-, i\omega\phi_-\rangle| < C(1)\big\|\check{\psi}_-\big\|_{L^2(\mathbb{R},dx)}\big\|\check{\phi}_-\big\|_{L^2(\mathbb{R},dx)}\frac{e^{-m\delta}}{m^{3/2}\delta^2}\left(1+\frac{1}{m\delta}\right).$$

Putting the first and fourth terms together, we write:

$$\begin{aligned}
|\langle \psi, \phi\rangle| < \frac{e^{-m\delta}}{m^{3/2}\delta^{1/2}}\Bigg[&C(-1/2)\big\|\check{\psi}_+\big\|_{L^2(\mathbb{R},dx)}\big\|\check{\phi}_+\big\|_{L^2(\mathbb{R},dx)} \\
&+ C(1)\frac{1}{\delta^{3/2}}\big\|\check{\psi}_-\big\|_{L^2(\mathbb{R},dx)}\big\|\check{\phi}_-\big\|_{L^2(\mathbb{R},dx)}\Bigg].
\end{aligned}$$

□

Therefore, we see that the spaces $H(I^{\perp\perp})$ and $H(J^{\perp\perp})$ are nearly orthogonal for disjoint intervals I and J. Furthermore, they become exponentially more orthogonal when the distance between the intervals increases and/or the mass increases. Let us compare this result with the general results in the last section:

- The exponential decay with a mass-dependent rate appears as a direct consequence of the decay properties of the Laplace-Beltrami operator.

- This estimate, however, has a disadvantage compared to the previous one: the distance also appears in the denominator, making the estimate less accurate at shorter distances.

Hence, we see that the results of this section, obtained with very different techniques, are in accordance with our abstract treatment in the last section. Furthermore, the abstract calculations give better estimates and more well-behaved bounds at smaller distances and are valid not only for irreducible representations but for arbitrary massive ones. These include Fock space representations and particles with arbitrary spin.

7.3.1 Comparison with Newton-Wigner

In this section, we want to compare our new approach to the Localizability Problem with the Newton-Wigner formalism. As we will see, the approaches are very different, but they converge in a certain sense (see below). A natural first question that arises is: if a state ψ is modular localized in $b^{\perp\perp} \in \mathcal{L}_\Sigma$ (meaning $\psi \in H(b^{\perp\perp})$), is it Newton-Wigner localized in b? Or, at least, is it close to being localized in this region? The following two propositions answer these questions. Remember that the Newton-Wigner unitary map $W : L^2(\mathbb{R}, dp/\omega) \to L^2(\mathbb{R}, dx)$ is given by:

$$(W\psi)(x) = \mathcal{F}^{-1}(\psi/\omega^{1/2})(x), \quad \psi \in L^2(\mathbb{R}, dp/\omega).$$

Proposition 7.17. *Let $\psi \in H(I^{\perp\perp})$, where I is a finite, closed interval. Let $W : L^2(\mathbb{R}, dp/\omega) \to L^2(\mathbb{R}, dx)$ be the Newton-Wigner unitary map, and denote by δ_x the distance between $x \in \mathbb{R}$ and I. Then, for $x \notin I$:*

$$|(W\psi)(x)| < \frac{2^{3/4}\sqrt{\pi}}{\Gamma(1/4)} \frac{|I| e^{-m\delta_x}}{m^{1/4}\delta_x^{1/2}} \left(\left\| \check{\psi}_+ \right\|_\infty + \left\| \check{\psi}_- \right\|_\infty \right). \tag{7.18}$$

Proof. Consider $\psi = \psi_+ + i\omega\psi_-$ as in equation (7.14). Then, we have that:

$$\begin{aligned}(W\psi) &= \mathcal{F}^{-1}\left(\frac{\psi_+}{\omega^{1/2}} + i\frac{\omega\psi_-}{\omega^{1/2}} \right) \\ &= \mathcal{F}^{-1}(\omega^{-1/2}) * \mathcal{F}^{-1}(\psi_+) + i\mathcal{F}^{-1}(\omega^{1/2}) * \mathcal{F}^{-1}(\psi_-) \\ &= f(x) + ig(x),\end{aligned}$$

where we defined $f(x) \doteq \mathcal{F}^{-1}(\omega^{-1/2}) * \mathcal{F}^{-1}(\psi_+)$ and $g(x) \doteq \mathcal{F}^{-1}(\omega^{1/2}) * \mathcal{F}^{-1}(\psi_-)$. We will look for bounds on $|f(x)|$ and $|g(x)|$ for $x \notin I$. Interpreting the Fourier transform in terms of distributions, we can express the inverse Fourier transform of the powers of ω as (check [107]

and [108]):

$$K_\nu(x) = \frac{\sqrt{\pi}x^\nu}{2^\nu\Gamma(\nu+1/2)}\int_1^\infty e^{-xt}(t^2-1)^{\nu-1/2}dt, \quad \nu > -1/2,$$

where K_ν is the modified Bessel function of the second kind. These functions respect the bound (see 8.4.5 in [109]):

$$K_\nu(x) \leq \sqrt{\frac{\pi}{2x}}e^{-x}\left(1+\frac{|4\nu^2-1|}{8x}\right).$$

Hence, we calculate:

$$\begin{aligned}|f(x)| &= \Big|\frac{2^{3/4}m^{1/4}}{\Gamma(1/4)}\int_{\mathbb{R}}\frac{K_{1/4}(m|x-\tau|)\check{\psi}_+(\tau)}{m|x-\tau|^{1/4}}d\tau\Big| \\ &\leq \frac{2^{3/4}m^{1/4}}{\Gamma(1/4)}\big\|\check{\psi}_+\big\|_\infty\int_I\frac{K_{1/4}(m|x-\tau|)}{m|x-\tau|^{1/4}}d\tau. \\ &\leq \frac{2^{1/4}\sqrt{\pi}}{\Gamma(1/4)}\big\|\check{\psi}_+\big\|_\infty|I|\frac{e^{-m\delta_x}}{m^{1/4}\delta_x^{1/2}}.\end{aligned}$$

Doing an analogous calculation for $g(x)$, we find that:

$$|g(x)| \leq \frac{2^{3/4}\sqrt{\pi}}{|\Gamma(-1/4)|}\big\|\check{\psi}_-\big\|_\infty|I|\frac{e^{-m\delta_x}}{m^{1/4}\delta_x^{1/2}}.$$

Gathering the two estimates, we obtain the result. □

Therefore, we see that a modular localized state is at least approximately Newton-Wigner localized since it has an exponential decay outside the localization region. However, can a modular localized state be exactly Newton-Wigner localized? This question is partially answered in our next proposition.

Proposition 7.18. *Let $b_1, b_2 \in \mathcal{B}(\mathbb{R})$, where b_1 is compact, and b_2 is any Borel subset contained in $\mathbb{R}_+$. Denote by $P^{NW}(b_1)\mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}, d\theta)$, the subspace of Newton-Wigner localized states in b_1. Then:*

$$P^{NW}(b_1)\mathcal{H} \cap H(b_2^{\perp\perp}) = \{0\}.$$

Proof. As we saw in Lemma 7.13, for a state to be in $H(b_2^{\perp\perp})$ it must be analytic in the strip S_π (since it is a subspace of $H(W_1)$). Consider a Newton-Wigner localized state $\psi \in P^{NW}(b_1)\mathcal{H}$. This state is in the image of $Z^{-1}L^2(b_1, dx)$, where $Z \doteq W \circ M$, and where M was defined in equation (7.13). Hence there is a function $f \in L^2(b_1, dx)$ such that $\psi = Z^{-1}f$. But then we

have:

$$(Z^{-1}f)(\theta) = \left(\sqrt{m\cosh(\theta)}\right)(\mathcal{F}f)(m\sinh(\theta)).$$

The function $(\mathcal{F}f)(m\sinh(\theta))$ is analytic on the whole complex plane, but the function $\sqrt{m\cosh(\theta)}$ has branch points that we need to examine carefully. The function $\cosh(\theta)$ has zeros in $\theta = i\frac{\pi}{2} + i\pi n$, for $n \in \mathbb{Z}$. Hence, we have a branch point inside the strip, namely, $\theta = \frac{i\pi}{2}$. This forces the choice of a branch cut inside the strip, for example, a half line on the complex axis starting at $i\pi/2$, which we will denote as $l_{i\pi/2}$. The theorem is still not proved, since $(\mathcal{F}f)(m\sinh(\theta))$ could vanish for every point in $l_{i\pi/2}$.

To conclude, we use the *Identity Theorem* for analytic functions that state the following. Given functions $Q(\theta)$ and $R(\theta)$ which are analytic in a domain $D \subseteq \mathbb{C}$, if $Q = R$ on some subset $S \subseteq D$, where S has an accumulation point in D, then $Q = R$ on D. In our case, $Q(\theta) = (\mathcal{F}f)(m\sinh(\theta))$, $R(\theta) = 0$, $D = \mathbb{C}$, $S = l_{i\pi/2}$, and every point in $l_{i\pi/2}$ is an accumulation point in $\mathbb{C}$. It follows from the Identity Theorem that if $(\mathcal{F}f)(m\sinh(\theta))$ vanishes on every point in $l_{i\pi/2}$, then it vanishes everywhere, and the proposition is proved since we showed that every non-zero function $(Z^{-1}f)(\theta)$ can not be analytic in S_π. □

The above proposition shows that Newton-Wigner localized states are essentially different from Modular localized states. This implies that the functions $W(H(b^{\perp\perp})) \subset L^2(\mathbb{R}, dx)$, $b \subseteq \mathbb{R}_+$, cannot have compact support. This is a direct consequence of the *anti-locality* property of the Laplace-Beltrami operator [110]. As an application of Proposition 7.17, we can compute how similar the Newton-Wigner probability distribution is to our quasi-probability measure μ_ψ.

Proposition 7.19. *Let $I = [\alpha, \beta] \subset \mathbb{R}$ be a finite, closed interval, and let $\epsilon > 0$. Then, for any $\psi \in H(I^{\perp\perp})$, $\|\psi\| = 1$, we have that:*

$$|\mu_\psi(I^{\perp\perp}) - \mu_\psi^{NW}(I_\epsilon)| < \left[\frac{2^{3/4}\sqrt{\pi}|I|}{\Gamma(1/4)m^{3/4}}\left(\left\|\check{\psi}_+\right\|_\infty + \left\|\check{\psi}_-\right\|_\infty\right)\right]^2 \times \frac{e^{-2m\epsilon}}{\epsilon},$$

where $I_\epsilon \doteq [\alpha - \epsilon, \beta + \epsilon]$.

