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Teoria de pertubação causal, produto estrela e amplitudes de espalhamento

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Causal perturbation theory, star products and scattering amplitudes

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Since all the members of that list speak Portuguese as their mother language , the dedication will be written in Portuguese.

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"This series is divergent; therefore we may be able to do something useful with it."

Oliver Heaviside

Abstract

Motivated by the lack of communication between the mathematical and the theoretical (especially high-energy theory) physics community, we develop the formalism of causal perturbation theory in a language adapted for high energy physics students perusing a better knowledge in mathematical physics. We start with a historical review to explain the necessity of a "new formulation" and proceed to some mathematical preliminaries. The quantization of the system is done by deformation quantization, and its relation with the "usual" Fock space quantization is briefly discussed. We proceed with the classical theory of fields and the classical retarded product. The classical theory is then used to impose axioms on the quantum retarded product, discussed in the next chapter. After that, we introduce the famous S- matrix and the recipe to compute its renormalization (in position space). The very end is devoted to a topic not present in the literature. We discuss a series of scattering amplitudes that are widely used in quantum field theory courses but are not explored in the mathematical physics community.

Keywords: quantum field theory; perturbation theory; causal perturbation theory; deformation quantization; scattering.

Resumo

Motivados pela falta de comunicação entre a comunidade da física matemática e a comunidade de físicos teóricos (principalmente de altas energias), desenvolvemos o formalismo da teoria de perturbação causal em uma linguagem acessível aos estudantes de física de altas energias que desejam obter um conhecimento mais sólido em física matemática. Começamos discutindo brevemente o desenvolvimento histórico que levou a formulação da teoria de perturbação causal justificando sua necessidade. Em seguida, introduzimos alguns preliminares matemáticos. A quantização do sistema é feito usando quantização por deformação e discutimos a relação entre este método e o "método usual" de quantização no espaço de Fock. Então prosseguimos para a teoria clássica de campos e a expansão retardadas de campos. Esta discussão é a base do homólogo quântico, tratado em sequência. Feito isso, discutimos a famosa matrix S- e a renormalização da mesma (feita no espaço de configurações). A parte final do trabalho é dedicada a um tema inédito na literatura: estudar diversas amplitudes de espalhamento usadas amplamente em cursos de teoria quântica de campos mas omitidos dentro da literatura da física matemática.

Palavras-chave: teoria quântica de campos; teoria de perturbação; teoria de perturbação causal; quantização por deformação; espalhamento.

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Chapter 1

Introduction

Before presenting the work, there are some questions that deserve to be answered. Every academic must make it clear:

- What is the work about.
- What are the necessary tools to understand the work.
- What is the relevance of the work.
- Who is the target audience

The goal of this section is to answer, at least partially, the above topics.

With regard to the content of the work, it is basically a different approach to perturbation theory in quantum field theory (QFT). Most of the content here was written based on the wonderful book by M. Dütsch [24]. More concerning the first question will be explained in the historical introduction. The reader of the present work is expected to have some knowledge of QFT.

The last two topics were the ones that gave a direction to the project and therefore deserve a better explanation.

The progress in science is not linear, and there is more than one way of thinking and expressing science, for instance, theoretical physics, mathematics, and mathematical physics. For this reason, it is not rare to find two different approaches to the same problem that cannot communicate with each other because the ways of attacking the obstacles that appear are too different. In mathematical physics, this problem is blatant. For example, the theme of this thesis (perturbation theory in quantum field theory) is part of a regular course in a graduate program in physics (sometimes even an undergraduate course). Nevertheless, most of the books that teach these methods (and consequently the scientists who use these methods in everyday life) very often have the basis of the theory fixed in "ideas that apparently make sense" (such as the path integral, whose existence is not trivial [53]) and do not even mention the existence of a more rigorous approach. When faced with other approaches that "solves problem" by fixing the theory on a solid mathematical basis, a common reaction of several members of this community is of surprise [31] but at the same time aversion to the theme. They say it is "interesting" but "way too distant" from the "standard approach". We believe that the problem is basically how one communicated with the rest of the community.

With this problem in mind, we were faced with a "dilemma": we could either take the standard approach in mathematical physics, using definitions, theorems, lemmas and so on or we could take the "physics" approach and distance ourselves from the rigor of mathematical text hoping to expand the set of possible readers. To solve the problem, we recall two sentences from Prof. Jan von Delft [21]:

1. *''The style of theses is different than for papers – your typical target audience are your fellow diplom- or PhD-students (and Professors from a different field that have no clue about yours!). They should be able to learn, if they so wish, from your thesis precisely what you did and how you did it."*

2. *"Often, your thesis will be the starting point for the next person joining the group to 'learn the basics' and the 'tricks of the trade'. Therefore, somewhat more details than in a paper, and an attempt at pedagogical presentation, are appropriate."*

With this philosophy in mind, we decided to take the second path. The book from which most of the present dissertation is based [24] already does that, but we still think it is too mathematical for a physicist. The way in which we tried to write the thesis is as close as possible to a standard book in quantum field theory, omitting, when possible, technicalities and focusing on the calculations and the relations between different approaches. Our biggest fear in doing so (and honestly, the probability of it concretizing itself is considerable) is to combine the worst of both worlds: Writing a text that is too technical for a physicist and not rigorous enough for a mathematician. We hope that this fear is just a delusion from an inexperienced writer and that the material can be useful for those who are trying to understand how perturbative quantum field theory is done in this particular way.

Chapter 2

Historical review

This section is inspired by the articles [9, 34].

The story of causal perturbation theory began about a century ago. In 1929, Pauli and Heisenberg introduced a theory of quantum electrodynamics [44] that was very similar to that used in contemporary physics. In this paper, the quantization is done by using equal-time commutation relations and relevant quantities were computed using a perturbative expansion in the coupling constant, just as it is usually taught in a standard quantum field theory course. The problem back then was that the theory was not manifestly Lorentz invariant, since it distinguished between time and space arguments. This problem could be solved using very sophisticated arguments, but the theory was still unsatisfactory due to another recurrent problem in quantum field theory: the divergent integrals.

The physics community at the time believed that the problems of infinities and covariance were deeply connected. Oppenheimer stated from this point of view in 1948 Solvay conference ([34],page4):

"One needs a covariant way to identify these [divergent] term; and for that, not merely the field equations themselves, but the whole method of approximation and solution must at all stages preserve covariance."

To get around the problem, Heisenberg proposed in the first half of the 1940s a completely new setting [41, 42, 43]. Heisenberg's approach was to avoid differential time evolution and instead use an operator (S-matrix) that maps asymptotic states at $t = -\infty$ to asymptotic states at $t = \infty$. The idea behind the S-matrix program was to impose restrictions directly on the $S-$matrix, avoiding, in this way, the need for a quantization procedure in the fields. Heisenberg imposed unitarity $SS^* = 1$ and Lorentz invariance $U_\Lambda S U_\Lambda^{-1} = S$ for the S matrix, but that was not enough to compute desired quantities.

During the same period as Heisenberg developed his theory of the $S-$ matrix, Feynman, Schwinger, Tomonaga, and Dyson insisted on the theory first introduced by Pauli and Heisenberg. The merit of the approach pursued by those authors was to fix the problem of infinities without changing the theoretical basis presented in the work of Pauli and Heisenberg. Within these frameworks, the $S-$ matrix proposed by Heisenberg could be derived, and it did not need to be imposed. This new approach was also not completely satisfactory since it introduces too much new structure to obtain the desired result.

The new procedure was based on the so-called interaction picture. The idea is to separate the time evolution into an hamiltonian representing the interacting part H_I and a free part H_0 (usually thought of as the state of the free particles). In this context, the Schrödinger equation was replaced by the Schwinger-Tomonaga equation.

$$i\frac{\delta\psi(\sigma)}{\delta\sigma(x)} = \mathcal{H}_I(x)\psi(\sigma) \tag{2.1}$$

where σ is a space-like cauchy surface containing the point x. The expression above is invariant under Lorentz transformations, and the perturbative expansion can be found by integrating the equation. The integration procedure automatically leads to the time-ordered product. In addition, the microcausality condition, also known as Einstein-Causality that is, $[\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0$ for space-like points, was imposed to make sense of the integration in the perturbative series. By doing so, one obtains the famous expression for the S matrix in terms of the Dyson series:

$$\begin{aligned} S(H_I) &= \sum_{i=1}^{n} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 ... \int_{-\infty}^{\infty} dt_n T(H_I(t_1)...H_I(t_n)) \\ T(H_I(t_1)...H_I(t_n)) &= \sum_{\pi\in S^n} H_I(t_{\pi(1)})...H_I(t_{\pi(n)})\theta(t_{\pi(1)} - t_{\pi(2)})...\theta(t_{\pi(n-1)} - t_{\pi(n)}) \end{aligned} \tag{2.2}$$

To tackle the infinities appearing in the expressions, the idea was to introduce counter-terms: a new set of (finite) parameters as mass and coupling constants together with (infinite) terms that cancel the infinities coming from the integrals. This recipe was incredibly successful in describing quantum electrodynamics, leading to the Nobel Prize for Feynman, Tomonaga, and Schwinger in 1965.

In spite of the empirical success, that was not the last word regarding the perturbative theory of quantum fields. One of the problems was the *"dubious mathematical rigor of the renormalization procedure"*. The manipulation of infinities just as a usual number and the addition of another infinity to cancel it to get a finite answer is not very reasonable. Another problem of the theory was made explicit by R.Haag in 1955 [39]: The interaction picture used to derive the formula does not exist.

The aforementioned facts brought renewed interest to Heisenberg's $S-$matrix program. During the period that Dyson, Feynman, Tomonaga and Schwinger were working in their perturbative program, Stückelberg continued the Heisenberg philosophy of $S-$matrix. He added the causality axiom and worked to construct a perturbative theory of the $S-$matrix. The biggest problem in this approach was how to identify and eliminate acausal terms. Stückelberg published a solution in 1948 [66]. The solution consisted in contracting field operators to obtain distributions in such a way that only creation or annihilation operators act separately on the vacuum. Although Stückelberg's theory could have replaced Dyson, Feynman, Tomonaga and Schwinger approach, it remained mostly unknown to the physical community at the time. According to A. Blum [9] page 17, this can be explained because "*From a calculational perspective, his formalism compared unfavorably to the compact, user-friendly, Feynman rules popularized by Dyson. The conceptual foundations of this earliest version of causal perturbation theory were also rather murky*".

In order to convince the physics community that his method was the best shot, Stückelberg explored another type of divergence in the theory of Dyson, Feynman, Tomonaga and Schwinger [74]. Instead of computing scattering states at $t = -\infty$ and $t = \infty$, Stückelberg tried to compute finite-time scatterings. He multiplied the interaction hamiltoninan by a function $g(t)$ that "switches on" the interaction at $t_i \neq -\infty$ and turn it off at $t_f \neq \infty$. The result he obtained is a new sort of divergence, called "boundary divergence". To cure the theory, one needs to consider only a class of functions that turns the interaction on and off smoothly. Stückelberg claimed that these "smooth functions" contradicted the equations of motion and, therefore, this approach should be discarded with his approach being adopted instead. Despite the fact that Stückelberg's argument is right, his work remained unknown to the western community. However it was appreciated in the Soviet Union by a very famous physicist Nicolay Bogouliubov.

In 1951 Bogoliubov published his first papers on the subject. He did not believe that the smooth function that switched on the interaction was incompatible with the equations of motion. He formulated an analog formula for the $S-$ matrix depending on those functions $S(g)$. He ended up giving up the attempt to unite the formalism using Dyson series with the smooth functions not because it was impossible to do it but because, diving deeper in the work of Stückelberg, he understood causality was more fundamental than the equations of motion. The functions g became more than a switch button and became a structural part of the theory. Using these functions, Bogouliobov imposed causality directly on the $S-$ matrix, recovering the original idea of Stückelberg with more clarity and mathematical rigor. Bogouliobv imposed that given two sets F, G and two smooth functions f, g with $f(x) \neq 0$ if and only if $x \in F$ and $g(x) \neq 0$ if and only if $x \in G$ and all points in G are in the causal future of F. Then, we define the S-matrix of $g(x)$ as $S(g) \equiv S(gH_I)$ and it holds:

$$S(g_1 + g_2) = S(g_2)S(g_1) \tag{2.3}$$

where the expression of S is given by 2.2. This formulation is the one still used today. The importance of the formulation above is that using only Lorentz invariance, unitarity, and the causality condition, one can reconstruct the Dyson series inductively using it (see [9] page 24 for details).

Once the problem with causality was solved, the problems with infinity remained to be clarified. In order to fully understand the problem, a new mathematical language needed to be adopted.

Fortunately for those developing the theory at the time, parallel to the development of causal perturbation theory, the theory of distributions was also being developed. Roughly speaking, the problem of infinities can be seen as ambiguities in the coinciding points of Bogouliubov's S- matrix construction, as pointed out by Stückelberg in 1951 [73]. That non-uniqueness is due to the multiplication of singular distributions whose pointwise multiplication is not always well defined (for example, the multiplication $\delta(x) \cdot \delta(x)$ is not well defined). The concept of distributions was so influential in the development of quantum field theory that even non-perturbative approaches define fields as operator-valued distributions [76]. This feature was already present in the work of Schwinger-Tomonaga, whose correction was done by subtracting infinite quantities. Since Bogouliubov imposed causality as an axiom and not a consequence of construction, the procedure of removing the singularities could be made more carefully. Hence, as mentioned on [9] page 32: *"Renormalization was now a mathematically recast as a problem of determining the extension of this [time-order] product to the full space of test functions,i.e., to switching functions which are nonzero at coincident space-time points"*.

The problem now was the non-uniqueness of the extension. After all, if one obtains more than one result for something that can be measured in the laboratory, how can one distinguish between what is right and what is wrong? The problem was solved by Stückelbert and Petermann in 1953 with the introduction of the renormalization group [58]. They showed that the ambiguities correspond to different but equivalent definitions of the expansion parameter and Bogouliubov showed in 1957 that the ambiguities are exactly the ones found in the "usual perturbative approach".

Although the work of Bogouliobov and Stückelbert was much more careful mathematically speaking than the ones of Feynman, for example, they did not achieve the necessary high standard of mathematical rigor. It was only in 1973, with the seminal work of Epstein and Glaser [29], that causal perturbation theory was made mathematically clean.

After the work of Epstein and Glaser, other approaches and new refinements were made without losing the main idea of using causality to inductively derive the expansion of the S- matrix. One of the approaches, which will be worked out throughout the next chapters, is connecting algebraic quantum field theory and perturbative causality in what is called **perturbative algebraic quantum field theory**.

This new approach was developed in the beginning of the XXI century mostly by Romeo

Brunetti, Chris Fewster, Klaus Fredenhagen, Stefan Hollands, Benard Kay and Reiner Verch ([34]page 19). The idea was to use the existing formalism of quantum field theory in curved space-times as the starting point to construct perturbative schemes. More than “rediscovering” known results in the perturbative approach, the results obtained through this new formalism are often the starting point to (attempt to) construct non-perturbative quantum field theories [15] as well as perturbative models in curved space-times [13, 65].

We hope that with this small historical introduction we are able to convince the reader that perturbation theory is not as simple as it looks like, that it took many years for some of the most famous physicists to develop it and that there still is much work to be done regarding it. Although we have explained how one can deal with ultraviolet-divergencies, that is, divergences appearing when the momentum goes to infinity, there are other problems with the perturbative approach that were not mentioned here, mostly because they do not drive as much attention (yet they should!). As an example, we would like to mention only two: the first one is the infrared divergence, related to the adiabatic limit $g \to 1$ (i.e. the interaction is non-longer local). Not much has been done to solve that problem, but as a counterexample, we cite the work of Duch [23] and references therein. Another type of infinity that plagued the perturbative expansion is regarding the convergence of the series as a whole. That problem was already mentioned by Dyson in 1952 [28]. To the best of our knowledge, there is only one work that shows that the series is divergent in a very limited scenario [49]. The general result is not known, but it is widely believed that it diverges. A small discussion on that topic is given in [31]. Fortunately for us, apparently the series is meaningful until the term of order $\frac{1}{e}$, where e is the coupling constant (for QED, for example, it means that we can compute ≈ 137 terms until the convergence starts to be a problem, far beyond what is computable in practice).

Chapter 3

Propagators and conventions

The first section is a quick review of the underling structure of the theory. It is most written to fix the notation and conventions. We also decided to present the propagators in this section. The motivation of each of them will be explained briefly and fully deduced in the right context. For the physics student reading the section and thinking that it contradicts what has been said in the introduction: That is the only section that has the style of just defining things. The later chapters are more fluid.

3.1 Minkowski space-time

The physical space on which we will be working during the project, except when explicitly stated the opposite, is the Minkowski space-time in $d-$ dimensions denoted by $\mathbb{M}$. A vector $x \in \mathbb{M}$ is denoted by $x := (x^0, \vec{x})$. The sign convention is $g \equiv \eta = \mathrm{diag}(1, -1, -1, ..., -1)$. The inner product in Minkowski space-time is simply written as $px \equiv p_\mu x^\mu = p^0 x^0 - \vec{x} \cdot \vec{p}$. We define the **forward and backward light cones** as:

$$V_+ := \{x \in \mathbb{M} | x^2 > 0, x^0 > 0\} \quad V_- := \{x \in \mathbb{M} | x^2 > 0, x^0 < 0\}. \tag{3.1}$$

The sets $\overline{V}_\pm$ are the closures of the forward and backward light cones. The **thin diagonal** is defined as

$$\Delta_n := \{(x_1, ..., x_n) \in \mathbb{M}^n | x_1 = x_2 = ... = x_n\}. \tag{3.2}$$

In the present work, we use natural units:

$$\hbar = c = 1. \tag{3.3}$$

However, we introduce a parameter $\hbar$ later in the text. This parameter is not to be understood as the Planck constant. Instead, we fix $\hbar \in \{0,1\}$. The "quantum world" is obtained when $\hbar = 1$ and the "classical world" $\hbar = 0$[1].

3.2 Formal power series and notations

As usual in perturbative approaches, most of the measured quantities are written as formal power series in some constant λ. For $\mathcal{V}$ a $\mathbb{C}-$ vector space and $\lambda \in \mathbb{R}$, we define the **set of formal power series in** λ as:

$$\mathcal{V}[\![\lambda]\!] := \left\{ V \equiv \sum_{n=0}^{\infty} \lambda^n V_n \equiv (V_n)_{n\in\mathbb{N}} | V_n \in \mathcal{V} \right\}. \tag{3.4}$$

In general, in the cases studied in quantum field theory, the convergence of the series is not under control. A notation that will be widely used later in the text is:

$$\begin{aligned} e_{\otimes}^{V} :=& 1 \oplus \bigoplus_{n=1}^{\infty} \frac{V^{\otimes n}}{n!} \\ f(e_{\otimes}^{\lambda V}) :=& f_0(1) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} f_n(V_n), \end{aligned} \tag{3.5}$$

where $f_n : \mathcal{V}^{\otimes n} \to \mathcal{W}$ a vector space is linear. We write the argument of f_n as

$$f_n(v_1 \otimes v_2 \otimes ... \otimes v_n) \equiv f_n(v_1, ..., v_n). \tag{3.6}$$

We postpone the definition for another moment.

3.3 Multi-index notation

We will use a shorthand notation for higher derivatives. For $a := (a_1, ..., a_n) \in \mathbb{N}^n$ and $x := (x_1, ..., x_n) \in \mathbb{R}^n$ we define:

$$|a| := a_1 + ... + a_n \quad a! := a_1!...a_n! \quad x^a := x_1^{a_1}...x_n^{a_n} \quad \partial_x^a := \partial_{x_1}^{a_1}...\partial_{x_n}^{a_n}. \tag{3.7}$$

In the case of Minkowski spacetime, the notation is analogous $a := (a_1^{\mu}, ..., a_n^{\mu}) \in (\mathbb{N}^d)^n$:

[1] In the text we usually write $\hbar \to 0$ or $\hbar \to 1$ instead of $\hbar = 0, 1$. The idea is that we "take the classical / quantum limit", but since we have not specified the topology, the limit should be understood as substituting $\hbar = 0, 1$

$$|a| := \sum_{j=1}^{n}\sum_{\mu=0}^{d-1} a_j^\mu \quad a! := \prod_{j=1}^{n}\prod_{\mu=0}^{d-1} a_j^\mu! \quad x^a := \prod_{j=1}^{n}\prod_{\mu=0}^{d-1} (x_j^\mu)^{a_j^\mu}$$

$$\partial^a f(x) := \left(\prod_{j=1}^{n}\prod_{\mu=0}^{d-1} (\partial_{x_j^\mu})^{a_j^\mu}\right) f(x). \tag{3.8}$$

We also use a different notation for the d'Alembertian operator:

$$\partial^2 \equiv \Box \equiv \partial_\mu \partial^\mu = \partial_t^2 - (\vec{\nabla})^2. \tag{3.9}$$

3.4 Distributions

Distributions are probably the first topic that is not properly studied during the bachelor's degree in physics, but it is widely used. We will follow mostly [2] (in Portuguese). A classical reference to the subject is [64] and [47]. Although not very common as reference, there is also a very pedagogical playlist on *YouTube* by Stanford University [69].

We start with the definition of the **set of smooth functions** $f : \Omega \subseteq \mathbb{M} \to \mathbb{R}$ (that is, continuous infinity and differentiable functions). The set is denoted by:

$$\mathcal{C} := C^\infty(\mathbb{M}, \mathbb{R}) \equiv C^\infty(\mathbb{R}^d, \mathbb{R}). \tag{3.10}$$

The **support of a function** is defined as the set:

$$\mathrm{supp}(f) := \overline{\{x \in \Omega | f(x) \neq 0\}}. \tag{3.11}$$

We denote the set of all smooth functions with compact support by $\mathcal{D}(\Omega)$. If $f \in \mathcal{D}(\Omega)$, we say that f is a **test function**.