Proof. For $\psi \in H(I^{\perp\perp})$, define $f(x) \doteq (W\psi)(x) \in L^2(\mathbb{R}, dx)$. Then:

$$|\mu_\psi(I^{\perp\perp}) - \mu_\psi^{NW}(I_\epsilon)| = |1 - \langle P_{NW}(I_\epsilon)\psi, P_{NW}(I_\epsilon)\psi\rangle| \tag{7.19}$$

$$= |1 - \langle f, \chi_{I_\epsilon} f\rangle_{L^2(\mathbb{R},dx)}|. \tag{7.20}$$

Since $\|\psi\| = 1$, we have that:

$$1 = \langle f, f\rangle_{L^2(\mathbb{R},dx)} = \int_{-\infty}^{\alpha-\epsilon} |f(x)|^2 dx + \int_{\alpha-\epsilon}^{\beta+\epsilon} |f(x)|^2 dx + \int_{\beta+\epsilon}^{\infty} |f(x)|^2 dx.$$

For our estimate, the integral in the middle is the relevant one. Without loss of generality, we take $\alpha - \epsilon$ and $\beta + \epsilon$ to be positive, with $\beta > \alpha$. Using equation (7.18), we write:

$$\begin{aligned}
\int_{\alpha-\epsilon}^{\beta+\epsilon} |f(x)|^2 dx &= 1 - \int_{-\infty}^{\alpha-\epsilon} |f(x)|^2 dx - \int_{\beta+\epsilon}^{\infty} |f(x)|^2 dx \\
&\geq 1 - \left[\frac{2^{3/4}\sqrt{\pi}|I|}{\Gamma(1/4)m^{1/4}} \left(\left\|\check{\psi}_+\right\|_\infty + \left\|\check{\psi}_-\right\|_\infty\right)\right]^2 \\
&\times \left(\int_{-\infty}^{\alpha-\epsilon} \frac{e^{-2m(\alpha-x)}}{\alpha - x} dx + \int_{\beta+\epsilon}^{\infty} \frac{e^{-2m(x-\beta)}}{x-\beta} dx\right) \\
&\geq 1 - \left[\frac{2^{3/4}\sqrt{\pi}|I|}{\Gamma(1/4)m^{1/4}} \left(\left\|\check{\psi}_+\right\|_\infty + \left\|\check{\psi}_-\right\|_\infty\right)\right]^2 \\
&\times \frac{1}{\epsilon}\left(\int_{-\infty}^{\alpha-\epsilon} e^{-2m(\alpha-x)} + \int_{\beta+\epsilon}^{\infty} e^{-2m(x-\beta)}\right) \\
&= 1 - \left[\frac{2^{3/4}\sqrt{\pi}|I|}{\Gamma(1/4)m^{3/4}} \left(\left\|\check{\psi}_+\right\|_\infty + \left\|\check{\psi}_-\right\|_\infty\right)\right]^2 \\
&\times \frac{e^{-2m\epsilon}}{\epsilon},
\end{aligned}$$

where we did the estimate $\int_{-\infty}^{\alpha-\epsilon} \frac{e^{-2m(\alpha-x)}}{\alpha-x} dx \leq \frac{1}{\epsilon}\int_{-\infty}^{\alpha-\epsilon} e^{-2m(\alpha-x)} dx$, and similarly for the second integral. Substituting back into equation (7.19), we have the desired result. □

Our interpretation of the above result is the following. According to Proposition 7.17, the function $(W\psi)$ has an exponential decay outside I. Hence, the Newton-Wigner probability also decays outside the interval. However, the bound given in the mentioned proposition gets worse when we approach the boundary of the interval, where we have a $1/\delta^{1/2}$ divergence. Nonetheless, since the function $(W\psi)$ belongs to $L^2(\mathbb{R}, dx)$, it cannot grow too fast near this boundary. We choose to evaluate the Newton-Wigner probability measure on the expanded interval I_ϵ to ignore this divergence of the estimate. Again, the question of how big this ϵ must be such that the probability measures $\mu_\psi(I^{\perp\perp})$ and $\mu_\psi^{NW}(I_\epsilon)$ are "sufficiently" close is related with the Compton wavelength. This can be seen from the mass dependence of the exponential decay. For instance, if we are doing a position measure in the region I of the system in the state $\psi \in H(I^{\perp\perp})$, and the Compton wavelength is much smaller than the precision of our apparatus, then these two probability measures will give very close results. We discuss more about the scale of the system in the next section.

7.4 Modular Localization as an Approach to the Localizability Problem

In this section, we want to summarize our new approach to the Localizability Problem of relativistic systems and give a physical interpretation of our results. The core idea is the following:

- We assume that the set of all experimentally verifiable propositions about a quantum system has the mathematical structure of a logic. To each individual (experimental) observable in the sense of Definition 1.1, there is an associated sublogic, encoding the way the measuring instruments can be used. A very important example was given in Section 5.3, where the "big" logic of a quantum system is chosen to be $\mathcal{P}(\mathcal{H})$. Given a logic, we can define a notion of states (Definition 5.8), observables (Definition 5.6), and probability measures (Definition 5.9).

- Position observables have the advantage of being more intuitive, as the results of experiments are necessarily directly related to regions of space/spacetime. For the Newton-Wigner approach, the relevant regions are spatial regions for a fixed time, and the relevant logic is $\mathcal{L} = (\mathcal{B}(\Sigma), \cup, \cap, ^c)$. For our approach, the relevant logic is the spacetime logic $\mathcal{L}_\Sigma = \{D \in \mathcal{L}_M | D = b^{\perp\perp}, \text{ where } b \in \mathcal{B}(\Sigma)\}$.

- The next step is to make contact with the quantum. We want to associate each algebraic state (Definition 6.10) with a probability measure on the relevant logic. For the Newton-Wigner approach, this is done by finding a system of imprimitivity, and the measure is given by the expectation values on the orthogonal projections. For our approach, the contact with the quantum is given by the modular map in equation (6.16), and the (quasi-) probability measure is given in Definition 7.12. All of our results are valid for general positive energy, positive mass, (anti-) unitary representations of $\mathcal{P}_+$.

Our approach has one distinguished feature when compared to Newton-Wigner: we do not have an exact probability measure, since the additivity property is not entirely satisfied. Let us try to gain some intuition on the importance of this property.

Suppose a particle is produced (by a decay process, for instance) in the spacetime point A, as in Figure 7.5. It propagates in the forward light cone until it reaches the Cauchy surface Σ_t at time t where we are performing the position measurement. Let $b_i^{\perp\perp}$, $i \in \{1, 2, 3, 4\}$, be the regions of spacetime corresponding to each measurement apparatus, as shown in the figure. Let $S \subset \Sigma_t$ denote the intersection of the forward light cone with the Cauchy surface. The importance of the additivity property is the following: if we cover S completely with

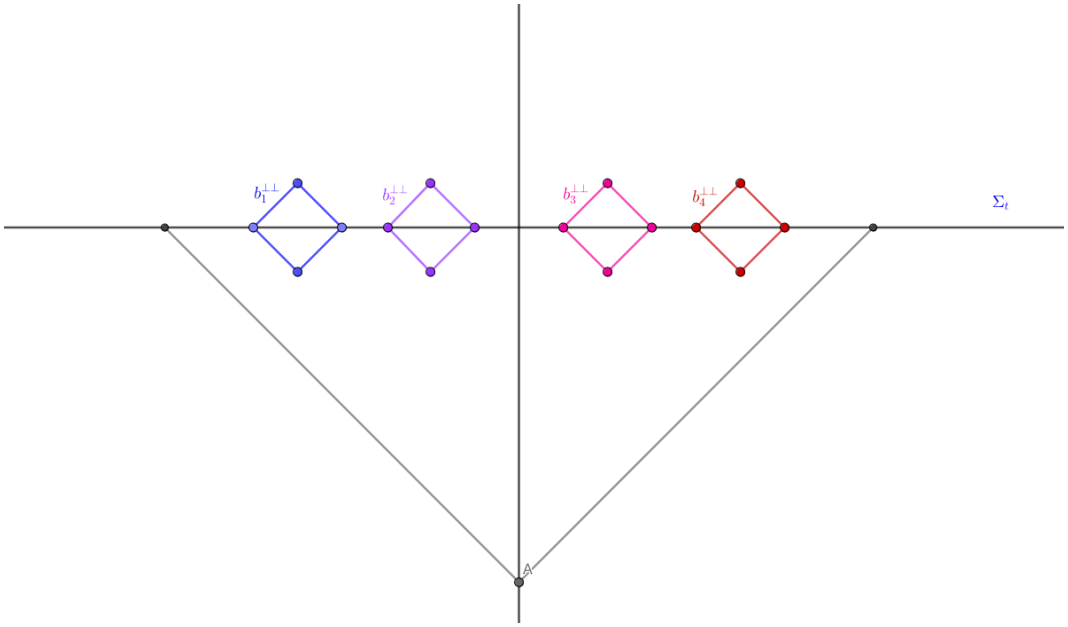


Figure 7.5: Particle produced in A and particle detectors in Σ_t.

disjoint measurement apparatuses (a finite number of them in a real scenario), the sum of the probabilities of detection in each individual apparatus should sum up to 1. The fact that our quasi-probability measure is not exactly additive implies that this will not be the case. However, the scale of the system plays an important role here. Let us analyze this more carefully.

The approximate additivity property, as given in Lemma 7.10, depends crucially on the exponential decay $e^{-m\delta}$, where m is the mass and $\delta > 0$ the distance between two regions. Let us include the fundamental constants such that the argument of the exponential is dimensionless. In this case, the decay is given by:

$$e^{-\frac{m\delta c}{\hbar}} \simeq e^{-m\delta\times 10^{42}},$$

where $c = 299.792.458\, m/s$ is the speed of light, and $\hbar = 6,62607015 \times 10^{-34}\, m^2 kg/s$ is the (reduced) Planck's constant. As an example, let $m_e = 9,109 \times 10^{-31} kg$ be the mass of the electron. The decay is then of the order of $e^{-\delta\times 10^{11}}$. Note, however, that the decay rate gets worse when we increase the number N of measuring apparatuses since there is an N^2 factor in equation (7.11). The volume of each region $b_i^{\perp\perp}$ representing a measurement, and the distance between them, is directly related to the precision of the measurement. Hence, for the non-additivity of the quasi-probability measurement to be visible in experiments, both the number of apparatuses and their precision must be extremely large. Furthermore, given that the mass of the electron is very small, the decay is significantly faster for heavier particles.