As an example of test function $f \in \mathcal{D}'(\mathbb{R})$, consider:

$$f(x) := \begin{cases} \exp\left(-\frac{1}{(x+5)^2} - \frac{1}{(x-5)^2}\right) & |x| < 5 \\ 0 & |x| \geq 5 \end{cases} \tag{3.12}$$

.We can plot this function:

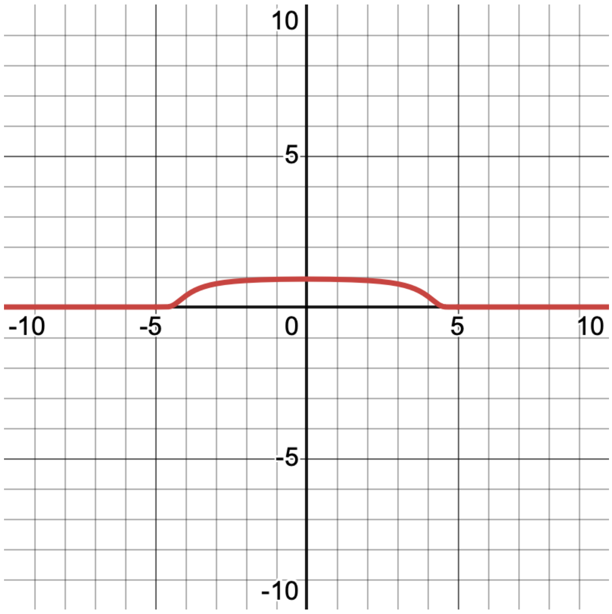


Figure 3.1: Test function: a smooth function with compact support

We say $t : \mathcal{D}(\Omega) \to \mathbb{R}$, t linear is a **distribution** if t is continuous (the notion of continuity for linear functionals can be found in the Appendix). The set of all continuous linear functions from $\mathcal{D}(\Omega) \to \mathbb{R}$ is denoted by $\mathcal{D}'(\Omega)$. The application of $t \in \mathcal{D}'(\Omega)$ to a test function $g \in \mathcal{D}(\Omega)$ is denoted by:

$$t(g) \equiv \langle t, g \rangle \equiv \langle t(x), g(x) \rangle_x \equiv \int_\Omega d^{dn}x\, t(x)g(x). \tag{3.13}$$

When working with many variables, we use the notation:

$$dx_1 ... dx_n \equiv dX_n. \tag{3.14}$$

When the domain of integration is not written, it is implicit that the integral is over the entire space. The support of a functional $t \in \mathcal{D}'(\Omega)$ is defined as the smallest closed subset $K \subseteq \Omega$ such that $t|_{\mathcal{D}(\Omega \setminus K)} = 0$. The support of t is also defined as supp(t). Usually we "abuse the notation" and write $t(x) = 0$ to denote $t|_{\mathcal{D}(\Omega)} = 0$.

Remark: The notation of an integral is only symbolic and must not always be understood as an integration over a certain domain.

3.5 Examples of distributions and propagators

Some distributions that every physics student has used are $\delta(x) \in \mathcal{D}'(\mathbb{R})$ and $\theta(x) \in \mathcal{D}(\mathbb{R})$. They are defined simply by:

$$\langle\delta(x), f(x)\rangle \equiv \int dx\, \delta(x) f(x) = f(0)$$
$$\langle\theta(x), f(x)\rangle \equiv \int dx\, \theta(x) f(x) = \int_0^\infty dx\, f(x). \tag{3.15}$$

Remark: we can write $\theta(x)$ as a function:

$$\theta(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{3.16}$$

.An important theorem states that if $t \in \mathcal{D}'(\mathbb{R})$ and $\text{supp}(t) \subseteq \{0_d\}$ where $0_d = (0, 0, ..., 0)$ is the null-vector in d dimensions. Then

$$t(x) = \sum_a C_a \partial^a \delta_d(x). \tag{3.17}$$

Where δ_d is the delta distribution in $d-$ dimensions, $C_a \in \mathbb{C}$ and the sum is finite. [47] page.46 Theorem 2.3.4

The next examples of distributions are known as **propagators** due to their physical interpretation. The ones presented here are propagators for the free theory of neutral scalar field ϕ. We will return to them in the near future for a proper introduction.

The **retarded propagator** $\Delta^{\text{ret}}(x) \in \mathcal{D}'(\mathbb{R}^d)$ is defined by:

$$\Delta^{\text{ret}}(x) := \frac{1}{(2\pi)^d} \int d^d p\, \frac{e^{-ipx}}{p^2 - m^2 + ip^0 0} \tag{3.18}$$

where $px = p_\mu x^\mu$ and $ip^0 0 \equiv \lim_{\epsilon \to 0^+} ip^0 \epsilon$. The retarded product appears naturally when one tries to find a perturbative solution to an interacting field.

The **Jordan-Pauli function** or **commutation function** is defined by:

$$\Delta(x) := \Delta^{\text{ret}}(x) - \Delta^{\text{ret}}(-x) = \frac{-i}{(2\pi)^{d-1}} \int d^d p\; \text{sgn}(p^0) \delta(p^2 - m^2) e^{-ipx}. \tag{3.19}$$

This propagator is mostly defined for convenience in practical calculations.

The **Wightman two-point function** also known as the positive part of $i\Delta$ is defined as the propagator of the scalar field $\langle\Omega|\phi(x)\phi(y)\Omega\rangle$ (see chapter about the Fock space). After some work, we can write it as follows:

$$\Delta^+(x) := \frac{1}{(2\pi)^{d-1}} \int d^d p \, \theta(p^0)\delta(p^2 - m^2)e^{-ipx}. \tag{3.20}$$

The Wightman two-point function is important for quantizing the theory.

The last propagator that we define is the **Feynman propagator**:

$$\Delta^F(x) := \theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^+(-x) = \frac{i}{(2\pi)^d} \int d^d p \frac{e^{ipx}}{p^2 - m^2 + i0} \tag{3.21}$$

The Feynman propagator is important in the construction of the so-called $S-$ matrix, probably the most important object in scattering theory.

We will briefly summarize some properties of the propagators (page 470 [24]). Almost all of them are classical exercises in the quantum field theory course.

I)

$$(\partial^2 + m^2)\Delta^{\text{ret}}(x) = -\delta(x) \quad \text{supp}(\Delta^{\text{ret}}) \subseteq \overline{V}^+$$

II)

$$\Delta(x) = -\Delta(-x) \quad (\partial^2 + m^2)\Delta^+(x) = 0$$
$$\text{supp}(\Delta^+) = (\overline{V}^+ \cup \overline{V}^-) \quad \Delta(x)\theta(x^0) = \Delta^{\text{ret}}(x)$$

III)

$$-i(\Delta^+(x) - \Delta^+(-x)) = \Delta(x) = \Delta^{\text{ret}}(x) - \Delta^{\text{ret}}(-x)$$
$$(\partial^2 + m^2)\Delta^+(x) = 0$$

IV)

$$\Delta^F(x) = \theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^+(-x) = i\Delta^{\text{ret}}(x) + \Delta^+(x)$$
$$\Delta^F(x) = \Delta^F(-x)$$

V)

$$(\partial^2 + m^2)\Delta^F(x) = -i\delta(x)$$

VI)

$$\rho^{d-2}\Delta^{+}_{\frac{m}{\rho}}(\rho x) = \Delta^{+}_{m}(x), \quad \rho > 0.$$

In the last relation, the subscript m from Δ^{+}_{m} indicates the mass of the field.

3.6 Technical remarks

An important feature to emphasize is that even if distributions are worked as if they were a usual function, they are not. Hence, we point out some technical details that are going to be used. For a shallow but didactical introduction to the theme, we recommend [69].

3.6.1 Derivative of distributions

The spirit of the derivative of a distribution is to imagine that the inner product with a test function is indeed an integral and not just notation. If that were the case, we could calculate the derivative of a distribution $t \in \mathcal{D}'(\mathbb{R})$ acting on $g \in \mathcal{D}(\mathbb{R})$ using integration by parts:

$$\begin{aligned}\int dx\,(\frac{d}{dx}t(x))g(x) \text{ `` $=$ '' } t(x)g(x)\Big|_{-\infty}^{\infty} - \int dx\,t(x)(\frac{d}{dx}g(x))\\ = -\int dx\,t(x)(\frac{d}{dx}g(x)).\end{aligned} \tag{3.22}$$

In the last step, we have used that g is compactly supported and therefore $\lim_{x\to\pm\infty} g(x) = 0$.

Unfortunately, the integral above is just a notation. Luckily, we define the derivative of a distribution to do exactly what is mentioned above:

$$\langle t', g\rangle \equiv \langle \frac{d}{dx}t, g\rangle := -\langle t, \frac{d}{dx}g\rangle. \tag{3.23}$$

A classical example is the derivative of $\theta(x)$:

$$\begin{aligned}\langle \theta'(x), g(x)\rangle = -\langle \theta(x), g'(x)\rangle = -\int_0^{\infty} dx\,g'(x)\\ = -(g(\infty) - g(0)) = g(0) = \langle \delta, g\rangle\end{aligned} \tag{3.24}$$

and the derivative of $\delta(x)$:

$$\langle \delta', g\rangle = -\langle \delta, g'\rangle = -g'(0) \Rightarrow \delta'(x) \equiv -\delta(x)\frac{d}{dx}. \tag{3.25}$$

More generally, we define the derivative of a distribution as:

$$\langle \partial^a t, g\rangle := (-1)^{|a|}\langle t, \partial^a g\rangle. \tag{3.26}$$

3.6.2 Fourier transform

We define the Fourier transformation of a distribution $u \in \mathcal{D}'(\mathbb{M}^n)$, $\mathcal{F}(u)(k) \equiv \tilde{u}(k)$ analogously to the derivative:

$$\langle \mathcal{F}(u)(k), f(k)\rangle_k = \langle u(x), \mathcal{F}(f)(x)\rangle \tag{3.27}$$

where

$$\mathcal{F}(f)(x) = \frac{1}{(2\pi)^{\frac{dn}{2}}}\int d^{dn}k\, e^{ikx} f(k). \tag{3.28}$$

Remark: Actually, the Fourier transformation is defined for a slightly "bigger" space, $\mathcal{J}(\mathbb{M}^n)$, the space of "rapidly decaying functions" or "Schwartz functions" see the Appendix and [2]

Remark 2: The relation (3.27) is just a fancy way of writing:

$$\langle \mathcal{F}(u)(k), f(k)\rangle_k = \int d^{dn}k \left(\int \frac{d^{dn}x}{(2\pi)^{\frac{dn}{2}}} e^{ikx} u(x)\right) f(k)$$
$$= \int d^{dn}x\, u(x) \left(\int \frac{d^{dn}k}{(2\pi)^{\frac{dn}{2}}} e^{ikx} f(k)\right) = \langle u(x), \mathcal{F}(f)(x)\rangle. \tag{3.29}$$

As an example, we can calculate the Fourier transformation of the delta function in $\mathbb{R}$ ([69] Lectures 13 and 14):

$$\langle \mathcal{F}(\delta)(k), f(k)\rangle_k = \langle \delta(x), \mathcal{F}(f)(x)\rangle = \int \frac{dk}{(2\pi)^{\frac{1}{2}}} e^{ik0} f(k) = \langle \frac{1}{\sqrt{2\pi}}, f\rangle. \tag{3.30}$$

Hence:

$$\mathcal{F}(\delta)(k) = \frac{1}{\sqrt{2\pi}}. \tag{3.31}$$

We also introduce the inverse Fourier transformation:

$$\mathcal{F}^{-1}(f)(k) := \int \frac{d^{dn}x}{(2\pi)^{dn/2}} e^{-ikx} f(x) = \tilde{f}(-k). \tag{3.32}$$

The proof of $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = 1$ can be found in [2] Theorem 39.3 (The inverse Fourier Transform) (In Portuguese).

With the definition of the inverse Fourier transform, we can obtain a very important result:

$$\begin{aligned}
\langle \mathcal{F}^{-1}(1), f\rangle = \langle 1, \mathcal{F}^{-1}(f)\rangle = \sqrt{2\pi}\langle \frac{1}{\sqrt{2\pi}}, \mathcal{F}^{-1}(f)\rangle \\
= \sqrt{2\pi}\langle \mathcal{F}(\delta), \mathcal{F}^{-1}(f)\rangle = \sqrt{2\pi}\langle \mathcal{F}^{-1}(\mathcal{F}(\delta)), f\rangle = \sqrt{2\pi}\langle \delta, f\rangle \\
\Rightarrow \int dx\, \frac{1}{\sqrt{2\pi}} e^{-ikx} = \int dx \frac{1}{\sqrt{2\pi}} e^{ikx} = \sqrt{2\pi}\delta(k) \Rightarrow \int dx\, e^{ikx} = 2\pi\delta(k).
\end{aligned} \tag{3.33}$$

The above result will be used a lot to compute scattering amplitudes.

3.6.3 Multiplication of distributions: The wave-front set

To introduce the problem, we try naively to calculate $\delta(x)\theta(x)$ applied to $g(x) \in \mathcal{D}(\mathbb{R})$ (this argument was constructed by prof. João Barata in a private conversation). To do it, we consider the relation $\theta'(x) = \delta(x)$ discussed above and $\theta^2(x) = \theta(x)$ (3.16). Hence, using the "product rule"(Leibniz rule) for derivatives, we can write:

$$\begin{aligned}
\langle \frac{d}{dx}\theta^2(x), g(x)\rangle = 2\langle \delta(x)\theta(x), g(x)\rangle \stackrel{!}{=} \langle \frac{d}{dx}\theta(x), g(x)\rangle = \langle \delta, g\rangle \\
\Rightarrow \langle \delta(x)\theta(x), g(x)\rangle = \frac{1}{2}\langle \delta, g\rangle = \frac{1}{2}g(0).
\end{aligned} \tag{3.34}$$

So far so good. The problem is that we can repeat the same calculation by changing $\theta^2(x)$ to $\theta^n(x), n > 2$ and the result would be:

$$\langle \delta(x)\theta(x), g(x)\rangle = \frac{1}{n}\langle \delta, g\rangle = \frac{1}{n}g(0). \tag{3.35}$$

That is absurd. What we learned from this example is that one has to be careful when multiplying distributions. To guide us in the search for a general rule for multiplying distributions, we have to introduce the so-called Wave-front set. We will follow mostly [11] and[2]. A complete exposition about the subject can be found in [47], Chapters 7 and 8.

Before attacking the problem, let us construct some intuition about the subject. If we want to multiply $u, v \in \mathcal{D}'(\mathbb{R}))$ at a point x we want to mimic the point-wise distribution, that is, we are not interested in what happens away from x and for that reason we multiply u, v by $f \in \mathcal{D}(\mathbb{R})$ so that $f((x-\epsilon, x+\epsilon)) = 1$. Since the multiplication of Fourier transformation is given by its convolution, if

$$\left|\left(\widetilde{fu} \cdot \widetilde{fv}\right)(k)\right| = \left|\int dq\, \widetilde{fu}(q)\widetilde{fv}(q-k)\right| < \infty \tag{3.36}$$

exists, then we can compute $uv(x)$ using the inverse Fourier transformation. In order for the integral above to converge, we need that in the directions of growth of $u(q)$, $v(q-k)$ decays faster in the same direction. If we want to impose derivatives, we need a stronger condition:

$$\left| \int dq\, \widetilde{fu}(q)\widetilde{fv}(q-k) \right| < \frac{C_n}{1+|k|^n}\, \forall n \in \mathbb{N}. \tag{3.37}$$

That means we need to find constants C_n such that the product decays faster than any polynomial in k. The reason for this statement is that a derivative is equivalent to a multiplication by k. Hence, if we want the product to be well defined, we need that for every direction of growth $q^n u(q)\ v(q-k)$ decays faster in the same direction for all n. To better visualize the discussion, let us give an example with distributions given by smooth functions:

Let $u \in \mathcal{D}'(\mathbb{R}^d)$ be a smooth function with compact support, that is, $u \in \mathcal{D}(\Omega \subseteq \mathbb{R}^d)$. In this case, we can prove that

$$\forall N \in \mathbb{N}: \quad |\tilde{u}(k)| < C_n(|1+|k|)^{-N}. \tag{3.38}$$

And the reverse is also true! If $\tilde{u}(k) < C_n(1+|k|)^{-N}$, then u test function(Paley-Wiener-Schwartz Theorem, page 181 [47]). The formal proof of it is very technical, but we can give a "hand-waving argument" that shows why it must be the case. For simplicity we will restrict ourselves to $\mathbb{R}$ but the generalization to $\mathbb{R}^d$ is immediate.

Take $u \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(u) = [a, a+L], L > 0$ and $\sup_{x\in[a,a+L]}(u) = u_M$. Then:

$$\begin{aligned} |\tilde{u}(k)| = |\int_a^{a+L} \frac{dx}{\sqrt{2\pi}} e^{ikx} u(x)| &\leq \int_a^{a+L} \frac{dx}{\sqrt{2\pi}} |e^{ikx} u(x)| \\ &\leq \int_a^{a+L} \frac{dx}{\sqrt{2\pi}} u_M = \frac{u_M L}{\sqrt{2\pi}}. \end{aligned} \tag{3.39}$$

Now we integrate by parts:

$$\begin{aligned} |\tilde{u}(k)| &= \left| \frac{1}{\sqrt{2\pi}} \frac{e^{ikx}}{ik} u(x) \Big|_a^{a+L} - \int \frac{dx}{\sqrt{2\pi}} \frac{e^{ikx}}{ik} u'(x) \right| \\ &\leq \frac{1}{k} \int \frac{dx}{\sqrt{2\pi}} |e^{ikx} u'(x)| \leq \frac{u'_M L}{\sqrt{2\pi} k}. \end{aligned} \tag{3.40}$$

In the first inequality we have used $u(a) = u(a+L) = 0$ and $\sup_{x\in[a,a+L]} u'(x) \equiv u'_M$. Note that since $|\tilde{u}(k)| \leq \frac{u_M L}{\sqrt{2\pi}}$, we can write the above equation as:

$$|\tilde{u}(k)| \leq \frac{C_1}{1+|k|} \tag{3.41}$$

where $C_1 = \max\{\frac{u'_M L}{\sqrt{2\pi}}, \frac{u_M L}{\sqrt{2\pi}}\}$. Repeating the procedure, we can derive:

$$|\tilde{u}(k)| < C_n \frac{1}{(1+|k|)^n}. \tag{3.42}$$

□

Hence, as expected, we can multiply a distribution u by a smooth function g and it will once again be a distribution because g decays faster than any polynomial. More than that, that new distribution is given by:

$$\begin{aligned}\langle gu, f\rangle &= \int dx\ (g(x)u(x))\, f(x) = \int dx\, u(x)(g(x)f(x)) \\ &= \langle u, gf\rangle \Rightarrow \langle gu, f\rangle := \langle u, gf\rangle.\end{aligned} \tag{3.43}$$

The objective of the rest of the section is to characterize when two distributions obey the decay property mentioned. We call the set of points where $u \in \mathcal{D}'(\mathbb{R}^d)$ can be seen as a smooth function with compact support **regular support**, denoted by reg supp(u). The complement of this set is called the **singular support** and is denoted by sing supp(u). For example, in one dimension:

$$\text{sing supp}(\delta(x)) = \{0\} \quad \text{reg supp}(\delta(x)) = \mathbb{R} \setminus \{0\}. \tag{3.44}$$

The first case where the multiplication of the distributions $u, v \in \mathcal{D}'(\mathbb{R}^d)$ is well defined is when sing supp$(u) \cap$sing supp$(v) = \emptyset$. To prove this assertion, let $u, v \in \mathcal{D}'(\mathbb{R}^d)$, sing supp$(u) = \Omega \subset \mathbb{R}^d$. Since sing supp$(u) \cap$ sing supp$(v) = \emptyset$, $\Omega \subseteq$ reg supp(v), that is, $v|_\Omega$ can be seen as a compactly supported smooth function. Hence, the multiplication of distributions in Ω is well defined and is given by:

$$\langle uv, f\rangle = \langle u, vf\rangle. \tag{3.45}$$

The product above also respects the Leibniz rule (for simplicity we assume $\partial^a = \partial_{x_1}$, higher derivatives require more complicated combinatorial but does not give deeper insights):

$$\begin{aligned}\langle \partial_{x_1}(uv), f\rangle &= -\langle uv, \partial_{x_1} f\rangle = -\langle u, v\partial_{x_1} f\rangle = -\langle u, \partial_{x_1}(vf)\rangle + \langle u, (\partial_{x_1} v) f\rangle \\ &= \langle \partial_{x_1} u, vf\rangle + \langle u\partial_{x_1} v, f\rangle = \langle (\partial_{x_1} u)v, f\rangle + \langle u(\partial_{x_1} v), f\rangle. \end{aligned} \tag{3.46}$$

The final step to prove that the multiplication is well defined is to extend the domain from Ω to $\mathbb{R}^d$. That is simple since

$$\mathbb{R}^d = \text{sing supp}(u) \cup \text{sing supp}(v) \cup (\text{reg supp}(u) \cap \text{reg supp}(v)). \tag{3.47}$$

In each of these domains, the new distribution is well defined.

The complication appears when sing supp$(u) \cap$ sing supp$(v) \neq \emptyset$. In this case, we want to mimic the behavior of a smooth function regarding its Fourier transformation. Unfortunately, there is no direct path to follow as in the other cases. However, we can present the answer (Theorem 8.2.10 p.267 [47]) and to those seeking a deeper understanding, we recommend Chapter 8 from [47].