This picture and our mathematical analysis so far suggest that we can associate a theoretical

observable with these position measurements, and that cannot be the traditional self-adjoint operator since there is no spectral measure involved. Remember that in the language of logics, an observable (Definition 5.6) is a map $x : \mathcal{B}(\mathbb{R}) \ni b \mapsto x(b) \in \mathcal{L}$, where $\mathcal{L}$ is the logic of measurements of the quantum system. Let us, for this discussion, restrict our attention to the $1+1$ dimensional Minkowski spacetime, for the sake of simplicity. In this case, when it comes to position observables, the set b corresponds to a possible value that a position measurement can give (this is also the case for the Newton-Wigner approach, where the set b belongs to the spectrum of the Newton-Wigner operator and is interpreted as a possible outcome of an experiment). We can think of this observable, both in the Newton-Wigner and in the Modular Localization approach, as taking values in two possible logics: a "purely classical" and a "purely quantum" one. Let us compare both of these approaches.

In the Newton-Wigner case, the classical logic is the logic of the background (spatial) space, namely, $\mathcal{L} = (\mathcal{B}(\mathbb{R}), \cup, \cap, ^c)$. Then, the classical position observable is the map:

$$x_c^{NW} : \mathcal{B}(\mathbb{R}) \ni b \mapsto b \in \mathcal{L}.$$

The quantum logic is $\mathcal{P}(\mathcal{H})$, and the quantum position observable is the map:

$$x_q^{NW} : \mathcal{B}(\mathbb{R}) \ni b \mapsto P^{NW}(b) \in \mathcal{P}(\mathcal{H}).$$

It is easy to check that conditions 1-3 in Definition 5.6 are satisfied. For the Modular Localization case, the classical logic is $\mathcal{L}_\Sigma$, with the classical position observable:

$$x_c^{ML} : \mathcal{B}(\mathbb{R}) \ni b \mapsto b^{\perp\perp} \in \mathcal{L}_\Sigma,$$

while the quantum "logic" is $\mathcal{P}(\mathcal{H})_\mathbb{R}$ (this is actually only a pseudo-orthocomplemented lattice, as discussed in Definition 7.5)), with quantum position observable:

$$x_q^{ML} : \mathcal{B}(\mathbb{R}) \ni b \mapsto E(b^{\perp\perp}) \in \mathcal{P}(\mathcal{H})_\mathbb{R}. \tag{7.21}$$

Conditions 1 and 2 in the definition of an observable are straightforward, while condition 3 is a consequence of Proposition 7.6. Note that, even though μ_ω is only an approximate probability measure on $\mathcal{L}_\Sigma$, x_q^{ML} is *an exact observable*. Furthermore, it is easy to see from the definition of the spectrum of a logic-theoretic observable (Definition 5.7), that we have:

$$\sigma(x_q^{NW}) = \sigma(x_q^{ML}) = \mathbb{R},$$

which is exactly the spectrum that we expect of a position observable.

There is another dichotomy between Newton-Wigner and Modular Localization which is worth comparing. These two localization schemes seem to provide approximate quantities but in very different ways: on one side, the Newton-Wigner localization scheme gives an exact probability measure, but it allows a non-local spread of the probability of detection, even though the probability of detection outside the future light-cone of a compact region is extremely low, as demonstrated in [4]. Hence, Newton-Wigner gives an exact probability measure but is only approximately compatible with relativity. On the other side, the quasi-probability measure we constructed is never an exact measure (although it is one to a very high degree of precision), meaning we cannot give a full statistical treatment to this observable, but it is fully compatible with relativity. It is a curious fact that the inclusion of the causal structure of spacetime has the effect of "messing up" with the statistics of position measurements.

To conclude, let us go back to where we started in the Introduction 1, when we defined the Localizability Problem. Our proposed map $\mathcal{T}_{RQT}$ for position measurements is:

$$\text{Exp}(\mathfrak{I}_{RQT}) \ni \left([\alpha], [Q]_{\text{pos}}, w_{[\alpha]}^{[Q]_{\text{pos}}}\right) \xmapsto{\mathcal{T}_{RQT}} \left(\omega, x_q^{ML}, \mu_\omega\right) \in \text{The}(\mathfrak{I}_{RQT}).$$

Some comments are in order. The precise formulation of the theoretical component $\text{The}(\mathfrak{I}_{RQT})$ is still, of course, an under-development topic of research. It was not our goal to provide a $\mathcal{T}_{RQT}$ that maps every triple on the experimental component to every triple in the theoretical component. Instead, we focused on a specific experimental measurement, namely, position measurements, and provided a map for that in the theoretical part. An intriguing point is that the *input structure is purely classical*: we start with Minkowski spacetime, whose set of isometries has a group structure, namely, the Poincaré group; we select a (anti-) unitary massive representation U of $\mathcal{P}_+$ on a Hilbert space $\mathcal{H}$; this (quantum) Hilbert space gives rise to algebraic states ω on $B(\mathcal{H})$, the (pseudo-) orthocomplemented lattice $\mathcal{P}(\mathcal{H})_{\mathbb{R}}$, the (quasi-) probability measure μ_ω, and the quantum observable of position x_q^{ML}. In summary: the spacetime structure seems to contain information on both the classical and quantum worlds. However, the same cannot be expected for spacetimes with fewer symmetries if we want to include interactions, or if we aim to investigate observables other than the position observable.

Chapter 8

Conclusions

We shall not cease from exploration and the end of all our exploring will be to arrive where we started and know the place for the first time.

T. S. Elliot

The Localizability Problem is a fascinating challenge for theoretical Physics. It refers to the foundations of relativistic quantum systems models, and it shows how the introduction of the causal structure in quantum theory has the potential to change our perspectives. This adaptation led to the creation of Quantum Field Theory, which is nowadays considered to be the most fundamental theory in Physics. Nonetheless, it is somehow surprising that the Localizability Problem is still open. In this thesis, we attacked this problem on two fronts. Let us summarize our achievements.

In Part I, our primary goal was to formulate Newton-Wigner localization in the broadest context possible. We did it for homogeneous globally hyperbolic spacetimes, a framework where the group of isometries is "big enough" (it acts transitively) such that we can apply the techniques of induced representations and its connections with systems of imprimitivity, the main mathematical object in Newton-Wigner's formulation of localizability. Similar to what is done in flat spacetime, all the construction is based on group theory and group representation theory. Our second objective was to classify which representations of the spacetime isometry group G^{ST} are localizable. We show (Theorem 4.6) that our method allows us to classify all induced representations of G^{ST} that are induced from an arbitrary closed subgroup $Z \subset G^{ST}$. When $Z = \mathcal{K}^S$ and K^S is normal, we showed that all representations induced from it are localizable. Furthermore, if G^{ST} has the form of a regular semi-direct product, then all representations are classifiable through our method. In this classification, the stabilizer group $\mathcal{K}^S$ has a decisive role, analogous to the role played by $SO(3)$ in flat spacetime. In addition, we analyzed some applications and direct consequences of our abstract results. In Section 4.1, we

made a first investigation to understand which states in the representation space follow causal geodesics (in the sense that expectation values of the Newton-Wigner operators define a curve on M), inspired by the fact that in Minkowski spacetime all states have this property. In Section 4.2, we gave decompositions of $L^2(\Sigma_t, \nu_t)$ that are induced from the local position operator $M_{i,\alpha}$, namely: a decomposition into invariant subspaces with cyclic vector, and a decomposition into a direct integral. Finally, in Section 4.3, we analyzed the effects on the Newton-Wigner operator of a Gaussian perturbation on the Euclidean metric.

In Part II, our goal was to propose a new approach to the Localizability Problem, inspired by the fact that position measurements must follow logical rules. We did it by implementing the spacetime logic on the quantum Hilbert space by using techniques from Modular Localization. As a result, we constructed a position observable (in the context of logics), and a quasi-probability measure. In contrast to almost all attempted solutions to the Localizability Problem, our approach includes regions of spacetime, rather then just space, and is valid for all unitary positive mass representations of $\mathcal{P}_+$. All the causality problems in Newton-Wigner localization are solved. In addition, in Section 7.3 we did an explicit construction in 1+1 dimensional Minkowski spacetime, and a comparison with Newton-Wigner, showing some close connections with our new approach.

Although we do not claim to have provided a definite solution to the Localizability Problem, we hope to have offered a new look on the issue by deepening our understanding of Newton-Wigner localization and presenting a novel approach based on the logic of spacetime regions. We hope that, at the end of our exploration, we have arrived back where we started, yet now see the Localizability Problem with a new perspective.

Appendices

Appendix A

Functional Analysis Basics

In this Appendix, we define the basic mathematical concepts and explore the most important results, for our purposes, in the areas of Functional Analysis and Spectral Theory which plays an important role in this thesis. The first two subsections are dedicated to the most basic definitions in classifying linear operators acting on Hilbert spaces and analyzing their spectrum. We will skip most of the demonstrations in these sections since they can be found in standard textbooks such as [111–114]. Next, we explore the important connection between self-adjoint operators and spectral measures, given by the spectral theorem. Finally, we investigate and obtain results about a particular type of linear operator which will be the prototype of our generalized position operators: the multiplication operators.

A.1 Classification of Linear Operators on Hilbert Spaces

Let $\mathcal{H}$ be a separable Hilbert space, and $\mathscr{L}(\mathcal{H})$ denote the set of linear operators acting on this space. In this section, we will arrange these linear operators into several classes, which will be relevant in the future when we study operators which possess physical interpretation.

Our first classification is with respect to the "length" of these operators. We can endow $\mathcal{L}(\mathcal{H})$ with the *operator norm*

$$\|T\|_{\text{op}} = \sup_{\psi\neq 0}\frac{\|T\psi\|}{\|\psi\|}, \quad \psi \in \mathcal{D}(T) \subseteq \mathcal{H}, \tag{A.1}$$

where $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{D}(T) \subseteq \mathcal{H}$ is its domain, and $\|.\|$ denotes the norm of $\mathcal{H}$. We will eventually drop the subscript "op" when the context is clear enough. It follows that the set of linear operators is complete in the topology induced by this norm, and the pair $(\mathcal{L}(\mathcal{H}), \|.\|_{\text{op}})$ is a Banach space. We say that a linear operator is *bounded* if $\|T\|_{\text{op}} < \infty$, and *unbounded* otherwise. We

will denote the subset of bounded operators by $\mathcal{B}(\mathcal{H})$. These are the linear maps that are continuous in the norm topology of $\mathcal{H}$. There are fundamental differences between these two kinds of operators. For example, if T is a bounded operator and S is unbounded, both with a common dense domain $\mathcal{D}_0$, then we can always find an extension of T to the whole Hilbert space (that is, we can find another bounded operator $\tilde{T}$ with domain $\mathcal{D}(\tilde{T}) = \mathcal{H}$ which agrees with T on $\mathcal{D}_0$), while this is not true for S in general (see, for example, the Hellinger-Toeplitz theorem in the above cited literature). Hence, it is an unavoidable difficulty when dealing with unbounded operators to be always careful with the domain of the operator.