We define the **wave-front set** of a distribution u as ([24] page 474):

$$\begin{aligned} \text{WF}(u) := \{&(x, k) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} | x \in \text{sing supp}(u), \\ &\widetilde{uf} \text{ does not decay rapidly in direction } k \forall f \neq 0\}. \end{aligned} \tag{3.48}$$

Rapidly decaying is synonym that it decays faster than any polynomial (just like test functions).

Given $u, v \in \mathcal{D}'(\mathbb{R}^d)$ with:

$$\begin{aligned} &(x, 0) \notin \text{WF}(u) \oplus \text{WF}(v) \\ &:= \{(x, k_1 + k_2) | (x, k_1) \in \text{WF}(u)\, (x, k_2) \in \text{WF}(v)\} \end{aligned} \tag{3.49}$$

The uv exists and follows the Leibniz rule. The above theorem is called the “Hörmander criterion” in the literature. The proof of this important theorem can be found in (Theorem 8.2.10 page.267 [47]). We also recommend [11] and [24] pages 473-477.

The theorem requires a rich structure to be proven, but can be easily applied. An important remark: It may happen that two distributions do not fulfill the criterion but the product exists and follows the Leibniz rule, but it is not usual. There are also other possible definitions for the product of distributions [7], but they are not generally used in this context. Next, we calculate some examples to show the power of the criterion just mentioned.

3.6.4 Examples

Let us start with some simple examples of the distributions in $\mathbb{R}$. We mostly follow [11].

$\delta(x-a)$**:**

The singular support of $\delta(x-a)$ is simply $\{a\}$. To calculate the directions in which δ decays rapidly, we consider the following:

$$\widetilde{f\delta}(k) = \int \frac{dx}{\sqrt{2\pi}} e^{ikx} \delta(x-a) f(x) = \frac{1}{\sqrt{2\pi}} f(a) e^{ika}. \tag{3.50}$$

Note that $\widetilde{f\delta}$ is periodic in k, hence it does not decay fast in any direction. Therefore:

$$\mathrm{WF}(\delta(x-a)) = \{(a,k) | k \in \mathbb{R} \setminus \{0\}\}. \tag{3.51}$$

$\theta(x-a)$**:**

We repeat the procedure:

$$\widetilde{f\theta}(k) = \int \frac{dx}{\sqrt{2\pi}} e^{ikx} \theta(x-a) f(x) = \int_a^\infty \frac{dx}{\sqrt{2\pi}} e^{ikx} f(x). \tag{3.52}$$

We claim that this function behaves as $\frac{1}{k}$ for $k \to \infty$. To prove it, we take a test function with $f(a) = 1$ and integrate by parts twice:

$$\begin{aligned} \int_a^\infty \frac{dx}{\sqrt{2\pi}} f(x) e^{ikx} &= \frac{f(a)}{\sqrt{2\pi} ik} e^{ika} - \int_a^\infty \frac{dx}{\sqrt{2\pi}} f'(x) \frac{e^{ikx}}{ik} \\ &= \frac{f(a)}{\sqrt{2\pi} ik} e^{ika} - f'(a) \frac{e^{ika}}{-\sqrt{2\pi} k^2} + \int_a^\infty \frac{dx}{\sqrt{2\pi}} f''(x) \frac{e^{ikx}}{ik^2 \sqrt{2\pi}}. \end{aligned} \tag{3.53}$$

Let $\sup_{x \in \mathrm{supp}(f)} f'(x) = f'_M$ and $\sup_{x \in \mathrm{supp}(f)} f''(x) = f''_M$. From the above equation, we read:

$$|\widetilde{f\theta}(k) - \frac{e^{ika}}{ik}| \leq \frac{f''_M + f'_M}{k^2}. \tag{3.54}$$

In order for this equation to be satisfied for large k,

$$\widetilde{f\theta}(k) \sim \frac{e^{ika}}{ik} + O(k^{-2}). \tag{3.55}$$

And this completes the proof. Hence:

$$\mathrm{WF}(\theta(x-a)) = \{(a,k) | k \in \mathbb{R} \setminus \{0\}\}. \tag{3.56}$$

$\frac{1}{x-i0}$**:**

This one is a little more tricky than the others. We start by calculating the following relation for $f, g \in \mathcal{S}(\mathbb{R})$:

$$\begin{aligned}
\mathcal{F}(fg)(k) &= \int \frac{dx}{\sqrt{2\pi}} f(x)g(x)e^{ikx} \\
&= \int \frac{dx}{\sqrt{2\pi}} \left(\int \frac{dk'}{\sqrt{2\pi}} \tilde{f}(k')e^{-ik'x}\right) \left(\int \frac{dk''}{\sqrt{2\pi}} \tilde{g}(k'')e^{-ik''x}\right) e^{ikx} \\
&= \left(\int \frac{dk'}{\sqrt{2\pi}} \tilde{f}(k')\right) \left(\int \frac{dk''}{\sqrt{2\pi}} \tilde{g}(k'')\right) \int \frac{dx}{\sqrt{2\pi}} e^{i(k-k'-k'')x} \\
&= \left(\int \frac{dk'}{\sqrt{2\pi}} \tilde{f}(k')e\right) \left(\int \frac{dk''}{\sqrt{2\pi}} \tilde{g}(k'')\right) \delta(k-k'-k'') \\
&= \int \frac{dk'}{2\pi} \tilde{f}(k')\tilde{g}(k-k').
\end{aligned} \tag{3.57}$$

That is, the Fourier transformation of a product of functions is the convolution of the functions (one can find a “less physical” deduction of this formula in [2] page 2156). the same formula holds for distributions.

Now, we calculate the Fourier transformation of $\frac{1}{x-i0}$:

$$\int \frac{dx}{\sqrt{2\pi}} \frac{e^{ikx}}{x-i0} = \lim_{\epsilon \to 0^+} \int \frac{dx}{\sqrt{2\pi}} \frac{e^{ikx}}{x-i\epsilon}. \tag{3.58}$$

The integrant above has a pole in $x = i\epsilon$. We will use this fact to calculate the integral using the residue theorem. We have to divide our analyses into two cases. First, consider $k > 0$. The contour we have to take is the following:

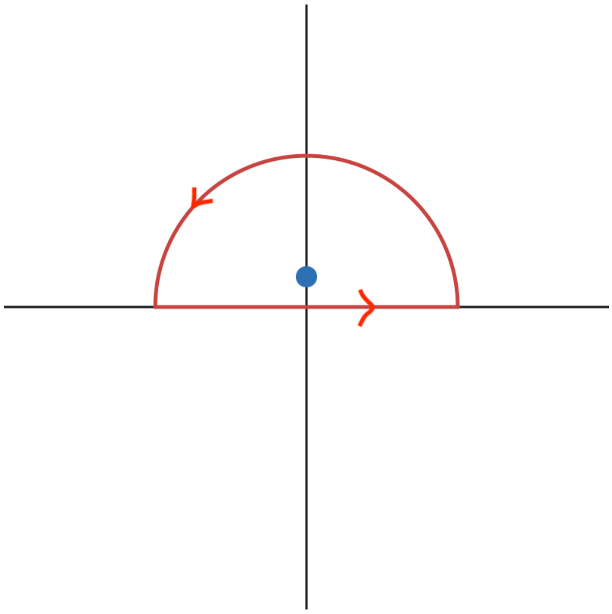


Figure 3.2: The contour for $k > 0$. The blue dot represent the residue.

Then the integral is simply

$$\int \frac{dx}{\sqrt{2\pi}} \frac{e^{ikx}}{x - i\epsilon} = \frac{2\pi i}{\sqrt{2\pi}} e^{ik\epsilon}. \tag{3.59}$$

If $k < 0$, then the contour is:

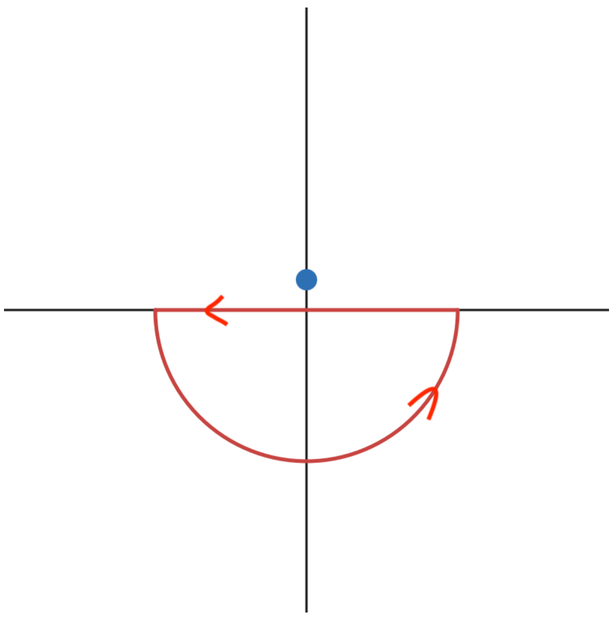


Figure 3.3: The contour for $k < 0$.The blue dot represent the residue.

and the integral is simply

$$-\int \frac{dx}{\sqrt{2\pi}} \frac{e^{ikx}}{x - i\epsilon} = 0. \tag{3.60}$$

Hence, in the limit $\epsilon \to 0$, we can write:

$$\widetilde{\frac{1}{x - i0}} = \sqrt{2\pi} i\theta(k). \tag{3.61}$$

Using the results discussed above:

$$\widetilde{f\frac{1}{x - i0}}(k) = \int \frac{dk'}{2\pi} \left(\sqrt{2\pi} i\theta(k - k')\right) \tilde{f}(k') = i \int_{-\infty}^{k} \frac{dk'}{\sqrt{2\pi}} \tilde{f}(k'). \tag{3.62}$$

To complete the proof, we use the fact that f is compactly supported, therefore:

$$\forall\, n \in \mathbb{N}\, |\tilde{f}(k)| < \frac{C_n}{(1 + |k|)^n} \tag{3.63}$$

where $C_n \in \mathbb{R}$. If $k < 0$ the integral decays faster than any polynomial, since

$$\begin{aligned} &|i \int_{-\infty}^{k} \frac{dk'}{\sqrt{2\pi}} \tilde{f}(k')| < \frac{C_n}{\sqrt{2\pi}} \int_{-\infty}^{k} dk' \frac{1}{(1 - k')^n} \\ &= \frac{C_n}{(n-1)\sqrt{2\pi}} \frac{1}{(1 - k')^{n-1}}\Bigg|_{-\infty}^{k} = \frac{\tilde{C}_{n-1}}{(1 + |k|)^{n-1}}. \end{aligned} \tag{3.64}$$

On the other hand, if $k \geq 0$, the argument does not hold since:

$$\begin{aligned} \int_{-\infty}^{k} dk' \frac{1}{(1 + |k'|)^n} &= \int_{-\infty}^{0} dk' \frac{1}{(1 - k')^n} + \int_{0}^{k} dk' \frac{1}{(1 + k')^n} \\ &= \frac{1}{n-1} \frac{1}{(1 - k')^{n-1}}\Bigg|_{-\infty}^{0} + \frac{1}{n-1} \frac{-1}{(1 + k')^{n-1}}\Bigg|_{0}^{k} \\ &= \frac{2}{n-1} - \frac{1}{(n-1)(1 + k)^{n-1}}. \end{aligned} \tag{3.65}$$

From the above analyses, we conclude

$$\mathrm{WF}\left(\frac{1}{x - i0}\right) = \{(0, k) | k < 0\}. \tag{3.66}$$

$\delta(x)\delta(y)$

As a last example, we consider some distributions defined in $\mathcal{D}'(\mathbb{R}^2)$. There is no big deal, but this example is instructive because the notation may be tricky, so it is good to explicit the difference. Consider $\delta(x), \delta(y) \in \mathcal{D}'(\mathbb{R}^2)$ given by:

$$\langle \delta(x), f(x,y) \rangle = f(0,y) \quad \langle \delta(y), f(x,y) \rangle = f(x,0). \tag{3.67}$$

Then, the product $\delta(x)\delta(y)$ exists and respects the Leibniz rule. To prove it, we can use the Hörmander criterion. To do it, consider:

$$\begin{aligned} \mathrm{WF}(\delta(x)) &= \{(0,y;k,0) | y \in \mathbb{R}, k \in \mathbb{R} \setminus \{0\}\} \\ \mathrm{WF}(\delta(y)) &= \{(x,0;0,q) | x \in \mathbb{R}, q \in \mathbb{R} \setminus \{0\}\} \\ \Rightarrow \mathrm{WF}(\delta(x)) \oplus \mathrm{WF}(\delta(y)) &= \{(0,0,k,q) | k, q \neq 0\}. \end{aligned} \tag{3.68}$$

Since we do not have a vector of the form $(x,y,0,0) \in \mathrm{WF}(\delta(x)) \oplus \mathrm{WF}(\delta(y))$, the product $\delta(x)\delta(y)$ exists. In that special case, it is easy to see that:

$$\begin{aligned} \langle \delta(x)\delta(y), f(x,y) \rangle &= f(0,0) \\ \mathrm{WF}(\delta(x)\delta(y)) &= \{(0,0,k,q) | k, q \neq 0\}. \end{aligned} \tag{3.69}$$

The example above is very simple, but the action of the product of two distributions $u, v \in \mathcal{D}'(\mathbb{R}^d)$ in $f \in \mathcal{D}(\mathbb{R}^d)$ is, in general, not that simple. The wave front set of the product is also not always $\mathrm{WF}(u) \oplus \mathrm{WF}(v)$. Nevertheless, we can give general statements on the wave front set of the product of distributions as long as they respect the Hörmander criterion:

$$\mathrm{WF}(uv) \subseteq \left(\mathrm{WF}(u) \cup \mathrm{WF}(s) \cup \left(\mathrm{WF}(u) \oplus \mathrm{WF}(v)\right)\right). \tag{3.70}$$

The proof of the statement can be found at [11], page11 and references therein.

Using the Hörmander criterion, we conclude that

- $\left(\frac{1}{x-i\epsilon}\right)^n$ follows the criterion, thus is well define and follows the Leibniz rule.

- $\delta^2(x), \delta(x)\theta(x)$ does not follow the criterion and does not exist.
- $\delta(x-a)\theta(x-b)$ respects the criterion as long as $b \neq a$.
- $\theta^2(x)$ does not respect the criterion, but it exists. The Leibniz rule does not apply for it.

3.6.5 List of Wave-front sets

For the distributions with which we will work (the so-called "propagators"), it is not easy to find the wave front set. For being a very technical calculation, we present the wave-front set of the propagators and where one can find the proof of it.

Wightman two point function

$$\mathrm{WF}(\Delta^{+}(x)) = \{(x,k)|x^2 = 0, k^2 = 0, x = \lambda k, k^0 > 0, \lambda \in \mathbb{R}^d\}. \tag{3.71}$$

The "square" in the above definition is to be understood as the product in Minkowski space-time $x^2 = x_\mu x^\mu, k^2 = k_\mu k^\mu$. The proof can be found on [63] page 106 (Theorem IX.48).

Retarded product

$$\mathrm{WF}(\Delta^{\mathrm{ret}}(x)) = \Big(\{0\} \times (\mathbb{R}^d \setminus \{0\})\Big) \cup \{(x,k)|x^2 = 0, k^2 = 0, x = \lambda k, \lambda > 0\}. \tag{3.72}$$

The proof can be found on [24] page 477.

Feynman Propagator

$$\begin{aligned}\mathrm{WF}(\Delta^{F}(x)) = &\Big(\{0\} \times (\mathbb{R}^d \setminus \{0\})\Big)\\ &\cup \{(x,k)|x^2 = 0, k^2 = 0, x^0 > 0, k^0 \neq 0 x = \lambda k, \lambda \in \mathbb{R}^d\}.\end{aligned} \tag{3.73}$$

The proof can be found on [24] page 477 or on [11] page 22, preposition 26.

Jordan-Pauli function

$$\mathrm{WF}(\Delta(x)) = \{(x,k)|x^2 = 0, k^2 = 0, k^0 \neq 0\, x = \lambda k,\ \lambda \in \mathbb{R}^d\}. \tag{3.74}$$

The proof can be found on [24] page 30.

Chapter 4

Fock space: Just the basics

4.1 Introduction

To motivate the discussion of causal perturbative quantum field theory, we take a step back and discuss the Fock space. The discussion is also fruitful for fixing notation. For now, we work only with the scalar field, since it is the simplest choice. Once the concepts are clear, we introduce fermionic fields and gauge fields.

4.2 The bosonic Fock space

The quantum mechanics of a fixed (and finite) number of particles is defined in a Hilbert space. In quantum field theory, particles can be created or annihilated; therefore, the structure of Hilbert space by itself is not enough. For that reason the concept of Fock space was first introduced in the early 30's [32]. Since the introduction here is very superficial,for mathematical details, we recommend [4]. We will follow a mixture of [33, 67, 35].

We start our discussion with [24] in Appendix A5.

Given $\mathbb{M}^*$ the momentum space belonging to the Minkowski space $\mathbb{M}$ equipped with the mass shell and a Lorentz invariante measurement:

$$\begin{aligned} &\mathcal{H}_m^+ := \{p = (p^0, \vec{p}) \in \mathbb{M}^* | p^2 = m^2, p^0 > 0\}, m > 0 \\ &\frac{d^{d-1}\vec{p}}{2\sqrt{(\vec{p})^2 + m^2}} \equiv \frac{d^{d-1}\vec{p}}{2\omega_{\vec{p}}} \equiv d\mu_p. \end{aligned} \tag{4.1}$$

The Hilbert space of one particle system is defined as:

$$\mathcal{H} := \{\phi : \mathcal{H}_m^+ \to \mathbb{C} | \, \|\phi\|_{\mathcal{H}} < \infty\}, \quad \|\psi\|_{\mathcal{H}}^2 := \hbar \int d\mu_p |\psi(\vec{p})|^2. \tag{4.2}$$

The wave function $\psi(\vec{p})$ is related to the wave function $\psi(\vec{x})$ by Fourier transformation:

$$\psi(\vec{x}) = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{p}\, e^{i\vec{p}\vec{x}} \psi(\vec{p}). \tag{4.3}$$

The bosonic Fock space $\mathfrak{F}$ can be defined as a direct sum of the bosonic Hilbert spaces of n identical particles with mass m:

$$\begin{aligned} \mathfrak{F} &:= \bigoplus_{n=0}^{\infty} \mathfrak{S}\mathcal{H}^{\otimes n} \\ \mathcal{H}^0 &:= \mathbb{C} \end{aligned} \tag{4.4}$$

Here $\mathfrak{S}$ means symmetrization:

$$\mathfrak{S}\psi_1 \otimes \psi_2 \otimes ... \otimes \psi_n = \sum_{\pi \in S^n} \frac{1}{n!} \psi_{\pi(1)} \otimes ... \otimes \psi_{\pi(n)}. \tag{4.5}$$

We also introduce the vectors $\Phi_n \in \mathfrak{F}$ corresponding to a system with n particles:

$$\Phi_n := \sqrt{n!}\mathfrak{S}\psi_1 \otimes \psi_2 \otimes ... \otimes \psi_n. \tag{4.6}$$

Remark:The factor $\sqrt{n!}$ is to obtain the right normalization since $\|\mathfrak{S}\psi_1 \otimes ... \otimes \psi_n\|^2 = \frac{1}{n!}$. We could have defined $\mathfrak{S}$ preserving the norm, but we would lose $\mathfrak{S}^2 = \mathfrak{S}$.

The inner product of $\Phi, \Psi \in \mathfrak{F}$ (in momentum space) is defined by:

$$\langle \Phi, \Psi \rangle := \sum_n \langle \overline{\Phi}_n, \Psi_n \rangle_{\mathcal{H}^n}. \tag{4.7}$$

Where $\langle\rangle_{\mathcal{H}^n}$ is the inner product in the subspace of n- relativistic particles given by:

$$\langle \Phi, \Psi \rangle_{\mathcal{H}^n} := \hbar^n \int \frac{d^{d-1}\vec{p}_1}{2\omega_{\vec{p}_1}} ... \frac{d^{d-1}\vec{p}_n}{2\omega_{\vec{p}_n}} \overline{\Phi_n(p_1, .., p_n)} \Psi_n(\vec{p}_1, ..., \vec{p}_n). \tag{4.8}$$

With $\omega_{\vec{p}} := \sqrt{p^2 + m^2}$. Just as an example, let us compute $\langle \Phi = \sqrt{2}\mathfrak{S}(\phi_1 \otimes \phi_2) | \Psi = \sqrt{2}\mathfrak{S}\psi_1 \otimes \psi_2 \rangle$:

$$
\begin{aligned}
&\left\langle \frac{\phi_1\otimes\phi_2+\phi_2\otimes\phi_1}{\sqrt{2}} \middle| \frac{\psi_1\otimes\psi_2+\psi_2\otimes\psi_1}{\sqrt{2}} \right\rangle \\
=&\frac{1}{2}\bigg(\langle\phi_1|\psi_1\rangle_{\mathcal{H}_1}\langle\phi_2|\psi_2\rangle_{\mathcal{H}_1} + \langle\phi_1|\psi_2\rangle_{\mathcal{H}_1}\langle\phi_2|\psi_1\rangle_{\mathcal{H}_1} \\
&+\langle\phi_2|\psi_1\rangle_{\mathcal{H}_1}\langle\phi_1|\psi_2\rangle_{\mathcal{H}_1} + \langle\phi_2|\psi_2\rangle_{\mathcal{H}_1}\langle\phi_2|\psi_2\rangle_{\mathcal{H}_1}\bigg). \qquad (4.9)
\end{aligned}
$$

We introduce the "**creation operator**" (in momentum space) $a^*(\psi_f) : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n+1}$ that act in $\psi_1 \otimes ...\psi_n$ by creating a particle with wave function ψ_f:

$$
\begin{aligned}
&a^*(\psi_f)\psi_1\otimes...\psi_n := \sqrt{n+1}(\psi_f\otimes\psi_1\otimes...\psi_n) \in \mathcal{H}^{\otimes n+1} \\
&a^*(\psi_f)\psi_0 = \psi_f \qquad (4.10)
\end{aligned}
$$

where $\psi_0 \in \mathbb{C} \equiv \mathcal{H}^0$. The factor $\sqrt{n+1}$ can be deduced imposing normalization on the state $a^*(\psi_f)\psi_1 \otimes ...\psi_n$. We can deduce the adjoint operator $a(\psi_f)$ using:

$$
\begin{aligned}
&\langle\psi_1\otimes...\psi_{n-1}|a(\psi_f)\chi_1\otimes...\chi_n\rangle = \langle a^*(\overline{\psi_f})\psi_1\otimes...\psi_{n-1}|\chi_1\otimes...\chi_n\rangle \\
&= \sqrt{n}\langle\psi_f\otimes\psi_1\otimes...\psi_{n-1}|\chi_1\otimes...\chi_n\rangle = \sqrt{n}\langle\psi_f|\chi_1\rangle\langle\psi_1\otimes...\psi_n|\chi_2\otimes...\chi_n\rangle. \qquad (4.11)
\end{aligned}
$$

Comparing the results, we conclude:

$$
a(\psi_f)\chi_1\otimes...\otimes\chi_n = \sqrt{n}\langle\psi_f|\chi_1\rangle\chi_2\otimes...\otimes\chi_n \in \mathcal{H}^{\otimes n-1}. \qquad (4.12)
$$

A special case is $a(\psi_f)\psi_0, \psi_0 \in \mathbb{C}$:

$$
a(\psi_f)\psi_0 := 0. \qquad (4.13)
$$

Note that the definitions above imply:

$$
\begin{aligned}
&[a(\psi_g), a^*(\psi_f)]\psi_1\otimes...\otimes\psi_n := (a(\psi_g)a^*(\psi_f) - a^*(\psi_f)a(\psi_g))\psi_1\otimes...\otimes\psi_n \\
&= \sqrt{n+1}a(\psi_g)\psi_f\otimes\psi_1\otimes...\psi_n - \sqrt{n}\langle\psi_g|\psi_1\rangle a^*(\psi_f)\psi_2\otimes...\otimes\psi_n \\
&= (n+1)\langle\psi_g|\psi_f\rangle\psi_1\otimes...\otimes\psi_n - n\langle\psi_g|\psi_1\rangle\psi_f\otimes\psi_2...\otimes\psi_n. \qquad (4.14)
\end{aligned}
$$

The operator $a(\psi_f)$ is called the **annihilator operator**. It has this name because, when acting on a state of $n-$ particles, it returns a state with $n-1$ particles. Note, however, that this is not the operator acting in the Fock space since the Fock space is constituted by symmetric products. We can extend the definition to the Fock space by first symmetrizing $\psi_1 \otimes ...\psi_n$ so that it becomes an element of the Fock space, applying $a(\psi_f)$ or $a^*(\psi_f)$ and then symmetrizing the result once again. In mathematical terms:

$$a^*_{\mathfrak{F}}(\psi_f) := \mathfrak{S}a^*(\psi_f)\mathfrak{S} \tag{4.15}$$
$$a_{\mathfrak{F}}(\psi_f) := \mathfrak{S}a(\psi_f)\mathfrak{S}. \tag{4.16}$$

Although the formula above is well defined, there is some redundancy in it. For example, in the definition of $a^*_{\mathfrak{F}}(\psi_f)$ we consider all the permutations of $\psi_1 \otimes .... \otimes \psi_n$, add $\psi_f\otimes$ to the first entry, and then consider all the permutations once again. We can simply add ψ_f to the first entry and then take the permutations. Hence, we conclude:

$$a^*_{\mathfrak{F}}(\psi_f) = \mathfrak{S}a^*(\psi_f). \tag{4.17}$$

Regarding the operator $a_{\mathfrak{F}}(\psi_f)$, we could do a similar analysis as in [33] or we can simply take the adjoint:

$$a_{\mathfrak{F}}(\psi_f) = (a^*_{\mathfrak{F}}(\overline{\psi_f}))^* = a(\psi_f)\mathfrak{S}. \tag{4.18}$$

These operators are well defined in the Fock space of bosons. A similar construction can be made for the fermionic Fock space [33]. A special case to look at is the action of $a_{\mathfrak{F}}(\psi_f)$ in the vacuum defined as:

$$\Omega := (1, 0, ...). \tag{4.19}$$

We expect $a_{\mathfrak{F}}\Omega = 0$. That is indeed the case since $\forall n$:

$$\langle \Omega a_{\mathfrak{F}}(\overline{\psi_f})|\Phi_n\rangle = \langle \Omega|a^*_{\mathfrak{F}}(\psi_f)\Phi_n\rangle = \langle 1|0\rangle_{\mathcal{H}^0} + \lambda_f\langle 0|\mathfrak{S}(f\otimes\Phi_n)\rangle_{\mathcal{H}^{\otimes n+1}} = 0. \tag{4.20}$$

Using the properties above, we can calculate the commutators (we omit the symbol for the tensor product in the following calculation):

$$\begin{aligned}[][a_{\mathfrak{F}}(\psi_g), a^*_{\mathfrak{F}}(\psi_f)]\psi_1...\psi_n &= \left(a_{\mathfrak{F}}(\psi_g)a^*_{\mathfrak{F}}(\psi_f) - a^*_{\mathfrak{F}}(\psi_f)a_{\mathfrak{F}}(\psi_g)\right)\psi_1...\psi_n \\ &= \left(a(\psi_g)\mathfrak{S}\mathfrak{S}a^*(\psi_f) - \mathfrak{S}a^*(\psi_f)a(\psi_g)\mathfrak{S}\right)\psi_1...\psi_n \end{aligned} \tag{4.21}$$

To make reading easier, we perform the calculation separately. In the first term, we use $\mathfrak{S}^2 = \mathfrak{S}$.

$$
\begin{aligned}
a(\psi_g)\mathfrak{S}a^*(\psi_f)\psi_1...\psi_n &= a(\psi_g)\mathfrak{S}\sqrt{n+1}\psi_f\psi_1...\psi_n \\
&= \frac{\sqrt{n+1}}{(n+1)!}a(\psi_g)\sum_{\pi\in S^{n+1}}\psi_{\pi(f)}\psi_{\pi(1)}...\psi_{\pi(n)} \\
&= \frac{n+1}{(n+1)!}\langle\psi_g|\psi_f\rangle\sum_{\pi\in S^n}\psi_{\pi(1)}...\psi_{\pi(n)} \\
&+ \frac{n+1}{(n+1)!}\sum_{\pi\in S^{n+1},i\neq f}\langle\psi_g|\psi_{\pi(i)}\rangle\psi_{\pi(f)}...\widehat{\psi_{\pi(i)}}..\psi_{\pi(n)}.
\end{aligned} \tag{4.22}
$$

Where $\widehat{\psi_{\pi(i)}}$ means that the term is not part of the tensor product. In the above equation, we divided the sum in two terms, one proportional to $\langle\psi_g|\psi_f\rangle$ and the other excluding exactly that product. The reason for doing it will become clear once we have calculated the next term.

$$
\begin{aligned}
\mathfrak{S}a^*(\psi_f)a(\psi_g)\mathfrak{S}\psi_1...\psi_m &= \frac{1}{n!}\mathfrak{S}a^*(\psi_f)a(\psi_g)\sum_{\pi\in S^n}\psi_{\pi(1)}...\psi_{\pi(n)} \\
&= \frac{\sqrt{n}}{n!}\mathfrak{S}a^*(\psi_f)\sum_{\pi\in S^n}\langle\psi_g|\psi_{\pi(1)}\rangle\psi_{\pi(2)}...\psi_{\pi(n)} \\
&= \frac{n}{n!}\mathfrak{S}\sum_{\pi\in S^n}\langle\psi_g|\psi_{\pi(1)}\rangle\psi_f\psi_{\pi(2)}...\psi_{\pi(n)} \\
&= \frac{n(n-1)!}{n!n!}\sum_{\pi\in S^{n+1},i\neq f}\langle\psi_g|\psi_{\pi(i)}\rangle\psi_{\pi(f)}...\widehat{\psi_{\pi(i)}}..\psi_{\pi(n)}.
\end{aligned} \tag{4.23}
$$

The last factorial term is far from obvious and deserves a better explanation. For that, consider a fix term $\pi(1)$ from the combination $\pi\in S^n$. We want to calculate $\mathfrak{S}\sum_{\pi\in S^{n-1}}\langle\psi_g|\psi_{\pi(1)}\rangle\psi_f...\psi_{\pi(n)}$. Note that since it is a finite sum and $\psi_{\pi(1)}$ is fixed, we can write:

$$
\begin{aligned}
&\mathfrak{S}\sum_{\pi\in S^{n-1}}\langle\psi_g|\psi_{\pi(1)}\rangle\psi_f...\psi_{\pi(n)} \\
&= \langle\psi_g|\psi_{\pi(1)}\rangle\sum_{\pi\in S^{n-1}}\mathfrak{S}\psi_f...\psi_{\pi(n)} \\
&= \frac{(n-1)!}{n!}\langle\psi_g|\psi_{\pi(1)}\rangle\sum_{\pi\in S^n}\psi_{\pi(f)}...\psi_{\pi(n)}.
\end{aligned} \tag{4.24}
$$

A quick explanation of the combinatorial factor: The $n!$ term in the denominator is due to the definition of $\mathfrak{S}$. The $(n-1)!$ in the numerator is because $\mathfrak{S}$ acts on $(n-1)!$ terms in the sum over $\pi\in S^{n-1}$ and the action of $\mathfrak{S}$ is the same in all those terms.

Combining the results above, we conclude:

$$
\begin{aligned}
&[a_{\mathfrak{F}}(\psi_g), a^*_{\mathfrak{F}}(\psi_f)]\psi_1...\psi_n \\
&= \frac{n+1}{(n+1)!}\langle\psi_g|\psi_f\rangle \sum_{\pi\in S^n} \psi_{\pi(1)}...\psi_{\pi(n)} + \frac{n+1}{(n+1)!} \sum_{\pi\in S^{n+1}, i\neq f} \langle\psi_g|\psi_{\pi(i)}\rangle\psi_{\pi(f)}...\widehat{\psi_{\pi(i)}}..\psi_{\pi(n)} \\
&- \frac{n(n-1)!}{n!n!} \sum_{\pi\in S^{n+1}, i\neq f} \langle\psi_g|\psi_{\pi(i)}\rangle\psi_{\pi(f)}...\widehat{\psi_{\pi(i)}}..\psi_{\pi(n)} \\
&= \frac{n+1}{(n+1)!}\langle\psi_g|\psi_f\rangle \sum_{\pi\in S^n} \psi_{\pi(1)}...\psi_{\pi(n)} = \langle\psi_g|\psi_f\rangle \sum_{\pi\in S^n} \frac{1}{n!}\psi_{\pi(1)}...\psi_{\pi(n)} = \langle\psi_g|\psi_f\rangle\mathfrak{S}\psi_1...\psi_n. \quad (4.25)
\end{aligned}
$$

Hence:

$$[a_{\mathfrak{F}}(\psi_g), a^*_{\mathfrak{F}}(\psi_f)] = \langle\psi_g|\psi_f\rangle. \quad (4.26)$$

Using similar arguments, we can also prove:

$$[a^*_{\mathfrak{F}}(\psi_f), a^*_{\mathfrak{F}}(\psi_g)] = [a_{\mathfrak{F}}(\psi_f), a_{\mathfrak{F}}(\psi_g)] = 0. \quad (4.27)$$

In order to define a field, we "decompose" $a^*_{\mathfrak{F}}(\psi)$ and $a_{\mathfrak{F}}(\psi)$ in terms of operator-valued functionals that express the linear map $\psi \to a^*_{\mathfrak{F}}(\psi)$ as follows:

$$a^*(\psi) = \int \frac{d\vec{p}}{2\omega_p}\, a^*(\vec{p})\psi(\vec{p}). \quad (4.28)$$

Using (4.26), one can show that:

$$[a^*(\vec{p}), a(\vec{q})] = 2\hbar\omega_p\delta(\vec{p}-\vec{q}). \quad (4.29)$$

Last but not least, we introduce the time using the Schrödinger equation:

$$i\hbar\frac{d}{dt}\psi = H\psi \Rightarrow \psi(t) = e^{-iHt}\psi. \quad (4.30)$$

These functions are used as arguments for the creation and annihilation operators:

$$
\begin{aligned}
a^*_{\mathfrak{F}}(\psi(-t)) &= \int \frac{d\vec{p}}{2\omega_p}\hat{\psi}(-t,\vec{p})a^*(\vec{p}) = \int \frac{d\vec{p}}{2\omega_p}\psi(\vec{p})e^{i\omega_p t}a^*(\vec{p}) \\
&= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{x} \int \frac{d\vec{p}}{2\omega_p}\psi(\vec{x})e^{i\omega_p t - i\vec{p}\vec{x}}a^*(\vec{p}). \quad (4.31)
\end{aligned}
$$

Remark about notation: Based on the equation above, one usually defines

$$\begin{aligned}
&a^*(t,\vec{p}) := a^*(\vec{p})e^{i\omega_p t} \\
&a^*(x) \equiv a^*(t,\vec{x}) := \frac{1}{(2\pi)^{\frac{d-1}{2}}}\int \frac{d\vec{p}}{2\omega_p}e^{ipx}a^*(\vec{p})\bigg|_{p^0=\omega_p}.
\end{aligned} \tag{4.32}$$

Such that

$$\psi(t,\vec{x}) = \int d\vec{x}\, a^*(t,\vec{x})\psi(\vec{x}). \tag{4.33}$$

From the above definition, we also deduce easily

$$\begin{aligned}
[a(t,\vec{x}), a^*(t',\vec{y})] &= \frac{1}{(2\pi)^{d-1}}\int \frac{d\vec{p}d\vec{q}}{4\omega_p\omega_q}e^{-ipx+iqy}[a(\vec{p}), a^*(\vec{q})]\bigg|_{p^0=\omega_p, q^0=\omega_q} \\
&= \frac{1}{(2\pi)^{d-1}}\int \frac{d\vec{p}}{4\omega_p\omega_q}e^{-ipx+iqy}2\omega_p\hbar\delta(\vec{p}-\vec{q}) = \hbar\Delta^+(x-y).
\end{aligned} \tag{4.34}$$

4.2.1 Normal ordering

The last step we need to introduce is what is called **normal ordering** or **Wick product**. First, we define the field as:

$$\phi^{\mathrm{op}}(x) := a(x) + a^*(x) = \frac{1}{(2\pi)^{\frac{d-1}{2}}}\int \frac{d\vec{p}}{2\omega_p}a^*(\vec{p})e^{ipx} + a(\vec{p})e^{-ipx}\bigg|_{p^0=\omega_p}. \tag{4.35}$$

The sup-script "op" is to distinguish the field as an operator (valued-distribution) and the field as a distribution (will be introduced later). For now on, we use the notation $x = (t,\vec{x}) \equiv (x^0,\vec{x})$ and $px \equiv p_\mu x^\mu$.

The normal ordering is basically "ordering the creation and annihilation operators such that the annihilation operators are on the left and the creation operators are on the right". A formal definition can be written as [24]:

$$: \phi^{\mathrm{op}}(x_1)...\phi^{\mathrm{op}}(x_n) ::= \sum_{J\subset\{1,...,n\}}\prod_{j\in J}a^*(x_j)\prod_{k\in J^c}a(x_k). \tag{4.36}$$

The formula above is well defined, but it is pedagogical to calculate some examples:

$$
\begin{aligned}
&: a^*(x)a^*(y) := a^*(x)a^*(y) \\
&: a(x)a(y) := a(x)a(y) \\
&: a(x)a^*(y) := a^*(x)a(y) \\
&: a^*(x)a(y)a^*(w)a^*(z) := a^*(x)a^*(w)a(y)a(z).
\end{aligned}
\tag{4.37}
$$

We also define the product of normal ordering product following common sense:

$$
:(:\phi^{\mathrm{op}}(x_1)...\phi^{\mathrm{op}}(x_n):)(:\phi^{\mathrm{op}}(y_1)...\phi^{\mathrm{op}}(y_m):)::=:\phi^{\mathrm{op}}(x_1)...\phi^{\mathrm{op}}(x_n)\phi^{\mathrm{op}}(x_1)...\phi^{\mathrm{op}}(x_n):. \tag{4.38}
$$

The main reason to introduce the normal ordering is to avoid pathological product of fields such as:

$$
\begin{aligned}
\langle\Omega|(\phi^{\mathrm{op}}(x))^2\Omega\rangle &= \langle\Omega|(a^*(x)a^*(x) + a(x)a(x) + a^*(x)a(x) + a^*(x)a(x))\Omega\rangle \\
&= \langle\Omega|a^*(x)a(x)\Omega\rangle = \langle\Omega|[a^*(x)a(x)]\Omega\rangle = \hbar\Delta^+(0) = \infty.
\end{aligned}
\tag{4.39}
$$

In the above equation, we used $a^*a\Omega = 0$. On the other hand, using the last line of (4.37):

$$
\langle\Omega| : (\phi^{\mathrm{op}}(x))^2 : \Omega\rangle = 0. \tag{4.40}
$$

Thus, the expectation values of the physical quantities are described by the normal ordering of those quantities and not the usual product.

Chapter 5

Classical fields

5.1 Introduction

Now we turn our attention to the formalism we will use for the rest of the work. The idea is that the fields in the Fock space are operators that when computed in some state return a distribution. We want to construct our formalism independent of the state, so we introduce the field as a functional: an object that receives a function as an argument and returns a number (in general a complex number, for the real scalar field a real number). The main difference from this formalism to the usual one (just discussed above) is that the field is not an operator and does not respect any kind of "field equation" in general. The goal is to make the construction precise and self-consistent. We will follow the first chapter of [24].

5.2 Basic structure

For our proposes we define the fields as functionals acting on the configuration space with image on $\mathbb{R}$ or $\mathbb{C}$. The **configuration space**, unless explicitly stated otherwise, is the set of smooth functions on Minkowski space-time in d dimensions, i.e., $C^\infty(\mathbb{M}) \equiv \mathcal{C}$. A **scalar field** is defined as

$$\phi(x) : \begin{cases} \mathcal{C} \to \mathbb{R} \\ h \mapsto h(x) \end{cases} \tag{5.1}$$

.We can then build the derivative of fields in the same way:

$$\partial^a \phi(x) : \begin{cases} \mathcal{C} \to \mathbb{R} \\ h \mapsto \partial^a h(x) \end{cases} \tag{5.2}$$

.Where a is an abbreviation for a multi-index. We denote the **set of polynomials** formed by $\partial^a \phi$ by $\mathcal{P}$.

5.2.1 The space of fields $\mathcal{F}$

We define the **space of fields** as the set of all functions $F \equiv F(\phi)$ of the following form:

$$F = f_0 + \sum_{n=1}^{N} \int dx_1 \cdots dx_n \, f_n(x_1, \cdots, x_n)\phi(x_1)\cdots\phi(x_n) \tag{5.3}$$

where $N < \infty$ and $f_n(x_1, \cdots, x_n)$ is a $\mathbb{C}$-valued distribution (i.e., $f_n \in \mathcal{D}'(\mathbb{M}^d, \mathbb{C})$) that is

(i) symmetric in its arguments: $f_n(x_{\pi(1)}, \cdots, x_{\pi(n)}) = f_n(x_1, \cdots, x_n)$ for all permutations $\pi \in S^n$

(ii) whose wave front set satisfies the following property:

$$WF(f_n) \subseteq \left\{ (x_1, \cdots, x_n, k_1, \cdots, k_n) | (k_1, \cdots, k_n) \notin \overline{V}_+^{\times n} \cup \overline{V}_-^{\times n} \right\}$$

.

Where $V_{+,-}$ is the forward/backyard light cone.

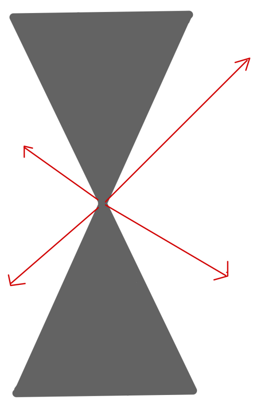


Figure 5.1: The red arrows indicate the allowed directions of singularity propagation

The set $\overline{V}_\pm$ is simply the closure of $V_\pm$. The first property is imposed by simplicity and the second one is a (very important) technicality needed to construct a consistent theory. We will not explore many of these technicalities, but we strongly recommend the discussion of [24]. A crucial remark is that since f_n has compact support, we do not have to impose any decay properties on $h \in \mathcal{C}$.

5.2.2 The set of local fields

We define the set of **local fields** as a subset of the set of fields formed by a linear combination of fields of the form (5.3). More explicitly:

$$\mathcal{F}_{\text{loc}} := \left\{ \sum_{i=1}^{K} \int dx A_i(x) g_i(x) \middle| A_i \in \mathcal{P}, g_i \in \mathcal{D}(\mathbb{M}), K < \infty \right\}. \tag{5.4}$$

Most of the development done in perturbative quantum field theory is done in the set of local fields.