Next, let us define a norm in the product space $\mathcal{H} \times \mathcal{H}$ as

$$\|(\psi, \phi)\|_{\mathcal{H}\times\mathcal{H}} = \|\psi\| + \|\phi\|,$$

where $\psi, \phi \in \mathcal{H}$. This norm inspires the following classification of operators in $\mathcal{L}(\mathcal{H})$.

Definition A.1. Let $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$ be a linear operator.

1. We define the **graph** of T as the set

$$G(T) = \{(\psi, T\psi) | \psi \in \mathcal{D}(T)\} \subseteq \mathcal{H} \times \mathcal{H}.$$

2. T is **closed** if $G(T) = \overline{G(T)}$ in the norm $\|.\|_{\mathcal{H}\times\mathcal{H}}$.

That is, T is closed if, and only if, every Cauchy sequence $(\psi_n, T\psi_n)$ in $G(T)$ converges in $G(T)$, which is equivalent to say that: if $\psi_n \to \psi$ *and* $T\psi_n \to \phi$, with $\psi_n \in \mathcal{D}(T)$, then this implies that $\psi \in \mathcal{D}(T)$ *and* $T\psi = \phi$. If T is injective and we define $\mathcal{D}(T^{-1}) = \text{Ran}(T)$, then T is closed if, and only if, T^{-1} is also closed. From the definition, we see that every bounded operator with a closed domain is closed: if $(\psi_n, T\psi_n)$ is a Cauchy sequence with $\psi_n \to \psi$ and $T\psi_n \to \phi$, then $\psi \in \mathcal{D}(T)$ (because the domain is closed) and $\phi = T\psi$ (because T is continuous). The converse relation is given by the following famous theorem.

Theorem A.2. *(Closed Graph Theorem) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $T : \mathcal{D}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ a closed operator. Then, T is bounded if, and only if, $\mathcal{D}(T) = \overline{\mathcal{D}(T)}$.*

Our next classification of linear operators is among the most relevant for Physics since, as we will see, they have many properties which advocate for them being a good prototype for physical quantities.

Definition A.3. Let $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$ be a linear operator with a dense domain. We define

the **adjoint** of T, $T^* : \mathcal{D}(T^*) \subseteq \mathcal{H} \to \mathcal{H}$, as

$$\begin{cases} \mathcal{D}(T^*) = \{\phi \in \mathcal{H} | \exists \eta \in \mathcal{H} \text{ such that for each } \psi \in \mathcal{D}(T) \text{ we have } \langle \phi, T\psi \rangle = \langle \eta, \psi \rangle\} \\ T^*\phi = \eta \end{cases}$$

That is, for all $\psi \in \mathcal{D}(T)$ and all $\phi \in \mathcal{D}(T^*)$, we have that $\langle \phi, T\psi \rangle = \langle T^*\phi, \psi \rangle$.

Note that if T is bounded, then Riesz's Theorem guarantees that $\mathcal{D}(T^*) = \mathcal{H}$: for any functional of the form $\langle \phi, T. \rangle : \mathcal{H} \to \mathbb{C}$ there is an unique vector $\eta \in \mathcal{H}$ such that the desired relation is satisfied. In the above definition, we demanded that the operator is densely defined in order that the element η, when it exists, is unique. Indeed, consider a vector $\phi \in \mathcal{H}$ e let $\eta, \eta' \in \mathcal{H}$ be such that

$$\langle \phi, T\psi \rangle = \langle \eta, \psi \rangle \text{ and } \langle \phi, T\psi \rangle = \langle \eta', \psi \rangle,$$

where $\psi \in \mathcal{D}(T)$. Then

$$\langle \eta - \eta', \psi \rangle = 0 \Rightarrow \eta = \eta' \text{ if } \overline{\mathcal{D}(T)} = \mathcal{H}.$$

It can be shown (see the above literature for proof) that the adjoint of any densely defined operator is closed. However, note that the double adjoint T^{**} does not necessarily exist because $\mathcal{D}(T^*)$ might not be dense. There is an interesting relation between the property of an operator being closed and the existence of its double adjoint.

Theorem A.4. *Let $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$ be a linear operator with a dense domain, and let T^* be its adjoint. Then, $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$ if, and only if, T is closable, that is, if it admits at least one extension which is a closed operator.*

Proof. See [114]. □

We are now able to define the important concept of self-adjoint operators.

Definition A.5. A densely defined linear operator $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$ is **self-adjoint** if

$$\mathcal{D}(T) = \mathcal{D}(T^*) \text{ and } T\psi = T^*\psi,$$

for all $\psi \in \mathcal{D}(T) = \mathcal{D}(T^*)$. That is, T is self-adjoint if $T = T^*$.

It is an obvious consequence of this definition that every self-adjoint operator is closed. We

will denote the set of all self-adjoint operators by $B(\mathcal{H})^{\mathbb{R}}$. In the following sections, most of our efforts will be to study the properties of this class of operators.

A.2 Spectrum of Linear Operators

One of the axioms of quantum mechanics is that the set of real numbers representing the results of experiments probing a quantum system coincides with the spectrum of the operator representing the quantum observable. Hence, it is of fundamental importance to understand the spectrum of operators acting on Hilbert spaces. This section is dedicated to investigating the basic definitions and results of this analysis. Let us begin with the following definition.

Definition A.6. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$ be a linear operator.

1. $\lambda \in \mathbb{C}$ is a **regular point** of T if the operator $(\lambda\mathbb{I} - T)^{-1}$ is bounded and defined in all $\mathcal{H}$. The set of all complex numbers of this kind is called the **resolvent set**, and denoted as $\rho(T)$.

2. If $\lambda \in \rho(T)$, we define the **resolvent operator** as $R_\lambda(T) = (\lambda\mathbb{I} - T)^{-1} \in \mathcal{B}(\mathcal{H})$.

3. The **spectrum** of T, $\sigma(T)$, is the complement of the resolvent set, that is

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

Let $\lambda \in \rho(T)$. This implies that $R_\lambda(T)$ is bounded and defined in the dense domain $\text{Ran}(\lambda\mathbb{I} - T)$, and that $\lambda\mathbb{I} - T$ is bijective in $\mathcal{D}(T)$. It follows that $\lambda\mathbb{I} - T$ is also closed (see discussion after definition A.1). Since $T = -((\lambda\mathbb{I} - T) - \lambda\mathbb{I})$, and the sum of a closed operator with a bounded operator is closed (in the intersection of the domains), then we conclude that T is closed. In other words, the existence of a regular point for any operator implies that this operator is closed. Hence, the non-closed operators have a trivial spectrum, namely, the whole complex plane, and, for this reason, we will only be interested in closed operators. For this class, we could alternatively write the resolvent set as

$$\rho(T) = \{\lambda \in \mathbb{C} | \lambda\mathbb{I} - T : \mathcal{D}(T) \to \mathcal{H} \text{ is bijective}\}.$$

Since this condition implies that $\mathcal{D}(R_\lambda(T)) = \mathcal{H}$, the closed graph theorem (see theorem A.2) guarantees that $R_\lambda(T)$ is bounded. Furthermore, the spectrum of any operator is always closed and, if the operator is bounded, it is also bounded (and hence compact) and non-empty (see the above-cited literature for proof of these statements).

The spectrum of any closed operator can be divided into the following subsets:

- *Point spectrum*:

$$\sigma_p(T) = \{\lambda \in \mathbb{C} | \lambda\mathbb{I} - T \text{ is not injective}\} .$$

- *Continuous spectrum*:

$$\sigma_c(T) = \{\lambda \in \mathbb{C} | \lambda\mathbb{I} - T \text{ is injective, } \mathrm{Ran}(\lambda\mathbb{I} - T) \text{ is dense but not closed}\} .$$

- *Residual spectrum*:

$$\sigma_r(T) = \{\lambda \in \mathbb{C} | \lambda\mathbb{I} - T \text{ is injective, but } \mathrm{Ran}(\lambda\mathbb{I} - T) \text{ is not dense}\} .$$

These sets are clearly disjoint, and they enclose all the possibilities of a complex number failing to be in the resolvent set. Hence, $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. Self-adjoint operators have the interesting property that their spectrum is necessarily real and the residual component is empty. This motivates the use of these operators as modeling physical observables of quantum systems. Furthermore, transformations on quantum systems with physical meaning (such as symmetry transformations) are often implemented as linear maps acting on these self-adjoint operators. Hence, an interesting question in this context is to ask what are the linear maps which preserve the spectrum (including its individual components). First, note that we can define the following equivalence relation in $\mathcal{B}(\mathcal{H})$: $A \sim B$ if, and only if, $\sigma(A) = \sigma(B)$, for $A, B \in \mathcal{B}(\mathcal{H})$. Hence, any bijective map which maps each equivalence class into itself is spectrum preserving. However, these maps do not have a clean, closed form, and can be quite difficult to deal with (see [115]). Nonetheless, we are still able to work with fairly general maps, as shown in the following proposition (see [116] for proof).

Proposition A.7. *Let A and B be two similar linear operators acting on $\mathcal{H}$ (recall that two operators are similar if there is an invertible operator P such that $P^{-1}AP = B$). Then, $\sigma(A) = \sigma(B)$. In addition, the components of the spectrum are also preserved.*

Suppose that T is a self-adjoint operator modeling a physical observable of a quantum system. Then, $\langle \psi, T\psi \rangle$ is a real number representing the expected value of the reading of this observable if repeated experiments are performed with the quantum system in the normalized state $\psi \in \mathcal{H}$. Therefore, it is interesting to understand the mathematical structure of the set of these expected values when we vary over all possible normalized states.

Definition A.8. Let T be a bounded operator acting on $\mathcal{H}$. We define the **numerical range** of T as the set

$$N(T) = \{\langle\psi, T\psi\rangle | \psi \in \mathcal{H}, \|\psi\| = 1\}\,.$$

The numerical range is closely connected with the spectrum, but it also might present quite different properties. For example, the numerical range is not always a closed set. In addition, while the spectrum is invariant under general similarity maps, the same is not true for the numerical range. However, both these sets are invariant under unitary similarity. A further contrast is that there are only a few general characteristics of the numerical range. One of these is the famous *Toeplitz-Hausdorff theorem*, which asserts that $N(T)$ is always a convex subset of the complex plane (see [117]). The connection with the spectrum is illustrated by the following property: for any bounded operator T, $\sigma(T) \subseteq N(T)$ (see [116]). Yet, these sets can still be very different. For example, the spectrum of the matrix

$$M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{A.2}$$

is $\sigma(M) = \{0\}$, while the numerical range is $N(M) = \{z \in \mathbb{C} | |z| \leq 1/2\}$.

Since the spectrum is contained in the numerical range, and this last is a convex set, the convex hull of the first must be contained in the numerical range. Furthermore, the convex hull of the spectrum must be contained in the closure of the numerical range of all operators which are similar to the one in question. In fact, a very elegant result, known as the *Hildebrandt theorem*, asserts that the convex hull of the spectrum is *precisely* the intersection of the closure of the numerical ranges of all similar operators. We will be mainly interested in self-adjoint operators and, fortunately, the numerical range of these operators is much more treatable, as a particular consequence of the following result (see [116] for a proof).

Lemma A.9. *Let T be a bounded, normal operator (that is, $TT^* = T^*T$). Then, $\overline{N(T)} = co(\sigma(T))$, where "co" denotes the convex hull.*

A.3 The many faces of the Spectral Theorem

In this section, we review the remarkable topic of the Spectral Theorem for self-adjoint operators. The importance of this subject can hardly be overestimated, finding uncountable applications in both mathematics and physics. In particular, this theorem lies at the core of the

mathematical formulation of quantum mechanics, and it plays an important role in the definition of the position operators. The spectral theorem has many faces, in the sense that it can be formulated in different, but equivalent, versions. Despite the equivalence of these versions, it will be interesting, and rewarding, to "see" each face of this theorem. Let us begin with some basic definitions and results. This section follows [57] and [58] closely. Some proofs are our own.

A.3.1 Spectral Measures and Spectral Integrals

Definition A.10. Let $\mathcal{H}$ be a Hilbert space, Ω a set, and $\mathfrak{A}$ a σ-algebra on Ω. A **spectral measure** on $\mathfrak{A}$ is a map E from $\mathfrak{A}$ to the set of orthogonal projections of $\mathcal{H}$ such that

1. $E(\Omega) = \mathbb{I}$,

2. For any sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}}$, whose union is also in $\mathfrak{A}$, we have that

$$E(\cup_{n\in\mathbb{N}} A_n) = \text{s-lim}_{k\to\infty} \sum_{n=1}^{k} E(A_n),$$

   where in the right hand side the limit is taken in the strong topology.

We can extract some properties of the spectral measures straightforwardly. Suppose that $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets, as in the second item above, but with $A_n = 0$ for every $n \geq k+1$, for some $k \in \mathbb{N}$. Then, it follows from the definition that

$$E(A_1 \cup ... \cup A_k) = E(A_1) + ... + E(A_k). \tag{A.3}$$

The left-hand side is also a projection, which means that $E(A_1 \cup ... \cup A_k)^2 = E(A_1 \cup ... \cup A_k)$. In particular, if $A_n = \emptyset$ for all n, then $E(\emptyset) = 0$.

Proposition A.11. *Let E be a spectral measure as above. Then, for any $A_1, A_2 \in \mathfrak{A}$, we have*

$$E(A_1)E(A_2) = E(A_1 \cap A_2).$$

In particular, for disjoint sets, $E(A_1)E(A_2) = 0$.

Proof. Let $A_1, A_2 \in \mathfrak{A}$ be disjoint sets. Then, by the equation (A.3), $E(A_1) + E(A_2)$ is also a projection. We will prove that if a vector $\psi \in \mathcal{H}$ belongs to the image of $E(A_1)$ (resp. $E(A_2)$),

then it necessarily belongs to the kernel of $E(A_2)$ (resp. $E(A_1)$). Suppose that $\psi \in \text{Im}(E(A_1))$ is a non-zero vector and $\psi \notin \text{Ker}(E(A_2))$. Define $\phi = E(A_2)\psi$. Then

$$\begin{aligned}\psi + \phi &= (E(A_1) + E(A_2))\psi \\ &= (E(A_1) + E(A_2))^2\psi \\ &= (E(A_1) + E(A_2))(\psi + \phi) \\ &= \psi + E(A_1)\phi + 2\phi.\end{aligned}$$

That is, we conclude that $E(A_1)\phi = -\phi$. But this implies that $E(A_1)^2\phi = \phi \neq E(A_1)\phi$, which is a contradiction to our hypotheses that $E(A_1)$ is a projection. Hence, there can be no non-zero element which is in the image of $E(A_1)$ and is not in the kernel of $E(A_2)$. Exchanging A_1 by A_2 in this proof, we have a similar argument, which allows us to conclude that $E(A_1)E(A_2) = E(A_2)E(A_1) = 0$.

Next, define $B_0 = A_1 \cap A_2$, $B_1 = A_1 \setminus B_0$, and $B_2 = A_2 \setminus B_0$. Since these are all pairwise disjoint sets, the product of their respective projections will be zero. Then, since $A_1 = B_1 \cup B_0$ and $A_2 = B_2 \cup B_0$, it follows from equation (A.3) that

$$\begin{aligned}E(A_1)E(A_2) &= \left(E(B_1) + E(B_0)\right)\left(E(B_2) + E(B_0)\right) \\ &= E(B_0)^2 \\ &= E(A_1 \cap A_2).\end{aligned}$$

□

Let us give an explicit example of a spectral measure.

Example A.12. Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space, and $L^2(\Omega, \mu)$ the space of square-integrable, measurable, complex-valued functions defined on Ω. Consider the following map: to every $A \in \mathfrak{A}$, we associate the operator $E(A)$ which acts as a multiplication operator by the characteristic function of A, that is,

$$\left(E(A)f\right)(s) = \chi_A(s)f(s), \tag{A.4}$$

where $f \in L^2(\Omega, \mu)$. Since $\chi_A^2 = \chi_A = \overline{\chi}_A$, it follows that $E(A)$ is an orthogonal projection.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoint sets of $\mathfrak{A}$ such that $A = \cup_{n\in\mathbb{N}} A_n$. Define the vector $f_k = \sum_{n=1}^{k} \chi_{A_n} f$, where $f \in L^2(\Omega, \mu)$. Note that the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges pointwise

to $\chi_A f$. Furthermore, we have

$$|f_k| = |\sum_{n=1}^{k} \chi_{A_n} f| \leq |\chi_A f|,$$

which means that f_k is dominated by $|\chi_A f|$ (which is an element of $L^2(\Omega, \mu)$. Therefore, we can apply the Lebesgue Dominated Convergence Theorem to conclude that

$$\lim_{k\to\infty} \|\chi_A f - f_k\| = 0,$$

that is,

$$\chi_A f = \lim_{k\to\infty} \sum_{n=1}^{k} \chi_{A_n} f,$$

and hence $E(A) = \sum_{n=1}^{\infty} E(A_n)$.

There is a direct relation between spectral measures and scalar measures defined on the same σ-algebra. This is the content of the following lemma.

Lemma A.13. *A map E from a σ-algebra into the set of orthogonal projections on $\mathcal{H}$ is a spectral measure if, and only if, $E(\Omega) = \mathbb{I}$ and for each $\psi \in \mathcal{H}$, the function $E_\psi(.) = \langle \psi, E(.)\psi \rangle$ is a measure.*

Proof. Let us begin by proving that if E is a spectral measure, then $E_\psi(.)$ is a measure. It is clear that $E_\psi(\emptyset) = \langle \psi, E(\emptyset)\psi \rangle = 0$. It is also straightforward to see that, if $(A_n)_{n\in\mathbb{N}}$ is a collection of disjoint sets in the σ-algebra, then

$$\begin{aligned} E_\psi(\cup_{n\in\mathbb{N}} A_n) &= \langle \psi, E(\cup_{n\in\mathbb{N}} A_n)\psi \rangle \\ &= \langle \psi, \sum_{n=1}^{\infty} E(A_n)\psi \rangle \\ &= \sum_{n=1}^{\infty} \langle \psi, E(A_n)\psi \rangle \\ &= \sum_{n=1}^{\infty} E_\psi(A_n). \end{aligned}$$

Let us now check that if $E(\Omega) = \mathbb{I}$ and $E_\psi(.) = \langle \psi, E(.)\psi \rangle$ is a measure, then E is a spectral measure. We only need to proof the second item in the definition of the spectral measure. Define $A = \cup_{n\in\mathbb{N}} A_n$, where the A_n's are pairwise disjoint and such that $A \in \mathfrak{A}$. Since $E_\psi(.)$ is a

measure, then it is countably additive and

$$\begin{aligned}\langle \psi, E(A)\psi \rangle &= E_\psi(A) \\ &= \sum_{n=1}^{\infty} E_\psi(A_n) \\ &= \sum_{n=1}^{\infty} \langle \psi, E(A_n)\psi \rangle \\ &= \left\langle \psi, \sum_{n=1}^{\infty} E(A_n)\psi \right\rangle .\end{aligned}$$

The strong limit of the sum in the last line exists and is an orthogonal projector since this is a sum of pairwise orthogonal projections (see [80] for a proof of this statement). This relations is valid for any $\psi \in \mathcal{H}$, then $E(A) = \sum_{n=1}^{\infty} E(A_n)$ and E is a spectral measure. □

One interesting question in this context is the following: if we have a pair, or a finite collection, of spectral measures, all defined in the same Hilbert space, can we construct a single spectral measure from these? The following result answers this question when the spectral measures pairwise commute, and are of fundamental importance in the definition of Newton-Wigner localizability.

Theorem A.14. *For $j = 1, ..., k$, let Ω_j be a locally compact Hausdorff space with a countable base of open sets. For each of these spaces, let E_j be a spectral measure defined on the Borel σ-algebra $\mathcal{B}(\Omega_j)$. Suppose that these spectral measures all act on the same Hilbert space $\mathcal{H}$, and pairwise commute, that is, for any $A \in \mathcal{B}(\Omega_j)$ and $B \in \mathcal{B}(\Omega_l)$, we have $E_j(A)E_l(B) = E_l(B)E_j(A)$. Then, there exists a unique spectral measure, E, which we call the* ***product spectral measure*** *(or the joint spectral measure), defined on the product Borel σ-algebra $\mathcal{B}(\Omega)$, where $\Omega = \Omega_1 \times ... \times \Omega_k$, such that*

$$E(A_1 \times ... \times A_k) = E_1(A_1)...E_k(A_k),$$

where $A_j \in \mathcal{B}(\Omega_j)$ and $j = 1, ..., k$.

Proof. See [57]. □

Our next goal is to define the so-called *spectral integrals*. These are operator-valued integrals of measurable functions defined with respect to a spectral measure. We will give a precise definition of spectral integrals of bounded (in the supremum norm), measurable functions. We do this by first defining the spectral integrals of simple functions, and latter extending

to bounded functions. Finally, we will comment on the generalization to spectral integrals of unbounded functions.