5.2.3 Derivative of functionals

The next tool we introduce is the (functional) derivative denoted by $\frac{\delta}{\delta\phi(x)}$. Given a field $F \in \mathcal{F}$,

$$F = f_0 + \sum_{n=1}^{N} \int dx_1 \cdots dx_n \, f_n(x_1, \cdots, x_n)\phi(x_1) \cdots \phi(x_n),$$

then its derivative with respect to the field ϕ is given by:

$$\begin{aligned} &\frac{\delta^k F}{\delta\phi(y_1) \cdots \delta\phi(y_k)} \\ &:= \sum_{n=k}^{N} \frac{n!}{(n-k)!} \int dx_1 \cdots dx_{n-k} \phi(x_1) \cdots \phi(x_{n-k}) f_n(y_1, \cdots, y_k, x_1, \cdots, x_{n-k}) \end{aligned} \tag{5.5}$$

Although not very illustrative at first, the formula is quite simple in practice. All you have to do is make the substitution

$$\int dx \, f(x) \frac{\delta\phi(x)}{\delta\phi(y)} = \int dx \, f(x) \delta(x-y) = f(y). \tag{5.6}$$

If we have more than one field:

$$\begin{aligned} &\frac{\delta}{\delta\phi(y)} \int dx_1 dx_2 \, f(x_1, x_2)\phi(x_1)\phi(x_2) \\ &= \int dx_1 dx_2 \, f(x_1, x_2) \left(\frac{\delta\phi(x_1)}{\delta\phi(y)} \phi(x_2) + \phi(x_1) \frac{\delta\phi(x_2)}{\delta\phi(y)} \right) \\ &= \int dx_1 \, f(x_1, y)\phi(x_1) + \int dx_2 \, f(y, x_2)\phi(x_2) = 2 \int dx f(x, y)\phi(x). \end{aligned} \tag{5.7}$$

In the last step, we used the symmetry of f. Using the properties above, we can also write:

$$\int dx\, g(x) \frac{\delta F}{\delta \phi(x)}(h) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} F(h+\epsilon g). \tag{5.8}$$

We could check the formula above explicitly, but it is more instructive to calculate an example since it generalizes trivially. Let us take as an example the field discussed above $F = \int dx_1 dx_2\, f(x_1, x_2)\phi(x_1)\phi(x_2)$. On the right-hand side of (5.8) we have:

$$\begin{aligned}\int dx\, g(x) \frac{\delta F}{\delta \phi(x)}(h) &= \int dx g(x) \left(2 \int dy\, f(x,y)\phi(y)\right)(h) \\ &= \int dx dy\, 2g(x)h(y)f(x,y).\end{aligned} \tag{5.9}$$

On the other hand,

$$\begin{aligned}&\frac{d}{d\epsilon}F(h+\epsilon g) = \int dx_1 dx_2 f(x_1,x_2) \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (h(x_1)+\epsilon g(x_1))(h(x_2)+\epsilon g(x_2)) \\ &= \int dx_1 dx_2 (g(x_1)h(x_2) + g(x_2)h(x_1)) f(x_1,x_2) = \int dx_1 dx_2\, 2h(x_1)g(x_2)f(x_1,x_2)\end{aligned} \tag{5.10}$$

and these prove the assertion. Using these properties, we can also prove the Leibniz rule:

$$\frac{\delta(F\cdot G)}{\delta\phi(x)} = \frac{\delta F}{\delta \phi(x)} \cdot G + F \cdot \frac{\delta G}{\delta\phi(x)} \quad \forall F, G \in \mathcal{F}. \tag{5.11}$$

That is a direct consequence of the properties of the derivative:

$$\begin{aligned}&\frac{d}{d\epsilon}(F\cdot G)(h+\epsilon g) = \frac{d}{d\epsilon}\left(F(h+\epsilon g)G(h+\epsilon h)\right) \\ &= \frac{dF(h+\epsilon g)}{d\epsilon} G(h+\epsilon) + F(h+g\epsilon)\frac{dG(h+\epsilon g)}{d\epsilon} \\ &= \int dx\, g(x) \left(\frac{\delta F}{\delta\phi(x)}(h)G(h) + F(h)\frac{\delta G}{\delta \phi(x)}(h)\right).\end{aligned} \tag{5.12}$$

If F is not only a function of the field but of $\partial^a \phi$, then we can compute the derivative of F using integration by parts:

$$F = \int dx\, f(x)\partial^a\phi(x) = (-1)^{|a|} \int dx\, (\partial^a f(x))\phi(x) \Rightarrow \frac{\delta F}{\delta\phi(y)} = (-1)^{|a|}\partial^a f(y). \tag{5.13}$$

A detail that we will not explore in this work but write just for completeness, we define the support of a functional using its derivative:

$$\operatorname{supp} F := \operatorname{supp} \frac{\delta F}{\delta\phi(\cdot)} \equiv \overline{\bigcup_{h\in\mathcal{C}} \operatorname{supp} \frac{\delta F}{\delta\phi(\cdot)}(h)}. \tag{5.14}$$

5.2.4 Classical product

From the above discussion, one can easily check that both $\mathcal{F}$ and $\mathcal{F}_{\text{loc}}$ form a vector space. Since we do not introduce a proper topology in these spaces, convergence is understood in the pointwise sense: For $F, F_n \in \mathcal{F}$, we say $F_n \to F$ if and only if

$$\lim_{n\to\infty} F_n(h) = F(h), \quad \forall h \in \mathcal{C}. \tag{5.15}$$

The product of fields is also understood as a pointwise product:

$$F \cdot G \equiv FG : h \mapsto F(h)G(h). \tag{5.16}$$

We also introduce a conjugation $F \mapsto F^*$:

$$F^* := \sum_n \langle \overline{f_n}, \phi^{\otimes n} \rangle \equiv \sum_n \int dx_1 \cdots dx_n \, \phi(x_1)\cdots\phi(x_n)\overline{f}_n(x_1, \cdots, x_n). \tag{5.17}$$

where $\overline{f}$ denotes the complex conjugation of f. And the parity transformation:

$$\begin{aligned} &\alpha(F) := \sum_n \int dx_1 ... dx_n \alpha(\phi(x_1)) ... \alpha(\phi(x_n)) f_n(x_1, ..., x_n) \\ &\alpha(\phi(x_1)) := -\phi(x_1). \end{aligned} \tag{5.18}$$

Technical note

To prove that the classical product is well defined, that is, $\cdot : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, we need to prove that the defining conditions (5.2.1) are satisfied. The symmetry is immediate from the definition. For $F = \int dX_n \, f(x_1, ..., x_n)\phi(x_1)...\phi(x_n)$ and $G = \int dY_m \, g(y_1, ..., y_m)\phi(y_1)...\phi(y_m)$:

$$FG(h) = \int dX_n dY_m f(x_1, ..., x_n) g(y_1, ...y_m) h(x_1)...h(x_n) h(y_1)...h(y_m). \tag{5.19}$$

The wave front set condition is also immediate. To simplify the notation, suppose that $F = \int dx \, f(x)\phi(x)$ and $G = \int dy \, g(y)\phi(y)$. The generalization to fields with higher powers is immediate:

$$\begin{aligned}
&\mathrm{WF}(f(x)\otimes g(y))\\
&=\{(x,y,k_x,k_y)|x\in \text{sing supp}(f), y\in \text{sing supp}(g), k_x^2<0, k_y^2<0\}.
\end{aligned} \tag{5.20}$$

(Note that we have a tensor product of distributions, i.e. $f\otimes g\in\mathcal{D}'(\mathbb{M}^2)$. That is different for the product of $fg\in\mathcal{D}'(\mathbb{M})$. The same argument holds for $\overline{f}(x_1,...,x_n)$)

5.2.5 Classical theory

Now that the technical structure has been introduced, we can start discussing physics. We start with the field equation for the free field:

$$S_0 := \int dx \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \equiv \int dx\, L_0(x). \tag{5.21}$$

Note that $S_0\notin\mathcal{F}$. Even though it seems counterintuitive at first, that is not a problem since we normally work with the derivative of the free action

$$\frac{\delta S_0}{\delta\phi(x)} = -\left(\partial^2+m^2\right)\phi(x)$$

Remark: we denote $\partial_\mu\partial^\mu\equiv\partial^2\equiv\Box$.

5.2.6 Generalized lagrangian

We define the generalized Lagrangian as the interacting part of the action but, differently from the free action, the interacting lagrangian is local. This means, it is defined through

$$\mathcal{L}:\mathcal{D}(\mathbb{M})\to\mathcal{F}_{\text{loc}}. \tag{5.22}$$

with the properties:

$$\begin{aligned}
&\operatorname{supp}\mathcal{L}(f)\subseteq\operatorname{supp} f,\quad \forall f\in\mathcal{D}(\mathcal{M}),\quad \mathcal{L}(0)=0\\
&\mathcal{L}(f+g+h)=\mathcal{L}(f+g)-\mathcal{L}(g)+\mathcal{L}(g+h) \text{ if } \operatorname{supp}(f)\cap\operatorname{supp}(h)=\emptyset.
\end{aligned} \tag{5.23}$$

Note that the property above basically states that we can factorize the Lagrangian if its arguments do not overlap. The idea becomes clear if we set $g(x)=0$:

$$\mathcal{L}(f+h)=\mathcal{L}(f)+\mathcal{L}(h). \tag{5.24}$$

We can give an explicit example. Consider $\mathcal{L}(\phi)=\int dx\, g(x)\phi^n(x)$,then:

$$\mathcal{L}(f+h) = \int_{\mathrm{supp}(f)} g(x)(f(x)+h(x))^n + \int_{\mathrm{supp}(h)} g(x)(f(x)+h(x))^n$$
$$= \int_{\mathrm{supp}(f)} g(x)(f(x))^n + \int_{\mathrm{supp}(h)} g(x)(h(x))^n = \mathcal{L}(f) + \mathcal{L}(h). \tag{5.25}$$

The interacting Lagrangian is local, therefore we do not have problems with convergence in the IR[1], since for large distances, all elements of $\mathcal{D}(\mathbb{M})$ are 0.

We are also able to define equivalence relations between different Lagrangians. $\mathcal{L}_1 = \int dx\, f(x) L_1(x)$ and $\mathcal{L}_2 = \int dx\, f(x) L_2(x)$ are called equivalent if and only if

$$\mathrm{supp}_x \left(\frac{\delta \mathcal{L}_1}{\delta \phi(x)}(f) - \frac{\delta \mathcal{L}_2}{\delta \phi(x)}(f) \right) \equiv \mathrm{supp}(\mathcal{L}_1(f) - \mathcal{L}_2(f)) \subseteq \bigcup_{\mu=0}^{d-1} \mathrm{supp}\, \partial_\mu f, \quad \forall f$$
$$\iff L_1 - L_2 = c + \partial_\mu A^\mu. \tag{5.26}$$

The free action of a real scalar field is then the equivalence class of a generalized Lagrangian $\mathcal{L}_0(f) := \frac{1}{2} \int dx\, L_0(x) f(x)$. Last but not least, we define the action of the interaction $S \in \mathcal{L}_{\mathrm{loc}}$:

$$S = \int dx \mathcal{L}_{\mathrm{int}}(x); \quad \mathcal{L}_{\mathrm{int}}(x) = -\kappa g(x) L_{\mathrm{int}}(x); \tag{5.27}$$

The field equation is:

$$\frac{\delta(S+S_0)}{\delta \phi(x)} = 0; \tag{5.28}$$

If S depends only on ϕ and $\partial^\mu \phi$, we recover the Euler-Lagrange equation:

$$\frac{\partial(\mathcal{L}+\mathcal{L}_0)}{\partial \phi} = \partial^\mu \frac{\partial(\mathcal{L}+\mathcal{L}_0)}{\partial(\partial^\mu \phi)}; \tag{5.29}$$

The space of smooth functions that solve the equation above is denoted by:

$$\mathcal{C}_{S+S_0} := \left\{ h \in \mathcal{C} \middle| \frac{\delta(S_0+S)}{\delta \phi(x)}(h) = 0 \quad \forall x \in \mathbb{M} \right\}. \tag{5.30}$$

The fields that solve the equation of motion are denoted by $F_S := F|_{\mathcal{C}_{S_0+S}}$ and are called the "on-shell fields". Fields that are not restricted to this domain are called off-shell fields.

[1]Convergence in the infrared is to be understood as convergence for large space-time arguments.

5.2.7 Poison algebra of the free theory

The main question of this section can be expressed as follows: we want to construct the poison algebra of the classical theory without mentioning the Hamiltonian and use the fact that our interaction is limited to a specific spacetime region. Such a construction has already been developed by Pierls in the early fifth's in [56]. The poison bracket of $F, G \in \mathcal{F}$ is simply given by:

$$\begin{aligned}\{F,G\} &:= \int dxdy\, \frac{\delta F}{\delta\phi(x)}(\Delta^{\text{ret}}(x-y) - \Delta^{\text{ret}}(y-x))\frac{\delta G}{\delta\phi(y)} \\ &\equiv \int dxdy\, \frac{\delta F}{\delta\phi(x)}\Delta(x-y)\frac{\delta G}{\delta\phi(y)}, \quad \Delta(x-y) = \Delta^{\text{ret}}(x-y) - \Delta^{\text{ret}}(y-x).\end{aligned} \tag{5.31}$$

$\Delta(x)$ is called the commutator function.

We can find a motivation for this definition in chapter 2 from [14]. The idea is simple: we start with an action S restricted to $[t_1, t_2] \times K$, K compact, and $\phi \in \mathcal{C}_S$. Then, we consider a new action given by $S + \lambda G$ and suppose that there is a function $r_{\lambda G}$ that maps solutions of S to solutions of $S + \lambda G$ such that the solutions coincide for $t < t_1$, before the interaction takes place. Similarly,we suppose the existence of another function $a_{\lambda F}$ that maps solutions of S to solutions of $S + \lambda F$ that coincide for $t > t_2$, after the interaction happened. We now calculate how $F(\phi)$ behaves when we change ϕ by $r_{\lambda G}(\phi)$ and how $G(\phi)$ changes when we change ϕ by $a_{\lambda F}(\phi)$ in the first order of λ:

$$\begin{aligned}D_G F(\phi) &:= \frac{d}{d\lambda}\bigg|_{\lambda=0} F(r_{\lambda G}(\phi)) \\ ᗡ_F G(\phi) &:= \frac{d}{d\lambda}\bigg|_{\lambda=0} G(a_{\lambda F}(\phi)).\end{aligned} \tag{5.32}$$

Then the Pierls bracket is simply the difference of the terms:

$$\{F,G\} := D_G F(\phi) - ᗡ_F G(\phi). \tag{5.33}$$

Physically, we are comparing small deviations for the trajectory for a particle that comes from $t = -\infty$ and passes through a region in which there is an interaction $S + \lambda G$ with the trajectories of a particle that comes from $t = \infty$ and passes through a region with interaction $S + \lambda F$. Hopefully, the concept will become clearer when we introduce the retarded expansion of fields.

Another possible explanation is that we are following the process in reverse: we already know the answer in the quantum case and are trying to recover the classical structure from it. Although not entirely sufficient, this can serve as a motivation for defining the Poisson bracket in this way. Now that we have defined it, we need to prove its properties to justify the name. The properties are[24]:

i) The pointwise product of distributions in (5.31) exists due to the wave front set properties of F and G.Moreover, $\{F, G\}$ again satisfies the wave front condition, hence $\{F, G\} \in \mathcal{F}$.

ii) $\{F, G\}$ is bilinear in F and G.

iii) $\{F, G\} = -\{G, F\}$.

iv) $\{F, GH\} = \{F, G\}H + \{F, H\}G$.

v) $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$.

The properties ii, iii, iv immediately follow from the definition. To prove i, we need a generalization of the Hörmander criterion for bi-distributions. The proof of this assertion will be omitted. The item v can be proved simply by applying the definition and using $\Delta(-x) = -\Delta(x)$. The computation can be found at chapter 1 from [24].

Remark: Although not obvious at first sight, the Pierls bracket is a generalization of the usual Poisson bracket of the classical theory. To prove the statement, we calculate $\{\phi(y), \partial_t\phi(x) \equiv \partial_{x^0}\phi(x)\}$.

$$\begin{aligned}\{\phi(y), \partial_t\phi(x)\} &= \int dx'dy' \, \frac{\delta\phi(y)}{\delta\phi(y')}\Delta(y'-x')\frac{\delta\partial_t\phi(x)}{\delta\phi(x')} \\ &= \int dx'dy' \, \delta(y-y')\Delta(y'-x')\partial_t\delta(x-x') \\ &= -\partial_{x^0}\Delta(y-x) = \partial_{x^0}\Delta(x-y). \end{aligned} \tag{5.34}$$

In the above equation we have used: $\Delta(-z) = -\Delta(z)$. Now we use:

$$\begin{aligned}\Delta(x-y) &= \frac{-i}{(2\pi)^{d-1}} \int dp \; \mathrm{sgn}(p^0)\delta(p^2-m^2)e^{-ip(x-y)} \\ \Rightarrow \partial_{x_0}\Delta(x-y) &= \frac{1}{(2\pi)^{d-1}} \int dp \, p^0 \, \mathrm{sgn}(p_0)\delta(p^2-m^2)e^{-ip(x-y)} \end{aligned} \tag{5.35}$$

Net we perform the integration on p_0. To do it, note that:

$$p^0 \, \mathrm{sgn}(p^0) = |p^0| \quad \delta(p^2-m^2) = \delta((p^0)^2-\omega_p^2) = \frac{\delta(p^0-\omega_p)}{2\omega_p} + \frac{\delta(p_0+\omega_p)}{2\omega_p}. \tag{5.36}$$

Thus,

$$
\begin{aligned}
&\int dp\, p^0 \,\mathrm{sgn}(p_0)\delta(p^2 - m^2)e^{-ipx} \\
&= \int \frac{d\vec{p}}{2\omega_p} \left(|\omega_p| e^{-i(x^0-y^0)\omega_p + i(\vec{x}-\vec{y})\vec{p}} + |-\omega_p| e^{i(x^0-y^0)\omega_p + i(\vec{x}-\vec{y})\vec{p}} \right) \\
&= \int d\vec{p}\, e^{i\vec{p}(\vec{x}-\vec{y})} \cos(\omega_p(x^0 - y^0)).
\end{aligned}
\tag{5.37}
$$

Hence, we obtained:

$$
\{\phi(y), \partial_t\phi(x)\}\Big|_{y^0=x^0} = \frac{1}{(2\pi)^{d-1}} \int d\vec{p}\, e^{i\vec{p}(\vec{x}-\vec{y})} = \delta^{d-1}(\vec{x} - \vec{y}). \tag{5.38}
$$

The formula above normally is the starting point to quantize the theory. Here we will keep the tradition alive and quantize the theory in the next chapter.

Chapter 6

Quantization

6.1 Introduction

The passage from the classical theory to the quantum theory is done by means of a quantization procedure. The most popular ones in the physics literature are probably the canonical quantization and the path integral quantization. The last one is polemical in the area of mathematical physics since it is not always a well-defined object (see, for example, [53] for a discussion on the subject).

Nevertheless, the canonical quantization is also not the best option for our formalism. The "problem" of canonical quantization is the promotion of fields to operators (there are other more technical problems that can be pointed out, but are not of relevance here, see, for example, [37]). In our formalism, the Fock space is constructed only in a later stage and therefore such a promotion is also not desired. We want the fields to remain fields! The solution of our problem is quantizing the theory using what is called "deformation quantization". This scheme has its roots in the work of von Neumann [54], but was developed formally only at the end of 1970s [6]. For a pedagogical review, we suggest [45, 40]. For (now not so) recent advances [22, 38, 78].

6.2 Deformation quantization

Just as the Pierls bracket is somehow a "generalization" of the usual Poisson bracket, the formalism used to quantized the theory is somehow a generalization of the usual commutators. It consists of expanding the usual commutators (for example $[q,p] = i\hbar$) in a power series in $\hbar$. To do it, we introduce a "star product" $\star_\hbar : \mathcal{F} \times \mathcal{F} \to \mathcal{F}[[\hbar]]$ that satisfies the following conditions:

a) bilinar in its arguments

b) associative

c) $F \star_\hbar G \to F \cdot G$ as $\hbar \to 0$

d) $\frac{F\star_\hbar G - G\star_\hbar F}{i\hbar} \equiv \frac{[F,G]_{\star_\hbar}}{i\hbar} \to \{F, G\}$ as $\hbar \to 0$

In the context presented here, the discussion is still abstract, but it can be naturally introduced in non-relativistic quantum mechanics. To do it, we strongly recommend [45, 40]. The main advantage of the deformation quantization method is that one can have complicated classical structures and still be able to perform the quantization (which is particularly useful in curved space times). Since we are working in Minkowski space-time with polynomial fields, we have some freedom to choose the star product as a series in $\hbar$ with more restrictive axioms. Without further ado, we define the star product $\star_\hbar : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ of two fields $F, G \in \mathcal{F}$ by:

$$\begin{aligned} F \star_\hbar G := &\sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \int dx_1 \cdots dx_n dy_1 \cdots dy_n \\ \times &\frac{\delta^n F}{\delta\phi(x_1)\cdots\delta\phi(x_n)} \prod_{l=1}^{n} H_m(x_l - y_l) \frac{\delta^n G}{\delta\phi(y_1)\cdots\delta\phi(y_n)}. \end{aligned} \tag{6.1}$$

The star product has to obey:

a) Bi-linearity in its arguments

b) Associativity

c) $F \star_\hbar G \to F \cdot G$ as $\hbar \to 0$

d) $\frac{1}{i\hbar}[F, G]_\star := \frac{1}{i\hbar}(F \star_\hbar G - G \star_\hbar F) \to \{F, G\}$ for $\hbar \to 0$

e) $(\partial^2 + m^2)H_m(x - y) = 0 \iff (\partial_x^2 + m^2)(\phi(x) \star_\hbar \phi(y))_0 = 0$ for $\phi(x) \in F_{\mathcal{C}_{S'}}$

f) Lorentz invariance: $H_m(\Lambda z) = H_m(z) \forall \Lambda \in \mathcal{L}_+^\uparrow$

g) All powers of $H_m^l(y - x)$ must exist

h) $\overline{H}_m(x) = H(-x)$ which is equivalent to $(F \star_\hbar G)^* = G^* \star_\hbar F^*$

Remark: The series above does not have infinitely many terms, since the fields in which we are working are polynomial. For non-polynomial fields one has to be more careful about the series, but it is still possible to define a suitable star-product.

6.2.1 Wightman two-point function

As discussed in the articles on quantization deformation in quantum mechanics, we have several distributions H_m that satisfy the axioms listed above. In the treatment of perturbative quantum field theory in Minkowski spacetime, we will use the Wightman two-point function:

$$H_m(z) \equiv \Delta_m^+(z) := \frac{1}{(2\pi)^{d-1}} \int d^d p\, \theta(p^0)\delta(p^2 - m^2)e^{-ipz}$$
$$= \frac{1}{(2\pi)^{d-1}} \int d^{d-1}\vec{p}\, \frac{e^{-i\omega_p z^0 - i\vec{p}\vec{z}}}{2\omega_p}, \quad \omega_p := \sqrt{(\vec{p})^2 + m^2}. \tag{6.2}$$

Note that

$$\Delta_m^+(z) - \Delta_m^+(-z) \equiv i\Delta_m(z) = i(\Delta^{\text{ret}}(z) - \Delta^{\text{ret}}(-z)) \tag{6.3}$$

We will remove $\hbar$ from $\star_\hbar$ to make the notation more transparent. The proof that the star product as defined above exists and fulfills all the desired properties can be found in Chapter 2 of [24]. We restrict ourselves to some comments. First of all, we will explain why one chooses the Wightman two point function instead of the commutator function of the classical product whose choice would be more intuitive. Let us consider fields of the form $F = \int dx\, g(x)\phi^2(x), n \geq 2$ and naively try to calculate $F(x) \star F(y)$ using the commutator function instead of the Wightman two-point function:

$$F(x) \star F(y) = \int dxdy\, g(x)g(y)\phi^2(x)\phi^2(y) + 4\hbar \int dxdy\, g(x)g(y)\phi(x)\Delta(x-y)\phi(y)$$
$$+ \frac{4\hbar^2}{2} \int dxdy\, g(x)g(y) \ \left(\Delta(x-y)\right)^2. \tag{6.4}$$

Note that we naturally have powers of $\Delta(x-y)$. An important question to be asked at this stage is whether the product is well defined. This question can be answered by studying the wave front set of the commutator function mentioned in the beginning of the dissertation:

$$\text{WF}(\Delta) = \{(x,k) | x^2 = k^2 = 0, x = \lambda k \text{ for some } \lambda \in \mathbb{R}, k^0 \neq 0\}. \tag{6.5}$$

The Hörmander criterion for the multiplication of distributions states that for $t, s \in \mathcal{D}'(\mathbb{R}^d)$, the multiplication st of distributions exists if the set

$$\text{WF}(t) \oplus \text{WF}(s) := \{(x, k_1 + k_2) | (x, k_1) \in \text{WF}(t), (x, k_2) \in \text{WF}(s)\}. \tag{6.6}$$

does not contain an element of the form $(x, 0)$. Hence, the product $\Delta(x) \cdot \Delta(x)$ does not follow the criterion and therefore is a bad choice for the propagator (as we have seen, the product does not actually exist). Hence, if we use the commutator in place of the Wightman functions, we would exclude fields of the form $F = \int dx\, g(x)\phi^n(x), n \geq 2$. To solve this problem, we consider only the "positive frequencies" of the commutator function (the Wightman function).The wave front set of this new distribution is:

$$\mathrm{WF}(\Delta^+) = \{(x,k)|x^2 = k^2 = 0, x = \lambda k \text{ for some } \lambda \in \mathbb{R}, k^0 > 0\}. \tag{6.7}$$

Hence, products of the form $(\Delta^+(x))^n$ exists. If we restrict the space of fields and consider only fields of the form $F(\phi) = \int dX_n f(x_1, ..., x_n)\phi(x_1)...\phi(x_n)$, both definitions of the star product work. More than that, they are, in fact, equivalent(more details can be found in [14] chapter 2 and references therein).

Note that the usual condition for canonical quantization is recovered in our formalism. Just as in the classical case, consider $F = \phi(y)$ and $G = \partial_t\phi(x)$. We can calculate $[F,G]_\star$ given by:

$$[F,G]_\star = \phi(y) \star \partial_{x^0}\phi(x) - \partial_{x^0}\phi(x) \star \phi(y). \tag{6.8}$$

Using

$$\begin{aligned}
\phi(y) \star \partial_{x^0}\phi(x) &= \phi(y)\partial_{x^0}\phi(x) + \hbar \int dx'dy' \, \frac{\delta\phi(y)}{\delta\phi(y')}\Delta^+(y'-x')\frac{\delta\partial_{x^0}\phi(x)}{\delta\phi(x')} \\
&= \phi(y)\partial_{x^0}\phi(x) + \hbar \int dx'dy' \, \delta(y-y')\Delta^+(y'-x')\partial_{x^0}\delta(x-x') \\
&= \phi(y)\partial_{x^0}\phi(x) - \hbar\partial_{x^0}\Delta^+(y-x)
\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}
\partial_{x^0}\phi(x) \star \phi(y) &= \partial_{x^0}\phi(x)\phi(y) + \hbar \int dx'dy' \, \frac{\delta\partial_{x^0}\phi(x)}{\delta\phi(x')}\Delta^+(x'-y')\frac{\delta\phi(y)}{\delta\phi(y')} \\
&= \partial_{x^0}\phi(x)\phi(y) + \hbar \int dx'dy' \, \partial_{x^0}\delta(x-x')\Delta^+(x'-y')\delta(y-y') \\
&= \partial_{x^0}\phi(x)\phi(y) - \hbar\partial_{x^0}\Delta^+(x-y)
\end{aligned} \tag{6.10}$$

we obtain:

$$[F,G]_\star = \hbar\partial_{x_0}(\Delta^+(x-y) - \Delta^+(y-x)) = i\hbar\partial_{x^0}\Delta(x-y). \tag{6.11}$$

In the above equation, we have used the III from (3.5). Hence:

$$[\phi(y), \partial_t\phi(x)]\Big|_{x^0=y^0} = i\hbar\delta^{d-1}(\vec{y}-\vec{x}). \tag{6.12}$$

The usual canonical commutation relation.

Remark:there are other quantization schemes that lead to interesting physics. We emphasize the quantization done for treating thermal states [50] and the quantization procedure to work with quantum field theory with curved background and/or external potential. [51, 12].

6.3 States

A good introduction to the formalism behind the idea of states we use here, called GNS construction, can be found (in Portuguese) at [5] (Representação de Álgebras C^*). The text presented here was extracted (almost literally) from Chapter 2.5 [24].

Let us introduce some notation: We denote the space of fields that are polynomials in $\hbar$ by $\mathcal{F}_\hbar$, more precisely:

$$\mathcal{F}_\hbar := \left\{ \sum_{s=0}^{S} F_s \hbar^s | F_s \in \mathcal{F}, S < \infty \right\}. \tag{6.13}$$

A state ω on the algebra $\mathcal{A} \equiv \mathcal{A}_\hbar := (\mathcal{F}_\hbar, \star)$ is a map:

$$\omega : \begin{cases} \mathcal{A} \to \mathbb{C} \\ F \mapsto \omega(F) \equiv \omega(F)_\hbar \end{cases} \tag{6.14}$$

ω itself my be a polynomial in $\hbar$, i.e., $\omega = \sum_n^N \omega_n \hbar^N$ for some $N < \infty$ and which:

- Linear: $\omega(F + \alpha G) = \omega(F) + \alpha\omega(G)$, $\forall F, G \in \mathcal{A}$, $\alpha \in \mathbb{C}$,
- Real: $\omega(F^*)_\hbar = \overline{\omega(F)}_\hbar$, $\forall F \in \mathcal{A}$, $\forall \hbar > 0$,
- Positive: $\omega(F^* \star F)_\hbar \geq 0$, $\forall F \in \mathcal{A}$, $\forall \hbar > 0$,
- Normalized: $\omega(1) = 1$, where $1 \in \mathcal{F}_\hbar$ is the functional $1(h) = 1 \forall h \in \mathcal{C}$

Note that

a) If $F^* = F$, $\omega(F) \in \mathbb{R}$,

b) Linearity implies $\omega(F) \equiv \omega\left(\sum_r F_r \hbar^r\right) = \sum_{r,s} \omega_s(F_r)\hbar^{r+s}$. and the sum is finite!

We can now define a vacuum state ω_0 and a coherent state. Given $F = f_0 + \sum_{n\geq 1}\langle f_n, \phi^{\otimes n}\rangle$, we define the vacuum state as:

$$\omega_0(F) := f_0. \tag{6.15}$$

The natural next step is to justify the name "states" relating the concepts presented here to the ones usually studied in quantum field theory in Fock space.

6.4 Bijection of on-shell quantized fields and normal order products

The goal of this section is to connect the star quantization of the fields and the Fock space representation of it. Before stating the general result, let us give an example of what we mean by it. Consider the following products:

$$\phi(x) \star \phi(y) = \phi(x)\phi(y) + \hbar\Delta^+(x-y). \tag{6.16}$$

and

$$\begin{aligned} &: \phi^{\text{op}}(x)\phi^{\text{op}}(y) :=: (a^*(x)+a(x))(a^*(y)+a(y)) : \\ &= a^*(x)a^*(y) + a^*(x)a(y) + a(x)a(y) + a^*(y)a(x). \end{aligned} \tag{6.17}$$

Using

$$[a^*(x), a(y)] = \hbar\Delta^+(x-y) \Rightarrow a(y)a^*(x) = a^*(x)a(y) + \hbar\Delta^+(x-y). \tag{6.18}$$

We can write:

$$: \phi^{\text{op}}(x)\phi^{\text{op}}(y) := \phi^{\text{op}}(x)\phi^{\text{op}}(y) + \hbar\Delta^+(x-y). \tag{6.19}$$

Hence, at least in the simpler case, the operator formalism and the quantization using star product are very similar. As one could already expect, it is not coincidence. The case we have just presented is a special case from a more general theorem page 65 [24]:

Let $\phi^m(x)$ be the free, real scalar field (for a given mass m) on the Fock space $\mathfrak{F}$ and let $\mathcal{F}^{(m)}_{0,h} := \mathcal{F}_\hbar|_{\mathcal{C}_{S_0}}$. Then the map

$$\Phi\colon \mathcal{F}^{(m)}_{0,h} \longrightarrow \Phi(\mathcal{F}^{(m)}_{0,h}) \subset \{\text{linear operators on } \mathfrak{F}\}$$

is given by

$$F_0 = \sum_{n=0}^{N} \int dx_1 \dots dx_n \, \phi_0(x_1)\dots\phi_0(x_n) f_n(x_1,\dots,x_n) \tag{6.20}$$

$$\Phi(F_0) = \sum_{n=0}^{N} \int dx_1 \dots dx_n \, : \phi^{\text{op}}(x_1)\dots\phi^{\text{op}}(x_n) \colon f_n(x_1,\dots,x_n) \tag{6.21}$$

is an algebra isomorphism

$$F_0 \star G_0 \longmapsto \Phi(F_0 \star G_0) = \Phi(F_0)\Phi(G_0) \tag{6.22}$$

for the star product on the left and the operator product on the right, which respects the $*$-operation:

$$\langle \psi_1, \Phi(F_0^*)\psi_2\rangle_{\mathfrak{F}} = \langle \Phi(F_0)\psi_1, \psi_2\rangle_{\mathfrak{F}}, \quad \forall F_0 \in \mathcal{F}_{0,h}^{(m)} \tag{6.23}$$

and for all ψ_1, ψ_2 in the domain of $\Phi(F_0)$ or $\Phi(F_0^*)$, respectively. The same map Φ also gives an algebra isomorphism

$$F_0 \cdot G_0 \longmapsto \Phi(F_0 \cdot G_0) =: \Phi(F_0)\Phi(G_0): . \tag{6.24}$$

We are not going to write a formal proof; rather we will simply indicate how one can prove it.

We want to show that given $F = \int dX_l\, \tilde{f}(x_1, ..., x_l)\phi(x_1)...\phi(x_k)$ and $G = \int dY_k\, \tilde{g}(y_1, ..y_k)\phi(y_1)....\phi(y_k)$, the bijection holds. The strategy is to write the star product and compare with the expression of normal ordering. The star product is given by:

$$F \star G = \sum_{n=0}^{\min\{l,k\}} \frac{\hbar^n}{n!} \int dU_n dV_n \frac{\delta^n F}{\delta\phi(u_1)...\delta\phi(u_n)} \prod_{j=1}^{n} \Delta^+(u_j - v_j) \frac{\delta^n G}{\delta\phi(v_1)...\delta\phi(v_n)}. \tag{6.25}$$

On the other hand, normal ordering can also be written as an exponential map via Wick's theorem [24] pg490:

$$: e^{i\phi^{\mathrm{op}}(f)} := e^{i\phi^{\mathrm{op}}(f)} e^{\frac{\hbar}{2}\Delta^+(f,f)}. \tag{6.26}$$

Where:

$$\begin{aligned} : e^{i\phi^{\mathrm{op}}(f)} : &:= 1 + \sum_{n=1}^{\infty} \int dX_n\, f(x_1)...f(x_n) : \phi^{\mathrm{op}}(x_1)...\phi^{\mathrm{op}}(x_n) : \\ \Delta^+(f,g) &:= \int dxdy f(x)\Delta^+(x-y)g(y). \end{aligned} \tag{6.27}$$

From which we deduce

$$: e^{i\phi^{\mathrm{op}}(f)} :: e^{i\phi^{\mathrm{op}}(g)} :=: e^{i\phi^{\mathrm{op}}(f+g)} : e^{-\hbar\Delta^+(f,g)}. \tag{6.28}$$

Now we compute $\left.\frac{\delta^{l+k}}{\delta f(x_1)...\delta f(x_k)\delta g(y_1)...\delta g(y_l)}\right|_{f=g=0}$ from the expression above leading to:

$$
\begin{aligned}
&:\phi^{\mathrm{op}}(x_1)...\phi^{\mathrm{op}}(x_k) :: \phi^{\mathrm{op}}(y_1)...\phi^{\mathrm{op}}(y_l) : \\
&= \sum_{n=0}^{\min\{l,k\}} \frac{\hbar^n}{n!} \int dU_n dV_n : \left(\frac{: \phi^{\mathrm{op}}(x_1)...\phi^{\mathrm{op}}(x_n) :}{\delta\phi^{\mathrm{op}}(u_1)...\delta\phi^{\mathrm{op}}(u_n)} \frac{: \phi^{\mathrm{op}}(y_1)...\phi^{\mathrm{op}}(y_n) :}{\delta\phi^{\mathrm{op}}(v_1)...\delta\phi^{\mathrm{op}}(v_n)} \right) : \prod_{j=1}^{n} \Delta^+(u_l - v_l).
\end{aligned}
\tag{6.29}
$$

Since the multiplication by test functions $\tilde{f}(x_1, ..., x_l)$ and $\tilde{g}(y_1, ..., y_k)$ do not change the results above, we conclude:

$$
\begin{aligned}
&: F :: G := \sum_{n=0}^{\min\{l,k\}} \frac{\hbar^n}{n!} \int dU_n dV_n : \\
&\left(\frac{: \delta^n F :}{\delta\phi^{\mathrm{op}}(u_1)...\delta\phi^{\mathrm{op}}(u_n)} \prod_{j=1}^{n} \Delta^+(u_l - v_l) \frac{: \delta^n G :}{\delta\phi^{\mathrm{op}}(v_1)...\delta\phi^{\mathrm{op}}(v_n)} \right) : .
\end{aligned}
\tag{6.30}
$$

The above formula is essentially the same as the star product. Since the classical product $\phi(x)\phi(y)$ agrees with $: \phi^{\mathrm{op}}(x)\phi^{\mathrm{op}}(y) :$ we have indeed a bijection. Probably the best way to understand the bijection is by using examples.

6.5 Some examples

6.5.1 Vacuum state

In the definition of star product we have defined the vacuum state as $\omega_0(f + \sum_{n=0}^{N} \int dX_n f(x_1, ..., x_n)\phi(x_1)...\phi(x_n)) = f$. The definition agrees with the vacuum of Fock space since:

$$
\langle \Omega | f + \sum_{n=0}^{N} \int dX_n f(x_1, ..., x_n) : \phi^{\mathrm{op}}(x_1)...\phi^{\mathrm{op}}(x_n) : \Omega \rangle = f.
\tag{6.31}
$$

6.5.2 Scattering of free theory

To motivate the discussion, consider the following problem: we have a particle described by $\psi(t, \vec{p})$ and another described by $\phi(t', \vec{q})$. We want to study the amplitude of one given one, so we can measure the other. For the free theory, it is simply given by $\langle \psi(t, \vec{x}), \phi(t', \vec{y}) \rangle$. In the formalism of Fock space, we can translate it by writing:

$$\psi(t,\vec{p}) = a^*_{\mathfrak{F}}(\psi(-t,\vec{p}))\Omega = \int \frac{d\vec{p}}{2\omega_p} e^{i\omega_p t}\overline{\psi}(\vec{p})a^*(\vec{p})\Omega$$
$$\phi(t',\vec{q}) = a^*_{\mathfrak{F}}(\psi(-t',\vec{q}))\Omega = \int \frac{d\vec{q}}{2\omega_q} e^{i\omega_q t'}\overline{\phi}(\vec{q})a^*(\vec{q})\Omega. \tag{6.32}$$

Hence:

$$\langle \psi(t,\vec{x}), \phi(t',\vec{y})\rangle = \int \frac{d\vec{p}d\vec{q}}{4\omega_p\omega_q} e^{-i\omega_p t + i\omega_q t'} \psi(\vec{p})\overline{\phi}(\vec{q})\langle \Omega a^*(p)|a^*(q)\Omega\rangle. \tag{6.33}$$

The object in which we will be most interested while the studding scattering process is the term $\mathcal{T}(p,q) := \langle \Omega a^*(p)|a^*(q)\Omega\rangle$. For example, considering a process of 2 incoming and n outgoing particles, the relation of the amplitude with the cross section is given by:

$$d\sigma = \frac{(2\pi)^2}{4\sqrt{(pq)^2 - m^4}}\delta\left(p + q - \sum_{j=1}^{m} p_j\right)|\mathcal{T}(p_1,...,p_m;p,q)|^2\, d\mu_{p_1}...d\mu_{p_n}. \tag{6.34}$$

for a deduction of the formula we [35] pg 76 or [16].

Hence, most of the energy devoted to this project is to show how to compute those amplitudes, the famous "Feynman rules" and show it is as easy as to compute using other methods usually thought in QFT courses.

We will construct the $S-$ matrix in a latter chapter, for now we consider the simplest possible problem: The "scattering" of 2 and 4 particles in the free theory. In the absence of interactions, the matrix $S-$ is trivial, i.e. $S = 1$.

2-particle "scattering" in free theory The amplitude of 1 incoming particle and 1 outgoing particle is given by:

$$\langle \Omega a^*(p)|a^*(q)\Omega\rangle \tag{6.35}$$

First, let us calculate the amplitude using the formalism of the Fock space. To calculate it, we first "take $a^*_{\mathfrak{F}}(p)$ to the other side":

$$\langle \Omega a^*(p)|a^*(q)\Omega\rangle = \langle \Omega|a(p)a^*(q)\Omega\rangle. \tag{6.36}$$

And use the commutation relation of $a(p)a^*(q)$ to write

$$a(p)a^*(q) = a^*(q)a(p) + [a(p), a^*(q)] = a^*(q)a(p) + 2\hbar\omega_p\delta(\vec{p} - \vec{q}). \tag{6.37}$$

Hence:

$$\langle \Omega a^*(p) | a^*(q)\Omega\rangle = \underbrace{\langle \Omega | a^*(q)a(p)\Omega\rangle}_{=0} + 2\hbar\omega_p\delta(\vec{p} - \vec{q}). \tag{6.38}$$

To compare to our formalism we need to write $a^*(p)\Omega$ as a function of the fields in the configuration space. To do it, let us compute $\phi^{\text{op}}(y)\Omega$

$$\phi^{\text{op}}(x)\Omega = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d\vec{p}}{2\omega_p}(e^{ipx}a^*(\vec{p}) + e^{-ipx}a(\vec{p}))\Omega = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d\vec{p}}{2\omega_p} e^{ipx}a^*(\vec{p})\Omega. \tag{6.39}$$

Now all we have to do is invert the above equation to get $a^*(\vec{p})\Omega$. To do it, we do the inverse Fourier transform:

$$\begin{aligned} \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{x}\, e^{i\vec{k}\cdot\vec{x}} \phi^{\text{op}}(x)\Omega &= \frac{1}{(2\pi)^{d-1}} \int \frac{d\vec{x}d\vec{p}}{2\omega_p}\, e^{i\omega_p x^0} e^{i(\vec{k}-\vec{p})\cdot\vec{x}} a^*(\vec{p})\Omega \\ &= \int \frac{d\vec{p}}{2\omega_p}\, e^{i\omega_p x^0} \delta(\vec{k} - \vec{p}) a^*(\vec{p})\Omega = \frac{e^{i\omega_k x^0}}{2\omega_k} a^*(\vec{k})\Omega. \end{aligned} \tag{6.40}$$

Hence:

$$a^*(\vec{p})\Omega = \frac{2\omega_p}{e^{i\omega_k x^0}} \int \frac{d\vec{x}}{(2\pi)^{\frac{d-1}{2}}} e^{i\vec{p}\vec{x}} \phi^{\text{op}}(x)\Omega \equiv \frac{2\omega_p}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{x}\, e^{-ipx} \phi^{\text{op}}(x)\Omega \bigg|_{p^0=\omega_p}. \tag{6.41}$$

Remark: even though the notation above suggested that the expression is dependent of time, it actually is not. Essentially, what we are computing is the **emission operator**[67] page 77. in the configuration space. The expression for the emission operator is:

$$\phi^{(+)}(x^0, \vec{x}) = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d\vec{p}}{2\omega_p} e^{i\omega_p x^0 - i\vec{p}\cdot\vec{x}} a^*(\vec{p}). \tag{6.42}$$

Note that the exponential containing x^0 cancels. Nevertheless, it is worth it to keep the exponential explicitly in the expressions. As we will see later, they act by canceling time factors coming from the propagators.