Let $\mathcal{B}$ be the space of all bounded, measurable (with respect to a σ-algebra $\mathfrak{A}$) functions from Ω to $\mathbb{C}$. We can equip this space with the *supremum norm*

$$\|f\|_s = \sup\{|f(s)| : s \in \Omega\},$$

such that $(\mathcal{B}, \|.\|_s)$ is a Banach space. Define $\mathcal{B}_s$ to be the subspace of simple functions in $\mathcal{B}$, that is, functions which have only a finite number of values. These can be written as

$$f = \sum_{r=1}^{N} c_r \chi_{A_r}, \tag{A.5}$$

where each c_r is a complex number and the A_r's are pairwise disjoint sets of the σ-algebra.

Definition A.15. Let E be a spectral measure defined on $\mathfrak{A}$, and $f \in \mathcal{B}_s$ be a simple function expressed by the sum (A.5). Then, we define the **spectral integral of f with respect to the spectral measure** E as the operator

$$\mathbb{I}(f) = \sum_{r=1}^{N} c_r E(A_r). \tag{A.6}$$

A possible source of ambiguity in this definition is that, in general, simple functions have different (but equivalent) "representations", that is, the expression of f in terms of the sum in (A.5) is in general not unique. Let us prove that the operator $\mathbb{I}(f)$ is independent of representation. Suppose that f can also be written as

$$f = \sum_{j=1}^{M} d_j \chi_{B_j},$$

where we are assuming that each d_j is a complex number, the B_j's are pairwise disjoint sets of $\mathfrak{A}$, and the sets $\{A_r\}$, $r = 1, ..., N$, and $\{B_j\}$, $j = 1, ..., M$, are partitions of a set $M \in \mathfrak{A}$, that is, $M = \cup_{r=1}^{N} A_r = \cup_{j=1}^{M} B_j$. Suppose there is some $r' \leq N$ and some $j' \leq M$ such that $A_{r'} \cap B_{j'} \neq \emptyset$. Then, for $s \in A_{r'} \cap B_{j'}$, we must have that

$$f(s) = c_{r'} = d_{j'}.$$

It follows that

$$\begin{aligned} E(A_r) &= E\left(A_r \cap \left(\cup_{j=1}^{M} B_j\right)\right) \\ &= E\left(\cup_{j=1}^{M} A_r \cap B_j\right) \\ &= \sum_{j=1}^{M} E(A_r \cap B_j), \end{aligned}$$

where in the first equality we used the fact that the A_r's and B_j's are partitions of M. It is now straightforward to check that

$$\begin{aligned} \sum_{r=1}^{N} c_r E(A_r) &= \sum_{r=1}^{N}\sum_{j=1}^{M} c_r E(A_r \cap B_j) \\ &= \sum_{r=1}^{N}\sum_{j=1}^{M} d_j E(A_r \cap B_j) \\ &= \sum_{j=1}^{M} d_j E\left(\cup_{r=1}^{N} A_r \cap B_j\right) \\ &= \sum_{j=1}^{M} d_j E(B_j). \end{aligned}$$

That is, the resulting operator is independent of the representation of f. The next step is to extend this definition of spectral integral to bounded functions. Before we do this, we need the following technical result.

Lemma A.16. *If $f \in \mathcal{B}_s$, then $\|\mathbb{I}(f)\| \leq \|f\|_s$.*

Proof. Let f be a simple function expressed by the sum in (A.5). By definition, the A_r's are pairwise disjoint and, according to our Proposition A.11, $E(A_r)\mathcal{H}$ and $E(A_l)\mathcal{H}$ are orthogonal

for $r \neq l$. Then, for any $\psi \in \mathcal{H}$

$$\begin{aligned}\|\mathbb{I}(f)\psi\|^2 &= \left\|\sum_r^N c_r E(A_r)\psi\right\|^2 \\ &= \sum_r^N |c_r|^2 \|E(A_r)\psi\|^2 \\ &\leq \sum_r^N \|f\|_s^2 \|E(A_r)\psi\|^2 \\ &= \|f\|_s^2 \left\|\sum_r^N E(A_r)\psi\right\|^2 \\ &\leq \|f\|_s^2 \|\psi\|^2.\end{aligned}$$

This result implies the desired inequality of the operator norm of $\mathbb{I}(f)$. □

We are now able to extend our construction to bounded functions. The subspace $\mathcal{B}_s$ is dense in $(\mathcal{B}, \|.\|_s)$, such that for every element $f \in \mathcal{B}$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging to f in the supremum norm. This sequence is necessarily a Cauchy sequence and, by our last result, $(\mathbb{I}(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space of bounded operators with respect to the operator norm. This space is complete, which means that there exists an operator, which we denote by $\mathbb{I}(f)$, which is the limit of this sequence. This operator is the spectral integral of f with respect to the spectral measure E. Let us prove some properties of these integrals.

Proposition A.17. *Let $f, g \in \mathcal{B}(\Omega, \mathfrak{A})$, $\psi, \phi \in \mathcal{H}$, and $\alpha, \beta \in \mathbb{C}$. Then:*

1. $\mathbb{I}(\alpha f + \beta g) = \alpha\mathbb{I}(f) + \beta\mathbb{I}(g)$, $\mathbb{I}(\overline{f}) = \mathbb{I}(f)^*$, *and* $\mathbb{I}(fg) = \mathbb{I}(f)\mathbb{I}(g)$.
2. $\langle\psi, \mathbb{I}(f)\phi\rangle = \int_\Omega f(s)d\langle\psi, E(s)\phi\rangle$.
3. $\|\mathbb{I}(f)\psi\|^2 = \int_\Omega |f(s)|^2 d\langle\psi, E(s)\psi\rangle$.
4. $\|\mathbb{I}(f)\| \leq \|f\|_s$.

Proof. Let us start by showing that the map $\mathcal{B}(\Omega, \mathfrak{A}) \ni f \to \mathbb{I}(f) \in B(\mathcal{H})$ is continuous: let $\epsilon > 0$. We will show that for every ϵ there is a $\delta > 0$ for which $\|f - g\|_s < \delta$ implies $\|\mathbb{I}(f) - \mathbb{I}(g)\| < \epsilon$. Let f, g be the limits of the sequences of simple functions $(f_n)_{n\in\mathbb{N}}$ and

$(g_n)_{n\in\mathbb{N}}$, respectively. Then

$$\begin{aligned}\|\mathbb{I}(f)-\mathbb{I}(g)\| &= \|\lim_{n\to\infty}(\mathbb{I}(f_n)-\mathbb{I}(g_n))\| \\ &= \lim_{n\to\infty}\|\mathbb{I}(f_n-g_n)\| \\ &\leq \lim_{n\to\infty}\|f_n-g_n\|_s \\ &= \|f-g\|_s,\end{aligned}$$

where we used the continuity of the norm and the obvious fact that the map is linear for simple functions. Hence, $\|f-g\|_s < \delta \equiv \epsilon$ implies that $\|\mathbb{I}(f)-\mathbb{I}(g)\| < \epsilon$ and the map is continuous.

Since the above map is continuous, it is sufficient to prove the properties for simple functions. In the following, we consider simple functions of the form (A.5).

1. The linearity property is obvious from the definition. The adjoint can also be straightforwardly computed

$$\begin{aligned}\mathbb{I}(f)^* &= \left(\sum_r^N c_r E(A_r)\right)^* \\ &= \sum_r^N (c_r E(A_r))^* \\ &= \sum_r^N \overline{c_r} E(A_r) \\ &= \mathbb{I}(\overline{f}).\end{aligned}$$

   Define $g = \sum_s b_s \chi_{N_s}$, where the N_s's are pairwise disjoint. Let us prove that $\mathbb{I}(fg) = \mathbb{I}(f)\mathbb{I}(g)$. We have that $fg = \sum_{r,s} c_r b_s \chi_{A_r\cap N_s}$. Then

$$\begin{aligned}\mathbb{I}(fg) &= \sum_{r,s} c_r b_s E(A_r\cap N_s) \\ &= \sum_{r,s} c_r b_s E(A_r)E(N_s) \\ &= \mathbb{I}(f)\mathbb{I}(g),\end{aligned}$$

   where we used Proposition A.11.

2. Computing directly the inner product we have that

$$\langle \psi, \mathbb{I}(f)\phi \rangle = \sum_{r}^{N} c_r \langle \psi, E(A_r)\phi \rangle.$$

This sum becomes an integral when f is a measurable function with the complex measure $E_{\psi,\phi}(A) = \langle \psi, E(A)\phi \rangle$, for $A \in \mathfrak{A}$.

3. This property follows from item 2 and the fact that $\|\mathbb{I}(f)\psi\|^2 = \langle \mathbb{I}(f)\psi, \mathbb{I}(f)\psi \rangle$.

4. This property is a direct application of Lemma A.16 and the continuity of $\mathcal{B}(\Omega, \mathfrak{A}) \ni f \to \mathbb{I}(f) \in B(\mathcal{H})$.

□

We have constructed, up to now, the spectral integral of bounded, complex-valued, measurable functions. However, this machinery can be generalized to a wider class of functions. Let E be a spectral measure. Then, we can define the spectral integral, with respect to this spectral measure, of measurable, complex-valued functions which are E-finite almost-everywhere, that is, possibly unbounded functions such that $E(\{s \in \Omega | f(s) = \infty\}) = 0$ (the zero operator). Let us denote this set of functions by $S(\Omega, \mathfrak{A}, E)$, and note that $\mathcal{B}(\Omega, \mathfrak{A}) \subset S(\Omega, \mathfrak{A}, E)$. This construction, however, is much more involved, since the resulting operators will be possibly unbounded. For this reason, we will not give the details of this definition and we will limit ourselves to comment on some general results. The interested reader is referred to [57] for a complete description. In the following proposition, we resume the main properties of these spectral integrals.