Last but not least, we compute:

$$\mathcal{T}_2 = \langle \Omega a^*(\vec{p}) | a^*(\vec{q})\Omega \rangle =$$
$$\left\langle \Omega \left(\frac{2\omega_p}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{x}\, e^{-ipx} : \phi^{\mathrm{op}}(x) : \Big|_{p^0=\omega_p} \right) \Bigg| \frac{2\omega_p}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{y}\, e^{-iqy} : \phi^{\mathrm{op}}(x) : \Omega \Big|_{q^0=\omega_q} \right\rangle$$
$$= \frac{4\omega_p\omega_q}{(2\pi)^{d-1}} \int d\vec{x}d\vec{y}\, \langle \Omega | e^{ipx-iqy} : \phi^{\mathrm{op}}(\vec{x}) :: \phi^{\mathrm{op}}(\vec{y}) : \Omega \rangle. \tag{6.43}$$

The translation to our formalism reads:

$$\mathcal{T}(p,q) = \frac{4\omega_p\omega_q}{(2\pi)^{d-1}} \int d\vec{x}d\vec{y}\, \omega_0(\phi(x) \star \phi(y)) e^{ipx-iqy} \Big|_{p^0=\omega_p, q^0=\omega_q}. \tag{6.44}$$

We can easily compute the star product:

$$\begin{aligned} \phi(x) \star \phi(y) = \phi(x)\phi(y) + \hbar\Delta^+(x-y) &\Rightarrow \omega_0(\phi(x) \star \phi(y)) \\ = \hbar\Delta^+(x-y) &= \frac{\hbar}{(2\pi)^{d-1}} \int d\mu_k e^{-ik(x-y)} \Big|_{k_0=\omega_k}. \end{aligned} \tag{6.45}$$

Now we need to work out the multiple integrals in the formula above. The strategy is to first integrate in the $\vec{x}$ and $\vec{y}$ then $\vec{k}$:

$$\begin{aligned} &\frac{4\omega_p\omega_q}{(2\pi)^{d-1}} \int d\vec{x}d\vec{y}\, e^{-ipx+iqy} \omega_0(\phi(x) \star \phi(y)) \\ &= \frac{4\omega_p\omega_q\hbar}{(2\pi)^{2(d-1)}} \int \mu_k d\vec{x}d\vec{y}\, e^{ipx-iqy} e^{-ik(x-y)} \Big|_{k_0=\omega_k} \\ &= \frac{2\omega_p\omega_q}{\omega_k} \frac{\hbar}{(2\pi)^{2(d-1)}} e^{-i\omega_k(x^0-y^0)+i\omega_p x^0 - i\omega_q y^0} \int d\vec{k}d\vec{x}d\vec{y}\, e^{-i(\vec{p}-\vec{k})\vec{x}} e^{i(\vec{q}-\vec{k})\vec{y}} \\ &= \frac{2\omega_p\omega_q}{\omega_k} \frac{\hbar}{(2\pi)^{2(d-1)}} e^{-i\omega_k(x^0-y^0)+i\omega_p x^0 - i\omega_q y^0} \int d\vec{k} (2\pi)^{2(d-1)} \delta(\vec{p}-\vec{k})\delta(\vec{k}-\vec{q}) \\ &= 2\omega_p \hbar \delta(\vec{p}-\vec{q}). \end{aligned} \tag{6.46}$$

In the above equation, we have used the fact that after the integration $\omega_k = \omega_p = \omega_q = \sqrt{\vec{p}^2 + m^2}$. We can interpret the result above using Feynman diagrams:

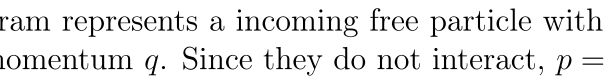


Figure 6.1: The diagram represents a incoming free particle with momentum p and outgoing particle with momentum q. Since they do not interact, $p = q$

4 particle "scattering" in the free theory The only allowed process involving the scattering of 4 particles in the free theory is 2 incoming particles and 2 outgoing particles. In the formalism of Fock space, the amplitude is given by:

$$\mathcal{T} = \langle \Omega a^*(p_1) a^*(p_2) | a^*(p_3) a^*(p_4) \Omega \rangle. \tag{6.47}$$

To calculate the above amplitude, we use a slightly different approach. Instead of working with the Wick theorem, we commute the annihilation and creation operators. Explicitly:

$$\begin{aligned}
&a(p_1)\left(a(p_2)a^*(p_3)\right)a^*(p_4) = a(p_1)\left(2\hbar\omega_{p_2}\delta(\vec{p}_2-\vec{p}_3) + a^*(p_3)a(p_2)\right)a^*(p_4)\\
&= 2\hbar\omega_{p_2}\delta(\vec{p}_2-\vec{p}_3)a(p_1)a^*(p_4) + a(p_1)a^*(p_3)\left(2\hbar\omega_{p_2}\delta(\vec{p}_2-\vec{p}_4) + a^*(p_4)a(p_1)\right)\\
&= 2\hbar\omega_{p_2}\delta(\vec{p}_2-\vec{p}_3)(2\hbar\omega_{p_1}\delta(\vec{p}_1-\vec{p}_4) + a^*(p_4)a(p_1))\\
&+ \left(2\hbar\omega_{p_1}\delta(\vec{p}_1-\vec{p}_3) + a^*(p_3)a(p_1)\right)\left(2\hbar\omega_{p_2}\delta(\vec{p}_2-\vec{p}_4) + a^*(p_4)a(p_1)\right).
\end{aligned} \tag{6.48}$$

Applying the above operator in vacuum and using $a\Omega = 0$ we obtain the following.

$$\mathcal{T} = 4\hbar^2\omega_{p_1}\omega_{p_2}\delta(\vec{p}_1-\vec{p}_3)\delta(\vec{p}_2-\vec{p}_4) + 4\hbar^2\omega_{p_1}\omega_{p_2}\delta(\vec{p}_1-\vec{p}_4)\delta(\vec{p}_2-\vec{p}_3). \tag{6.49}$$

To translate to our formalism,we repeat the same procedure of inverting the Fourier transform: Thus:

$$\begin{aligned}
\mathcal{T} &= \left(\prod_{i=1}^{4}\frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\int d\vec{X}_4\, e^{i(p_1x_1+p_2x_2-p_3x_3-p_4x_4)}\\
&\times \langle \Omega : \phi(x_1)\phi(x_2) : | : \phi(x_3)\phi(x_4) : \Omega\rangle\\
&= \left(\prod_{i=1}^{4}\frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\int d\vec{X}_4\, e^{i(p_1x_1+p_2x_2-p_3x_3-p_4x_4)}\omega_0(\phi(x_1)\phi(x_2)\star\phi(x_3)\phi(x_4)).
\end{aligned} \tag{6.50}$$

Using the result:

$$\begin{aligned}
&\phi(x_1)\phi(x_2)\star\phi(x_3)\phi(x_4) = \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\\
&+\hbar\int dY_2\,\frac{\delta\phi(x_1)\phi(x_2)}{\delta\phi(y_1)}\Delta^+(y_1-y_2)\frac{\delta\phi(x_3)\phi(x_4)}{\delta\phi(y_2)}\\
&+\frac{\hbar^2}{2}\int dY_4\frac{\delta^2\phi(x_1)\phi(x_2)}{\delta\phi(y_1)\delta\phi(y_2)}\Delta^+(y_1-y_3)\Delta^+(y_2-y_4)\frac{\delta^2\phi(x_3)\phi(x_4)}{\delta\phi(y_3)\delta\phi(y_4)}.
\end{aligned} \tag{6.51}$$

The only term that is important to us is the one involving the second derivative (the other terms are proportional to the fields and, when computed in the vacuum, do not contribute):

$$\begin{aligned}
&\int dY_4\frac{\delta^2\phi(x_1)\phi(x_2)}{\delta\phi(y_1)\delta\phi(y_2)}\Delta^+(y_1-y_3)\Delta^+(y_2-y_4)\frac{\delta^2\phi(x_3)\phi(x_4)}{\delta\phi(y_3)\delta\phi(y_4)}\\
&= 2(\Delta^+(x_1-x_3)\Delta^+(x_2-x_4) + \Delta^+(x_1-x_4)\Delta^+(x_2-x_3)).
\end{aligned} \tag{6.52}$$

Hence:

$$\omega_0(\phi(x_1)\phi(x_2) \star \phi(x_3)\phi(x_4))$$
$$= \hbar^2(\Delta^+(x_1 - x_3)\Delta^+(x_2 - x_4) + \Delta^+(x_1 - x_4)\Delta^+(x_2 - x_3)). \tag{6.53}$$

Finally, we can compute the amplitude:

$$\begin{aligned}\mathcal{T} = &\left(\prod_{i=1}^{4} \frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\hbar^2 \\ &\times \left\{ \int d\vec{X}_4 \left(\Delta^+(x_1 - x_3)e^{i(p_1x_1 - p_3x_3)}\right)\left(\Delta^+(x_2 - x_4)e^{i(p_2x_2 - p_4x_4)}\right)\right. \\ &+ \left.\int d\vec{X}_4 \left(\Delta^+(x_1 - x_4)e^{i(p_1x_1 - p_4x_4)}\right)\left(\Delta^+(x_2 - x_3)e^{i(p_2x_2 - p_3x_3)}\right)\right\}.\end{aligned} \tag{6.54}$$

The integrals above are analogous to the ones in the first example:

$$\int d\vec{x}_1 d\vec{x}_4\, \Delta^+(x_1 - x_4)e^{i(p_1x_1 - p_4x_4)} = \frac{(2\pi)^{d-1}}{2\omega_{p_1}}\delta(\vec{p}_1 - \vec{p}_4). \tag{6.55}$$

Thus:

$$\begin{aligned}\mathcal{T} &= \hbar^2\left(\prod_{i=1}^{4} \frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\frac{(2\pi)^{2(d-1)}}{4\omega_{p_1}\omega_{p_2}}\left(\delta(\vec{p}_1 - \vec{p}_3)\delta(\vec{p}_2 - \vec{p}_4) + \delta(\vec{p}_1 - \vec{p}_4)\delta(\vec{p}_2 - \vec{p}_3)\right) \\ &= 4\hbar^2\omega_{p_3}\omega_{p_4}(\delta(\vec{p}_1 - \vec{p}_3)\delta(\vec{p}_2 - \vec{p}_4) + 4\omega_{p_3}\omega_{p_4}\delta(\vec{p}_1 - \vec{p}_4)\delta(\vec{p}_2 - \vec{p}_3)) \\ &= \langle p_1|p_3\rangle\langle p_2|p_4\rangle + \langle p_1|p_4\rangle\langle p_2|p_3\rangle.\end{aligned} \tag{6.56}$$

In the above equation, we used $\langle p|q\rangle = 2\hbar\delta(\vec{p} - \vec{q})$.
The corresponding Feynman diagrams are:

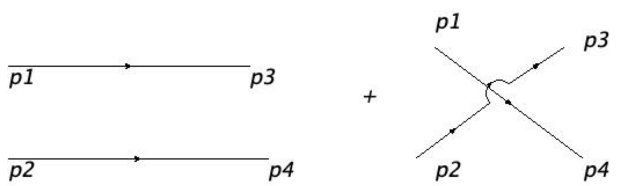


Figure 6.2: The diagrams represents free particles propagating. The "jump" in the second one is to make clear they do not interact.

The examples above show that both formalisms are compatible, at least in the free case.

From the examples above, we can already read some general features of the amplitude, even in the interacting case. The general formula for a scattering amplitude is given by

$$\langle \Omega a^*(p_1)...a^*(p_n)|Sa^*(q_1)...a^*(q_m)\Omega\rangle. \tag{6.57}$$

To translate $\langle \Omega : \phi^{\mathrm{op}}(x_1)...\phi^{\mathrm{op}}(x_n) : |S : \phi^{\mathrm{op}}(y_1)...\phi^{\mathrm{op}}(y_m) : \Omega\rangle$ into the formula $\mathcal{T}(p_1, ..., p_n, q_1, ..., q_m)$, we need to repeat the inverse Fourier transformation. Hence, every incoming particle contributes to the amplitude with:

$$\frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{x}_i e^{ip_i x_i}\bigg|_{p_i^0=\omega_{p_i}}. \tag{6.58}$$

The outgoing particle contribute to the amplitude with:

$$\frac{2\omega_{q_i}}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{y}_i e^{-iq_i y_i}\bigg|_{q_i^0=\omega_{q_i}}. \tag{6.59}$$

Chapter 7

Classical retarded product

7.1 Introduction

The main limitation of quantum field theory is that it is hard to exactly solve the theory and one has to use perturbation theory to extract results. The goal of the next section is to motivate the solution of a classical interacting field theory as a series in the coupling constant. The result obtained using this method will be the starting point for the quantum case, where the "intuition" used in the classical case is no longer valid and must be replaced by axioms. Most of the text is based on chapter 1 from [24]

7.2 Retarded fields

The philosophy of perturbation theory is to write the interacting quantities as a function of the non interacting ones. To do it, the first step is to construct a map $r_{S_0+S,S} : \mathcal{C} \to \mathcal{C}$ that has the following property: If h is a solution of the free field, i.e., $h \in \mathcal{C}_{S_0}$, then $r_{S_0+S,S}(h)$ is a solution of the interacting field, i.e., $r_{S_0+S,S}(h) \in \mathcal{C}_{S+S_0}$. This family of maps is called the retarded wave operator and they obey:

(i) $r_{S_0+S,S_0}(h)(x) = h(x)$ for x "sufficiently early", that is, before x "arrives" in support of the interaction.

(ii) $\frac{\delta(S_0+S)}{\delta\phi(x)} \circ r_{S_0+S,S_0} = \frac{\delta S_0}{\delta\phi(x)}$

Explicitly, an $h \in \mathcal{C}$ is mapped by r_{S_0+S,S_0} to $f(x)$, where $f(x)$ solves the equation:

$$\underbrace{-(\partial^2 + m^2)f(x) + \frac{\delta S}{\delta\phi(x)}(f)}_{\frac{\delta S_0+S}{\delta\phi(x)}} = \underbrace{-(\partial^2 + m^2)}_{\frac{\delta S_0}{\delta\phi(x)}} h(x). \tag{7.1}$$

Note that if $x \notin \text{supp}(S)$, we recover the equation:

$$-(\partial^2 + m^2)h(x) = -(\partial^2 + m^2)f(x). \tag{7.2}$$

which agrees with $h(x)$ for x "sufficiently early". The existence and uniqueness of r is discussed in [24] and references therein. In our work, it is not necessary to discuss the existence and uniqueness in the general case since we are able to construct such a map as a formal power series. We call a retarded field F_S^{ret} the field defined by:

$$F_S^{\mathrm{ret}} := F \circ r_{S_0+S,S_0} : \mathcal{C} \to \mathbb{C}. \tag{7.3}$$

7.3 An explicit formula for the classical retarded product

The goal of this section is to expand F_S^{ret} into a series in the coupling constant λ.A very important remark: The power series is to be understood as a formal power series, we can not guarantee the convergence of these series. We denote:

$$\left.\frac{d^n}{d\lambda^n}\right|_{\lambda=0} F_{\lambda\tilde{S}}^{\mathrm{ret}} =: R(\tilde{S}^{\otimes n}, F) \equiv R_{n,1}(\tilde{S}^{\otimes n}, F). \tag{7.4}$$

Using these notation, F_S^{ret} is understood as a formal power series:

$$F_S^{\mathrm{ret}} \simeq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} R(\tilde{S}^{\otimes n}, F) = \sum_{n=0}^{\infty} \frac{1}{n!} R(S^{\otimes n}, F) \equiv R(e_\otimes^S, F). \tag{7.5}$$

where

$$\begin{aligned} e_\otimes^S &:= 1 \oplus \bigoplus_{n=1}^{\infty} \frac{S^{\otimes n}}{n!} \equiv (1, S, \frac{1}{2!} S \otimes S, \cdots) \\ S &:= \lambda \tilde{S}. \end{aligned} \tag{7.6}$$

7.4 Some examples

Before "giving the answer", it is a good idea to construct an intuition about these series. To do it, we examine three examples. The goal is to solve the retarded interacting field equation

$$\frac{\delta(S + S_0)}{\delta\phi(x)} \circ r_{S_0+S,S_0} = \frac{\delta S_0}{\delta\phi(x)} \tag{7.7}$$

for a) $S = -\lambda \int dx g(x)\phi(x)$, b)$S = -\lambda \int dx g(x)\phi(x)^2$ and c) $S = -\lambda \int dx g(x)\phi(x)^{k+1}$. First, we define the retarded field $\phi^{\mathrm{ret}}(x) := \phi \circ r_{S_0+S,S_0}$. Now let us find what equation this field obeys. Using

$$\frac{\delta S_0}{\delta\phi(x)} = \frac{\delta}{\delta\phi(x)} \int dy \frac{1}{2}(\partial^\mu\phi(y)\partial_\mu\phi(y) - m^2\phi^2(y)) = -(\partial^2 + m^2)\phi(x) \tag{7.8}$$

we get

$$\begin{aligned}\frac{\delta(S+S_0)}{\delta\phi(x)} \circ r_{S_0+S,S_0} &= \frac{\delta S_0}{\delta\phi(x)} \circ r_{S_0+S,S_0} + \frac{\delta S}{\delta\phi(x)} \circ r_{S_0+S,S_0} \\ &= -(\partial^2 + m^2)\phi(x) \circ r_{S_0+S,S_0} + \frac{\delta S}{\delta\phi(x)} \circ r_{S_0+S,S_0} \\ &\stackrel{!}{=} \frac{\delta S_0}{\delta\phi(x)} = -(\partial^2 + m^2)\phi(x).\end{aligned} \tag{7.9}$$

Hence, we obtain the equation of motion for retarded fields:

$$(\partial^2 + m^2)\phi^{\mathrm{ret}}(x) = (\partial^2 + m^2)\phi(x) + \left(\frac{\delta S}{\delta\phi(x)}\right)^{\mathrm{ret}}. \tag{7.10}$$

Example a)

Now we can calculate the first example. For $S = -\lambda \int dx g(x)\phi(x)$, we get:

$$\frac{\delta S}{\delta\phi(x)} = \frac{\delta}{\delta\phi(x)} - \lambda \int dy\, g(y)\phi(y) = -\lambda g(x). \tag{7.11}$$

Hence, the retarded field equation is simply:

$$(\partial^2 + m^2)\phi^{\mathrm{ret}}(x) = (\partial^2 + m^2)\phi(x) - \lambda g(x). \tag{7.12}$$

The idea to solve this equation is basically to multiply both sides by "$(\partial^2 + m^2)^{-1}$" and obtain:

$$\phi^{\mathrm{ret}}(x) = \phi(x) - \lambda(\partial^2 + m^2)^{-1} g(x). \tag{7.13}$$

Of course now we have to show that such an inverse exists, and, more than that, we need to show that $\phi^{\mathrm{ret}}(x) = \phi(x)$ for "x" sufficiently early. The way to do it is via the retarded propagator. We define the retarded propagator $\Delta_m^{\mathrm{ret}} \in \mathcal{D}'(\mathbb{R}^d)$ as:

$$(\partial^2 + m^2)\Delta_m^{\text{ret}} = -\delta(x), \text{ satisfying } \operatorname{supp}(\Delta_m^{\text{ret}}) \subseteq x + \overline{V}_+. \tag{7.14}$$

It is easy to check (and also a classical exercise of quantum field theory) that for the scalar field:

$$\Delta_m^{\text{ret}}(x) = \frac{1}{(2\pi)^d} \int d^d p \, \frac{e^{-ipx}}{p^2 - m^2 + ip^0 0}. \tag{7.15}$$

A short remark about this specific propagator: It is a possible choice of the inverse "$(\partial^2 + m^2)^{-1}$". It is chosen such that the solution of the field equation is a free field in the distant past. We could also change $ip^0 0 \to -ip^0 0$ and it would also be an inverse "$(\partial^2 + m^2)^{-1}$" but with the free field as a solution in the distant future. There are other possible combinations that solve the problem. We have chosen that specific one because that is how quantum field theory was developed [8]. In this scenario, the incoming field is known and we want to connect the outgoing field with experimentally observable quantities. For this proposal, an inverse that respects the form of the incoming field is very useful.