Proposition A.18. *Let $f, g \in S(\Omega, \mathfrak{A}, E)$, $\psi \in \mathcal{D}(\mathbb{I}(f))$, $\phi \in \mathcal{D}(\mathbb{I}(g))$, and $\alpha, \beta \in \mathbb{C}$. Then*

1. *$\mathbb{I}(\overline{f}) = \mathbb{I}(f)^*$ and $\mathbb{I}(\alpha f + \beta g) = \overline{\alpha \mathbb{I}(f) + \beta \mathbb{I}(g)}$, where the overline denotes the closure of the operator.*

2. *$\langle \mathbb{I}(f)\psi, \mathbb{I}(g)\phi \rangle = \int_\Omega \overline{f(s)} g(s) d\langle \psi, E(s)\phi \rangle$.*

3. *Define the space $L^\infty(\Omega, E)$ of measurable functions on Ω that are bounded E-almost-everywhere, that is, they are bounded everywhere except on subsets of Ω for which the spectral measure associates the zero operator. Then, $\mathbb{I}(f)$ is bounded if, and only if, $f \in L^\infty(\Omega, E)$.*

4. The spectrum of $\mathbb{I}(f)$ is the essential range of f, that is,

$$\sigma(\mathbb{I}(f)) = \{\lambda \in \mathbb{C} | E(\{s \in \Omega : |f(s) - \lambda| < \epsilon\}) \neq 0 \text{ for all } \epsilon > 0\}.$$

Proof. See [57] for a proof. □

A.3.2 The Spectral Theorem for Self-Adjoint Operators

The spectral theorem for self-adjoint operators is among the most important results in Functional Analysis, and it lies at the bottom of the mathematical formulation of quantum mechanics. Our aim in this section is to present the different versions of this theorem, namely, the Spectral Theorem in the *spectral measure form*; in the *multiplication operator form* and in the *direct integral form*. For each version, we need to treat separately bounded and unbounded operators, since the statement and the proof of the theorem is different in each case. We will follow [58] closely in this section.

In the last section, we have shown how to construct linear operators acting on a Hilbert space when a spectral measure is defined in some σ-algebra. The first version of the Spectral Theorem that we present says that if the σ-algebra in question is the Borel σ-algebra of the real line, then there is a one-to-one correspondence between these spectral measures and self-adjoint operators.

Theorem A.19. *(Spectral Theorem for bounded self-adjoint operators- spectral measure form). Let A be a bounded, self-adjoint operator acting on a Hilbert space $\mathcal{H}$. Let $\mathcal{I} = [a, b]$ be a compact interval on $\mathbb{R}$ such that $\sigma(A) \subseteq \mathcal{I}$. Then, there exists a unique spectral measure E on the Borel σ-algebra $\mathcal{B}(\mathcal{I})$ such that*

$$A = \mathbb{I}(\lambda) \equiv \int_{\mathcal{I}} \lambda dE(\lambda),$$

where $\mathbb{I}(\lambda)$ denotes the spectral integral of the (bounded, real-valued) function $f(\lambda) = \lambda$ with respect to the spectral measure E.

Proof. See, for example, [57, 58, 114]. □

The idea of the proof is the following. If $p(\lambda)$ is a polynomial, we can define $p(A)$ in an obvious way and, by the density of the polynomials in $(C(\mathcal{I}), \|.\|_s)$, this definition can be extended to continuous, complex-valued functions in $\mathcal{I}$. Hence, if $f \in (C(\mathcal{I}), \|.\|_s)$, the functional $\langle ., f(A). \rangle : \mathcal{H} \to \mathbb{C}$ given by $\langle \psi, f(A)\psi \rangle$ is a continuous, positive linear functional

and, by the Riesz-Markov Theorem, there exists an unique measure $\mu_{\psi,A}$ on $\mathcal{B}(\mathcal{I})$ such that $\langle\psi, f(A)\psi\rangle = \int_{\mathcal{I}} f(\lambda)d\mu_{\psi,A}$. The next step is then to show that, for each $M \in \mathcal{B}(\mathcal{I})$, there exists an operator $E(M)$ such that $\mu_{\psi,A}(M) = \langle\psi, E(M)\psi\rangle$. This operator turns out to be an orthogonal projection and the collection of all these projections form a spectral measure E on $\mathcal{B}(\mathcal{I})$ such that, with respect to this spectral measure, $\mathbb{I}(\lambda) = A$. Furthermore, this spectral measure is unique.

If A is an unbounded operator, then its spectrum is not necessarily a bounded subset of $\mathbb{R}$ and we can not follow the same construction as above to associate a spectral measure to an unbounded, self-adjoint operator. Nonetheless, we can counter this obstacle in the following way. The map $t \to z_t = t(1+t^2)^{-1/2}$ is an homeomorphism from $\mathbb{R}$ on the interval $(-1, 1)$. Inspired by this fact, we can define the operator

$$Z_A = AC_A^{1/2},$$

where $C_A = (\mathbb{I} + A^*A)^{-1}$. It can be shown that Z_A is a bounded, self-adjoint operator with $\sigma(Z_A) \subseteq [-1, 1]$ (see [57]). Hence, by our Theorem A.19, there exists an unique spectral measure F on $[-1, 1]$ such that, with respect to F, $Z_A = \mathbb{I}(z)$, where $z \in [-1, 1]$. On this interval, define the function $\phi(z) = z(1-z^2)^{-1/2}$. This is a measurable, finite almost-everywhere function with respect to F. It can also be shown that, with respect to F, we have that $\mathbb{I}(\phi) = A$. Finally, we can associate to each $M \in \mathcal{B}(\mathbb{R})$ an operator given by

$$E(M) = F(\phi^{-1}(M)).$$

It turns out that this set of operators form a spectral measure in $\mathcal{B}(\mathbb{R})$ such that, with respect to E, $\mathbb{I}(\lambda) = A$, where $\lambda \in \mathbb{R}$. Furthermore, E is the unique spectral measure with this property. In summary, we have the following theorem.

Theorem A.20. *(Spectral Theorem for unbounded self-adjoint operators- spectral measure form). Let A be an unbounded, self-adjoint operator acting on a Hilbert space $\mathcal{H}$. Then, there exists a unique spectral measure E on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ such that*

$$A = \mathbb{I}(\lambda) \equiv \int_{\mathbb{R}} \lambda dE(\lambda),$$

where $\mathbb{I}(\lambda)$ denotes the spectral integral of the (unbounded, real-valued) function $f(\lambda) = \lambda$ with respect to the spectral measure E.

Proof. See, for example, [57, 58]. □

Our next goal is to present the multiplication operator form of the Spectral Theorem. But first, let us try to develop the intuition behind this theorem. Consider a separable Hilbert space $\mathcal{H}$, and let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis on this space. A linear operator A acting on $\mathcal{H}$ is called a *diagonal operator* (with respect to this basis) if $Ae_j = \alpha_j e_j$, where $\alpha_j \in \mathbb{C}$ for every j. Note that $\langle e_j, Ae_j\rangle = \alpha_j$, which implies that $A^*e_j = \overline{\alpha_j}$ and that A is normal. On the other hand, a normal operator whose eigenvectors span $\mathcal{H}$ is also a diagonal operator. Hence, this is a basis-independent manner of defining diagonal operators. It can be shown that if the sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ is bounded, then A is a bounded operator (see [116]). Since the e_j's form an orthonormal basis in $\mathcal{H}$, the operator A can be seen as a map from l^2 into itself: let $\psi = \sum_j \xi_j e_j$ be some vector in $\mathcal{H}$. Then, $A\psi \equiv \phi = \sum_j (\alpha_j\xi_j)e_j$ and A maps the sequence $(\xi_j)_{j\in\mathbb{N}}$ in the sequence $(\alpha_j\xi_j)_{j\in\mathbb{N}}$ in l^2.

It turns out that these diagonal operators are special cases of a more general construction. Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space, $\varphi : \Omega \to \mathbb{C}$ a bounded measurable function, and consider the Hilbert space $L^2(\Omega, \mu)$. We define the **multiplication operator with respect to** φ as the operator

$$(M_\varphi f)(s) = \varphi(s)f(s),$$

where $f \in L^2(\Omega, \mu)$. See section A.4 for properties of multiplication operators. Note that if Ω is the set of natural numbers and μ is the counting measure, then $L^2(\Omega, \mu) = l^2$ and the multiplication operators reduce to our diagonal operators. Therefore, the multiplication operators are a generalized version of the diagonal operators. Our next form of the Spectral Theorem asserts that every self-adjoint operator can be written in the form of a multiplication operator in a suitable Hilbert space.

Theorem A.21. *(Spectral Theorem for bounded self-adjoint operators- multiplication operator form). Let $A \in B(\mathcal{H})$ be a self-adjoint operator acting on a Hilbert space $\mathcal{H}$. Then, there exists a σ-finite measure space (Ω, μ), a bounded, measurable, real-valued function h on Ω, and a unitary map $U : \mathcal{H} \to L^2(\Omega, \mu)$ such that*

$$\left[UAU^{-1}\psi\right](\lambda) = h(\lambda)\psi(\lambda),$$

for all $\psi \in L^2(\Omega, \mu)$.

Proof. See, for example, [58, 111]. □

As we will see in the next section, a multiplication operator is bounded if, and only if, its

associated function is in $L^\infty(\Omega, \mu)$. Then, if the operator is unbounded, there can be no bounded function such that the above unitary equivalence holds. Despite this fact, the statement of the Spectral Theorem for unbounded operators is almost the same, except that we must be careful with the domain of the operator.

Theorem A.22. *(Spectral Theorem for unbounded self-adjoint operators- multiplication operator form). Let A be a self-adjoint operator acting on a Hilbert space $\mathcal{H}$. Then, there exists a σ-finite measure space (Ω, μ), a measurable, real-valued function h on Ω, and a unitary map $U : \mathcal{H} \to L^2(\Omega, \mu)$ such that*

$$U(\textit{Dom}(A)) = \left\{\psi \in L^2(\Omega, \mu) | h\psi \in L^2(\Omega, \mu)\right\},$$

and such that

$$\left[UAU^{-1}\psi\right](\lambda) = h(\lambda)\psi(\lambda),$$

for all $\psi \in U(\textit{Dom}(A))$.

Proof. See, for example, [58, 111]. □

In our above definition of a diagonal operator, the coefficients α_j are all eigenvalues of the diagonal operator. Hence, these operators can be seen as multiplication operators by the function $f(\lambda) = \lambda$, where $\lambda \in \sigma_p(A)$, the point spectrum of A. It is natural to ask if it is possible to refine further the multiplication operator form of the Spectral Theorem such that the function defining the multiplication operator is just $f(\lambda) = \lambda$, for λ in the spectrum of the operator. The answer is yes, and this is the content of our last version of the Spectral Theorem. However, before we state this theorem, we need to define the direct integral of Hilbert spaces.

Let $(\Omega, \mathfrak{A}, \mu)$ be a σ-finte measure space. Suppose that for each $\lambda \in \Omega$ we associate a separable Hilbert space $\mathcal{H}_\lambda$ with inner product $\langle , \rangle_\lambda$. We will define the direct integral of the $\mathcal{H}_\lambda$'s *with respect to the measure* μ, which will be also a separable Hilbert space. The elements of the direct integral will be called *sections*, $s(\lambda)$, and they will be functions from Ω with values in the union of the $\mathcal{H}_\lambda$'s, such that, for each λ, we have

$$s(\lambda) \in \mathcal{H}_\lambda.$$

The next step is to define the inner product and the norm in the resulting direct integral. Before we do this, we need some notion of measurability. We define a *simultaneous orthonormal basis* for the family of Hilbert spaces $\mathcal{H}_\lambda$'s as a collection of sections $\{e_j(.)\}_{j=1}^\infty$ where, for each

λ, we have that $\{e_j(\lambda)\}_{j=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}_\lambda$, in the sense that $\langle e_j(\lambda), e_k(\lambda)\rangle = 0$ for $j \neq k$, the norm of each e_j is either 0 or 1, and the closure of the span varying over all j's is equal to $\mathcal{H}_\lambda$. If the function $\lambda \to \dim\mathcal{H}_\lambda$ is a measurable function into $[0, \infty]$, then we can choose an orthonormal basis such that $\langle e_j(\lambda), e_k(\lambda)\rangle$ is measurable for all j and k. Then, we say that a *section s is measurable* if the function

$$\lambda \to \langle e_j(\lambda), s(\lambda)\rangle_\lambda$$

is a complex-valued measurable function for each j. Such a choice of simultaneous orthonormal basis is called a *measurability structure* on the collection of Hilbert spaces $\mathcal{H}_\lambda$. We are now ready to define the direct integral of this collection.

Definition A.23. Let $(\Omega, \mathfrak{A}, \mu)$ be a σ-finite measure space;$\{\mathcal{H}_\lambda\}$, for $\lambda \in \Omega$, a collection of separable Hilbert spaces for which $\lambda \to \dim\mathcal{H}_\lambda$ is measurable; and $\{e_j(.)\}_{j=1}^{\infty}$ a measurability structure. Then, we define the **direct integral** of the $\mathcal{H}_\lambda$'s with respect to μ, which we denote by

$$\int_\Omega^\oplus \mathcal{H}_\lambda d\mu(\lambda),$$

as the space of equivalence classes of almost-everywhere-equal measurable sections for which

$$\|s\|^2 \equiv \int_\Omega \langle s(\lambda), s(\lambda)\rangle_\lambda d\mu(\lambda) < \infty.$$

The inner product of two sections s_1 and s_2 is defined as

$$\langle s_1, s_2\rangle \equiv \int_\Omega \langle s_1(\lambda), s_2(\lambda)\rangle_\lambda d\mu(\lambda).$$

Example A.24. Let $(\Omega, \mathfrak{A}, \mu)$ be a σ-finite measure space, and suppose that to each $\lambda \in \Omega$ we associate the Hilbert space $\mathbb{C}$. Then, each $\mathcal{H}_\lambda$ is finite-dimensional and a measurability structure can be easily defined. The resulting direct integral is clearly

$$\int_\Omega^\oplus \mathcal{H}_\lambda d\mu(\lambda) = L^2(\Omega, \mu).$$

With these definitions at hand, we can now formulate the last version of the Spectral Theorem.

Theorem A.25. *(Spectral Theorem for bounded self-adjoint operators- direct integral form). Let $A \in B(\mathcal{H})$ be a self-adjoint operator. Then, there exists a σ-finite measure μ on $\sigma(A)$, a*

direct integral

$$\int_{\Omega}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda),$$

and a unitary map $U : \mathcal{H} \to \int_{\Omega}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$ *such that*

$$[UAU^{-1}s](\lambda) = \lambda s(\lambda), \quad \lambda \in \sigma(A),$$

for all sections in the direct integral.

Proof. See, for example, [58, 118]. □

Hence, this theorem guarantees that for each bounded, self-adjoint operator there exists a unitarily equivalent Hilbert space such that this operator acts as multiplication by the elements of the spectrum. For this reason, this theorem is often considered as the most refined version of the Spectral Theorem for a single operator. Then, except from a unitary map, the operator A acts as $\lambda\mathbb{I}$ on each $\mathcal{H}_\lambda$, and these are often considered as "generalized eigenspaces" of A. These are not necessarily real eigenspaces since the spaces $\mathcal{H}_\lambda$ are not, in general, subspaces of the direct integral. However, if $\lambda_0 \in \sigma(A)$ is an eigenvalue and $\mathcal{H}_{\lambda_0} \subset \mathcal{H}$ its eigenspace, we can embed it isometrically into the direct integral by defining the section

$$s(\lambda) = \begin{cases} \frac{1}{\sqrt{c}}\psi, \text{ if } \lambda = \lambda_0 \\ 0, \text{ if } \lambda \neq \lambda_0, \end{cases}$$

for each $\psi \in \mathcal{H}_{\lambda_0}$, and where $c \equiv \mu(\{\lambda_0\})$ (which will be nonzero, necessarily). In this case, $\mathcal{H}_{\lambda_0}$ is a true eigensubspace of the direct integral.

The statement of this version of the Spectral Theorem for unbounded, self-adjoint operators is almost the same, with the difference that we must be careful with the domains of the operators.

Theorem A.26. *(Spectral Theorem for unbounded self-adjoint operators- direct integral form). Let* A *be a self-adjoint operator on* $\mathcal{H}$*. Then, there exists a* σ*-finite measure* μ *on* $\sigma(A)$*, a direct integral*

$$\int_{\Omega}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda),$$

and a unitary map $U : \mathcal{H} \to \int_\Omega^\oplus \mathcal{H}_\lambda d\mu(\lambda)$ *such that*

$$U(\textit{Dom}(A)) = \left\{ s \in \int_\Omega^\oplus \mathcal{H}_\lambda d\mu(\lambda) \Big| \int_{\sigma(A)} \|\lambda s(\lambda)\|_\lambda^2 d\mu(\lambda) < \infty \right\},$$

and such that

$$[UAU^{-1}s](\lambda) = \lambda s(\lambda), \quad \lambda \in \sigma(A),$$

for all sections in $U(\textit{Dom}(A))$.

Proof. See, for example, [58, 118]. □

A.4 Multiplication Operators

The Spectral Theorem in the multiplication operator form presented above shows that the class of multiplication operators is highly relevant in Functional Analysis since any self-adjoint operator can be put in this form for a suitable Hilbert space. In addition, the position operators belongs to this class. Therefore, we dedicate this section to rigorously defining these operators and exploring some of their most basic and general properties.

Let $(\Omega, \mathfrak{A}, \mu)$ be a σ-finite measure space and $\phi : \Omega \to \mathbb{C} \cup \{\infty\}$ be a μ-a.e finite, measurable function. From this measure space, we construct the function space $L^2(\Omega, \mu)$. The **multiplication operator** M_ϕ, acting on this function space, is defined as

$$(M_\phi g)(s) \equiv \phi(s)g(s), \quad s \in \Omega \tag{A.7}$$

$$D(M_\phi) = \left\{ g(s) \in L^2(\Omega, \mu) \Big| \int_\Omega |\phi(s)|^2 |g(s)|^2 d\mu(s) < \infty \right\}. \tag{A.8}$$

The following result characterizes several important properties of these operators.

Lemma A.27. *The above defined multiplication operator has the following properties:*

1. *The adjoint of* M_ϕ *is given by* $(M_\phi)^* = M_{\overline{\phi}}$. *In particular, the multiplication operator is self-adjoint if, and only if,* ϕ *is a real function.*

2. *The operator* M_ϕ *is bounded if, and only if,* $\phi \in L^\infty(\Omega, \mu)$.

3. *The spectrum of the multiplication operator is given by*

$$\sigma(M_\phi) = sp(\phi), \tag{A.9}$$

where $sp(\phi)$ denotes the essential range of ϕ. In particular, the spectrum will equal the closure of the image of ϕ when the pre-images of any open set in the range of ϕ has non-zero measure.

4. *Let ϕ be a real function. The spectral measure defined on $\sigma(M_\phi)$, which is uniquely associated with M_ϕ by Theorem $A.19$ or A.20, is given by*

$$(P_{M_\phi}(B)g)(s) = \chi_{\phi^{-1}(B)}(s)g(s), \tag{A.10}$$

where $g \in L^2(\Omega, \mu)$ and $B \in \mathcal{B}(\sigma(M_\phi))$.

Proof. See [57] and [119]. □

Before we finish this chapter with mathematical preliminaries, let us prove the following result, which characterizes all the bounded multiplication operators acting on a general space $L^2(\Omega, \mu)$ in terms of a unique spectral measure.

Proposition A.28. *Let $(\Omega, \mathfrak{A}, \mu)$ be a σ-finite measure space and define the Hilbert space $\mathcal{H} = L^2(\Omega, \mu)$. Let E be the spectral measure defined in* (A.4)*, which acts as*

$$(E(A)\psi)(s) = \chi_A(s)\psi(s),$$

for $A \in \mathfrak{A}$ and $\psi \in \mathcal{H}$. Let $\mathcal{B}(\Omega, \mathfrak{A})$ denote the space of all bounded, measurable, complex-valued functions on Ω. Define the following subsets of $B(\mathcal{H})$

$$\mathfrak{I} \equiv \{\mathbb{I}(f) | f \in \mathcal{B}(\Omega, \mathfrak{A})\}$$
$$\mathcal{M} \equiv \{M_f | f \in \mathcal{B}(\Omega, \mathfrak{A})\},$$

where $\mathbb{I}(.)$ denotes the spectral integral with respect to the spectral measure E. Then, there exists a one-to-one correspondence between $\mathfrak{I}$ and $\mathcal{M}$ given by

$$\mathbb{I}(f) = M_f.$$

Proof. For any $f \in \mathcal{B}(\Omega, \mathfrak{A})$ we have that

$$\langle \psi, \mathbb{I}(f)\psi \rangle = \int_\Omega f(s) d\langle \psi, E(s)\psi \rangle,$$

and

$$\langle \psi, M_f \psi \rangle = \int_\Omega f(s)|\psi(s)|^2 d\mu(s),$$

where in the first equation we used proposition A.17. Note that

$$\nu_\psi(A) \equiv \int_A |\psi(s)|^2 d\mu(s)$$

defines a measure on $\mathfrak{A}$. By the other hand, we have

$$E_\psi(A) = \langle \psi, E(A)\psi \rangle = \int_\Omega \chi_A(s)|\psi(s)|^2 d\mu(s).$$

That is, $\nu_\psi(A) = E_\psi(A)$ for every $A \in \mathfrak{A}$. Hence, it follows that

$$\begin{aligned} \langle \psi, M_f \psi \rangle &= \int_\Omega f(s) d\nu_\psi(s) \\ &= \int_\Omega f(s) d\langle \psi, E(s)\psi \rangle \\ &= \langle \psi, \mathbb{I}(f)\psi \rangle, \end{aligned}$$

and therefore

$$\langle \psi, (M_f - \mathbb{I}(f)\psi) \rangle = 0, \quad \forall \psi \in \mathcal{H}.$$

Since this last relation is valid for every $\psi \in \mathcal{H}$, then $\mathbb{I}(f) = M_f$. □

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