Returning to the solution of interacting field equation, we claim that the solution to our problem is simply

$$\phi^{\text{ret}}(x) = \phi(x) + \lambda \int dy \, \Delta_m^{\text{ret}}(x-y) g(y). \tag{7.16}$$

To show that it fulfills the field equation is quite simple, one just has to apply $(\partial^2 + m^2)$ on both sides of the equation. To prove that $\phi^{\text{ret}}(x) = \phi(x)$ for "x" sufficiently early, we need to consider two cases. If x and y are space-like, then $\Delta_m^{\text{ret}}(x-y) = 0$ because $\operatorname{supp}(\Delta_m^{\text{ret}}) \subseteq \overline{V}_+$. If x and y are not space-like, we are interested in the case where $x < \operatorname{supp} g(y)$. The notation $a < b$ means that a is in the causal past of b. In that specific case, $\operatorname{supp} g \cap \operatorname{supp}(\Delta_m^{\text{ret}}) = \emptyset$ and therefore the integral is zero. One can visualize it better with a drawing. For simplicity, let us consider the integral over the space time as an integral over an axis:

Figure 7.1: The red regions represents $\operatorname{supp} \Delta^{\text{ret}}(x-y)$ and the blue one $\operatorname{supp} g(y)$. They do not intercept.

Hence, it is the desired solution. In this context, we want to emphasize what happens when $x > \operatorname{supp} g(y)$. If $\phi(x)$ solves the field equation, then:

$$(\partial^2 + m^2)\phi^{\rm ret}(x) = (\partial^2 + m^2)\phi(x) - \lambda g(x) \stackrel{!}{=} 0. \tag{7.17}$$

Hence, the retarded field is also a free field, but with different "initial conditions". Diagrammatically, our procedure can be sketched as follows:

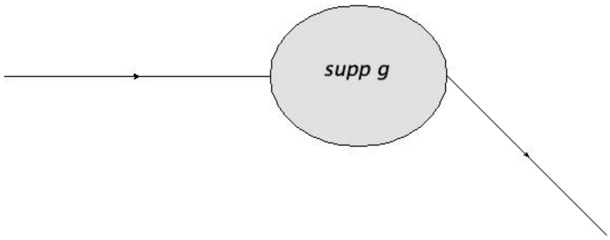


Figure 7.2: The line denotes a propagating free field and the gray dot the region in space time where $g \neq 0$

We can construct a physical picture of it: the interaction is restricted to a finite region of space-time; therefore, for times in the distant past, the interaction was not felt yet and for times in the distant future, the interaction has already been turned off.

Example b)

The first example was useful for introducing some of the concepts, but it was so simple that the expansion ended at the first order of the coupling constant. The second example aims to clarify the philosophy behind the expansion in the coupling constant, but it is also "exactly solvable" in the sense that we can find a simple closed formula for all orders in perturbation theory.

Once again, we start by calculating $\frac{\delta S}{\delta\phi(x)}$:

$$\frac{\delta}{\delta\phi(x)} - \lambda \int dy\, g(y)\phi^2(y) = -2\lambda g(x)\phi(x) \Rightarrow \left(\frac{\delta S}{\delta\phi(x)}\right)^{\rm ret} = -2\lambda g(x)\phi^{\rm ret}(x). \tag{7.18}$$

Plugging the above result into the retarded field equation (7.10):

$$\begin{aligned}\phi^{\rm ret}(x) &= \phi(x) - 2\lambda(\partial^2 + m^2)^{-1} g(x)\phi^{\rm ret}(x)\\ \phi^{\rm ret}(x) &= \phi(x) - 2\lambda \int dy\, \Delta_m^{\rm ret}(x-y) g(y)\phi^{\rm ret}(y).\end{aligned} \tag{7.19}$$

To pass from the first line to the second we used the same arguments as explained in the previous example. We will try to solve the integral equation following the usual strategy of perturbation theory. We start with the Ansatz:

$$
\begin{aligned}
\phi^{\text{ret}}(x) &= R_{0,1}\left(\tilde{S}^0, \phi(x)\right) - \lambda R_{1,1}\left(\tilde{S}, \phi(x)\right) + \frac{(-\lambda)^2}{2} R_{2,1}\left(\tilde{S}^{\otimes 2}, \phi(x)\right) + ... \\
&\equiv \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} R_{n,1}\left(\tilde{S}^{\otimes n}, \phi(x)\right). \qquad (7.20)
\end{aligned}
$$

Once again we stress that the expansion above must be understood as a formal power series whose convergence cannot be always guaranteed. Substituting into the equation, we get:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} R_{n,1}\left(\tilde{S}^{\otimes n}, \phi(x)\right) \\
&= \phi(x) - 2\lambda \int dy\, \Delta_m^{\text{ret}}(x-y) g(y) \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} R_{n,1}\left(\tilde{S}^{\otimes n}, \phi(x)\right). \qquad (7.21)
\end{aligned}
$$

Now, all we have to do is compare order by order in the coupling constant λ. Finally, we obtain:

$$
R_{0,1}\left(\tilde{S}^0, \phi(x)\right) = \phi(x) \qquad (7.22)
$$

$$
R_{1,1}\left(\tilde{S}, \phi(x)\right) = 2 \int dx_1 \Delta_m^{\text{ret}}(x-x_1) g(x_1)\, \overbrace{R_{0,1}\left(\tilde{S}^0, \phi(x)\right)}^{=\phi(x_1)} \qquad (7.23)
$$

$$
\begin{aligned}
&R_{2,1}\left(\tilde{S}^{\otimes 2}, \phi(x)\right) = 2 \cdot 2 \int dx_1\, \Delta_m^{\text{ret}}(x-x_1) g(x_1) R_{1,1}\left(\tilde{S}, \phi(x_1)\right) \\
&= 2 \cdot 2^2 \int dx_1 dx_2\, \Delta_m^{\text{ret}}(x-x_1) \Delta_m^{\text{ret}}(x_1-x_2) g(x_1) g(x_2) \phi(x_2) \qquad (7.24) \\
&... \\
&R_{n,1}\left(\tilde{S}^{\otimes n+1}, \phi(x)\right) = n \cdot 2 \int dx_1\, \Delta_m^{\text{ret}}(x-x_1) g(x_1) R_{n-1,1}\left(\tilde{S}^{n-1}, \phi(x_1)\right) \\
&\equiv 2^n n \int dX_n\, g(x_1)...g(x_n) \Delta^{\text{ret}}(x-x_1) \Delta^{\text{ret}}(x_1-x_2)...\Delta^{\text{ret}}(x_{n-1}-x_n) \phi(x_n). \qquad (7.25)
\end{aligned}
$$

Where we define $dX_n := dx_1...dx_n \equiv \prod_{k=1}^n dx_k$. There is an important feature on the above equation. Since $\text{supp}(\Delta_m^{\text{ret}}) \subseteq \overline{V}_+$, we have a causal ordering on the integral, i.e., $x^0 \geq x_1^0 \geq x_2^0 \geq .... \geq x_n^0$. Pictorially, we can imagine the series above as a sequence of interactions:

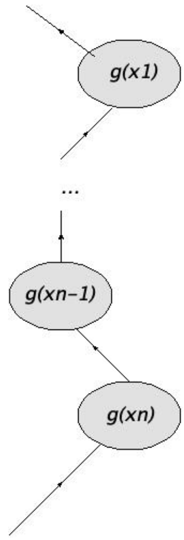


Figure 7.3: We can imagine the series as every interaction happening in a small space time region. The time goes from past (bottom) to future (top).

Example c)

The last example cannot be exactly solved as the other two, but it illustrates the usual difficulty when working with perturbation theory and it is also a good "prelude" to the most general solution. As usual, we start with $\frac{\delta S}{\delta\phi(x)}$:

$$\begin{aligned} \frac{\delta}{\delta\phi(x)} - \lambda \int dy\, g(y)\phi^{k+1}(y) &= -(k+1)\lambda g(x)\phi^k(x) \\ \Rightarrow \left(\frac{\delta S}{\delta\phi(x)}\right)^{\text{ret}} &= -(k+1)\lambda g(x)(\phi^{\text{ret}}(x))^k. \end{aligned} \tag{7.26}$$

Plugging the above result into the retarded field equation (7.10):

$$\begin{aligned} \phi^{\text{ret}}(x) &= \phi(x) - (k+1)\lambda(\partial^2 + m^2)^{-1} g(x)(\phi^{\text{ret}}(x))^k \\ \phi^{\text{ret}}(x) &= \phi(x) - (k+1)\lambda \int dy\, \Delta_m^{\text{ret}}(x-y) g(y)(\phi^{\text{ret}}(y))^k. \end{aligned} \tag{7.27}$$

We start with the Ansatz:

$$\begin{aligned} \phi^{\text{ret}}(x) &= R_{0,1}\left(\tilde{S}^0, \phi(x)\right) - \lambda R_{1,1}\left(\tilde{S}, \phi(x)\right) + \frac{(-\lambda)^2}{2} R_{2,1}\left(\tilde{S}^{\otimes 2}, \phi(x)\right) + ... \\ &\equiv \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} R_{n,1}\left(\tilde{S}^{\otimes n}, \phi(x)\right). \end{aligned} \tag{7.28}$$

Substituting the Ansatz into the field equation we obtain:

$$\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}R_{n,1}\left(\tilde{S}^{\otimes n},\phi(x)\right)$$
$$= \phi(x) - (k+1)\lambda\int dx_1\,\Delta^{\text{ret}}(x-x_1)\left[\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}R_{n,1}\left(\tilde{S}^{\otimes n},\phi(x_1)\right)\right]^k. \tag{7.29}$$

Unlike the examples that were worked on previously, in these cases, we have a combinatorial factor to compute in each order of perturbation theory. We can calculate $\phi^{\text{ret}}(x)$ to third order just to illustrate the problem.

λ^0 **order:**

The first case is trivial:

$$R_{0,1}(\tilde{S}^0,\phi(x)) = \phi(x). \tag{7.30}$$

λ^1 **order:**

The first non-trivial term is also not hard to calculate. Since

$$\left[\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}R_{n,1}\left(\tilde{S}^{\otimes n},\phi(x_1)\right)\right]^k = R_{0,1}(\tilde{S},\phi(x))^k(x_1) + O(\lambda). \tag{7.31}$$

The first term is simple:

$$R_{1,1}(\tilde{S}^1,\phi(x)) = (k+1)\int dx_1\,\Delta^{\text{ret}}(x-x_1)R_{0,1}(\tilde{S},\phi(x))^k(x_1)$$
$$= (k+1)\int dx_1\,\Delta^{\text{ret}}(x-x_1)\phi^k(x_1). \tag{7.32}$$

λ^2 **order:**

This term is already a bit tricky. We have to expand $(\phi^{\text{ret}})^k$ accounting for all terms of the order λ^2. The other terms can be neglected for the calculation:

$$
\begin{aligned}
&\left[\sum_{n=0}^{\infty}\frac{(-(k+1)\lambda)^n}{n!}R_{n,1}\left(\tilde{S}^{\otimes n},\phi(x_1)\right)\right]^k \\
&\overset{\lambda}{=}\left[R_{0,1}(x_1)-\lambda R_{1,1}(x_1)\right]^k=\sum_{j=1}^{k}\binom{k}{j}R_{0,1}^{k-j}(x_1)\left[-\lambda R_{1,1}(x_1)\right]^j \\
&\overset{\lambda}{=}-k\lambda R_{1,1}(x_1)R_{0,1}^{k-1}(x_1).
\end{aligned}
\tag{7.33}
$$

In the above equation, we have used the notation $R_{n,1}(x_1)\equiv R_{n,1}(\tilde{S}^{\otimes n},\phi(x_1))$ and the symbol $\overset{\lambda^n}{=}$ to mean that we are neglecting all terms that are not of order λ^n.

Hence:

$$
R_{2,1}(\tilde{S}^{\otimes 2},\phi(x))=2k(k+1)\int dx_1\,\Delta^{\mathrm{ret}}(x-x_1)g(x_1)R_{1,1}(x_1)R_{0,1}^{k-1}(x_1). \tag{7.34}
$$

λ^3 **order:**

That term is truly more complicated than the last ones to be computed but the way to do it is the same. We expand $(\phi^{\mathrm{ret}}(x_1))^k$ up to second order in λ:

$$
\begin{aligned}
&\left[\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}R_{n,1}\left(\tilde{S}^{\otimes n},\phi(x_1)\right)\right]^k \\
&\overset{\lambda^2}{=}\left[R_{0,1}(x_1)-\lambda R_{1,1}(x_1)+\frac{\lambda^2}{2}R_{2,1}(x_1)\right]^k \\
&=\sum_{j=0}^{k}\binom{k}{j}\left[\frac{\lambda^2}{2}R_{2,1}(x_1)\right]^{k-j}\left[R_{0,1}(x_1)-\lambda R_{1,1}(x_1)\right]^j \\
&=\sum_{j=0}^{k}\binom{k}{j}\left[\frac{\lambda^2}{2}R_{2,1}(x_1)\right]^{k-j}\sum_{l=0}^{j}\binom{j}{l}R_{0,1}^{l}(x_1)\left[-\lambda R_{1,1}(x_1)\right]^{j-l} \\
&\overset{\lambda^2}{=}k\frac{\lambda^2}{2}R_{2,1}(x_1)R_{0,1}^{k-1}(x_1)+\frac{k(k-1)}{2}R_{0,1}^{k-2}(x_1)\left[-\lambda R_{1,1}(x_1)\right]^2.
\end{aligned}
\tag{7.35}
$$

Finally:

$$
\begin{aligned}
&R_{3,1}(x)=-3!(k+1)\int dx_1\,\Delta^{\mathrm{ret}}(x-x_1)g(x_1) \\
&\times\left(\frac{k}{2}R_{2,1}(x_1)R_{0,1}^{k-1}(x_1)+\frac{k(k-1)}{2}R_{0,1}^{k-2}(x_1)R_{1,1}^{2}(x_1)\right).
\end{aligned}
\tag{7.36}
$$

Although this example is not very elucidating, it already shines a light on patterns that will lead us to the final form of the "R products". That is, every term $R_{n,1}$ has exactly n retarded products and a causal ordering. We will use it to construct the "close formula" for $R_{n,1}(F_1 \otimes F_2.... \otimes F_{n-1}, G)$ in the next sections, as well as some of its features. Later we will use those silly examples,as the main building block to construct the quantum version of the expansion. In the latter case, the philosophy described here must be imposed as axioms to the construction of the quantum retarded product. We will return to these questions in a further chapter.

7.5 The "close" formula for the retarded product

Using the structure above, we can explicitly construct a retarded product with the desired properties. Those properties will be imposed as axioms. These axioms will also be imposed for the quantum case together with some properties that can be deduced in the classical expansion but are not immediate in the quantum one. The axioms for the classical retarded product $R_{n,1}(F_1 \otimes F_2... \otimes F_n; F) \equiv R_{n,1}(F_1, ..., F_n; F)$ are:

- Symmetry in the first n arguments
- linearity
- factorization property, that is, $R(e^{\otimes S}; AB) = R(e^{\otimes S}; A)R(e^{\otimes S}; B)$
- off-shell field equation: $(\partial_x^2 + m^2)R(e_\otimes^S, \phi(x)) = (\partial^2 + m^2)\phi(x) + R(e_\otimes^S, \frac{\delta S}{\delta\phi(x)})$, with $R(e_\otimes^S; A(x)) = A(x)$ for sufficiently early x

The construction of the retarded product can be done as follows. Let $S \in \mathcal{F}_{\text{loc}}$ and define the operator:

$$\mathcal{R}_S \equiv \mathcal{R} := -\int dy \, \frac{\delta S}{\delta\phi(x)} \Delta_m^{\text{ret}}(y-x) \frac{\delta}{\delta\phi(y)}. \tag{7.37}$$

For all $F \in \mathcal{F}$, the pointwise product of distributions appearing in $\mathcal{R}(x)F$ exists and $\int dx \, \mathcal{R}(x)F$ lies again in $\mathcal{F}$. For $F \in \mathcal{F}_{\text{loc}}$, the retarded product $R(S^{\otimes n}, F)$ is obtained by the formula:

$$R_{n,1}(S^{\otimes n}; F) = n! \int_{x_1^0 \leq \cdots \leq x_n^0} dx_1 \cdots dx_n \mathcal{R}(x_1) \cdots \mathcal{R}(x_n) F. \tag{7.38}$$

The proof that the above retarded product solves the field equation is unique and coincide with the starting field in a sufficiently distant past can be found in [24] page 36 and [26]. Linearity and factorization properties can be shown by using the definition of a functional derivative. With respect to symmetry, the result is obvious when the first arguments n are equal. In case we have

$R_{n,1}(F_1, ..., F_n; F)$ with at least one $F_i \neq F_j$, we can derive a specific formula for these products. These formulas can be interpreted as "proto-"Feynman diagrams. The formula is as follows.

$$R_{n,1}(F_1 \otimes \cdots \otimes F_n; F) = \int_{x_1^0 \leq \cdots \leq x_n^0} dx_1 \cdots dx_n \sum_{\pi \in S_n} \mathcal{R}_{F_{\pi_1}}(x_1) \cdots \mathcal{R}_{F_{\pi_n}}(x_n) F. \tag{7.39}$$

To prove it, first we have to show that if given $\mathcal{V}, \mathcal{W}$ vector spaces, $f(v_1 \otimes ... \otimes v_n) : \mathcal{V}^{\times n} \to \mathcal{W}$ symmetric and linear, then f can be written as

$$f(v_1 \otimes ... \otimes v_n) = \frac{1}{n!} \frac{\partial^n}{\partial\lambda_1 ... \partial\lambda_n}\bigg|_{\lambda_1 = ... = \lambda_n = 0} f((\sum_{k=1}^{n} \lambda_k v_k)^{\otimes n}). \tag{7.40}$$

Where $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$.The proof is written in the Appendix.

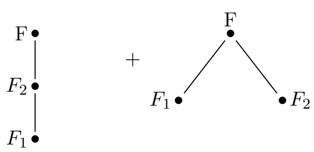
Returning to the original problem, since symmetry is imposed as an axiom, we can use the formula above and (7.38) to write:

$$\begin{aligned} R_{n,1}(F_1, ..., F_n; F) &= \frac{\partial^n}{\partial\lambda_1 ... \partial\lambda_n}\bigg|_{\lambda_1 = ... = \lambda_n = 0} n! \int_{x_1^0 \leq ... \leq x_n^0} dX_n \mathcal{R}_{F_\lambda}(x_1) ... \mathcal{R}_{F_\lambda}(x_n) F \\ &= \int_{x_1^0 \leq \cdots \leq x_n^0} dX_n \sum_{\pi \in S_n} \mathcal{R}_{F_{\pi_1}}(x_1) \cdots \mathcal{R}_{F_{\pi_n}}(x_n) F. \end{aligned} \tag{7.41}$$

where $F_\lambda = \sum_{k=1}^{n} \lambda_k F_k$. The above result has a diagrammatical interpretation. To make the construction clearer, consider an example: We want to calculate $\mathcal{R}_{F_1}(x_1)\mathcal{R}_{F_2}(x_2)F$:

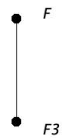
$$\begin{aligned} &\mathcal{R}_{F_1}(x_1)\mathcal{R}_{F_2}(x_2)F \\ &= \int dY_2 \left(\frac{\delta F_1}{\delta\phi(x_1)} \Delta^{\text{ret}}(y_1 - x_1) \frac{\delta}{\delta\phi(y_1)} \right) \frac{\delta F_2}{\delta\phi(x_2)} \Delta^{\text{ret}}(y_2 - x_2) \frac{\delta F}{\delta\phi(y_2)} \\ &= \int dY_2 \frac{\delta F_1}{\delta\phi(x_1)} \Delta^{\text{ret}}(y_1 - x_1) \frac{\delta^2 F_2}{\delta\phi(y_1)\delta\phi(x_2)} \Delta^{\text{ret}}(y_2 - x_2) \frac{\delta F}{\delta\phi(y_2)} \\ &+ \frac{\delta F_1}{\delta\phi(x_1)} \Delta^{\text{ret}}(y_1 - x_1) \frac{\delta F_2}{\delta\phi(x_2)} \Delta^{\text{ret}}(y_2 - x_2) \frac{\delta^2 F}{\delta\phi(y_1)\delta\phi(y_2)}. \end{aligned} \tag{7.42}$$

Diagrammatically, we can represent the result above as:

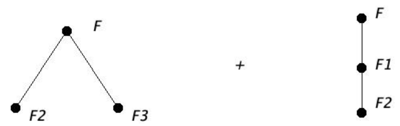


The first diagram represents the second line of (7.42) and the second diagram represents the third line of (7.42). The rule to construct the Feynman diagram and read it is to imagine the time direction flowing from the bottom to the top. The points represent the field. The inner lines represent the propagators Δ^{ret}. If a field F is connected to a field G and F is above G, the correspondent expression is $\frac{\delta F}{\delta\phi(y)}\Delta^{\text{ret}}(y-x)\frac{\delta G}{\delta\phi(x_2)}$. On the other hand, if G is above F, the correspondent expression is given by $\frac{\delta F}{\delta\phi(y)}\Delta^{\text{ret}}(x-y)\frac{\delta G}{\delta\phi(x_2)}$.

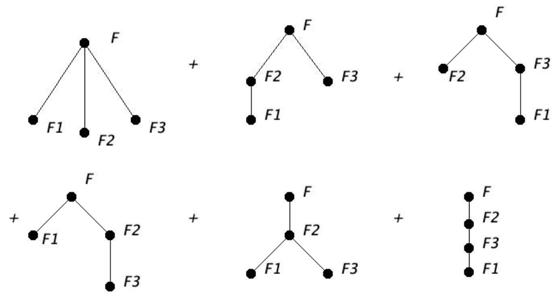
There are some rules that can be made explicitly to facilitate construction. Consider $\mathcal{R}_{F_1}(x_1)\mathcal{R}_{F_2}(x_2)...\mathcal{R}_{F_n}(x_n)F$.To build the diagrams we have to consider every connected diagram with n vertices and n inner line. The easiest way to construct such a diagram is by induction. PActing with $\mathcal{R}_{F_1}(x_1)$ on $\mathcal{R}_{F_2}(x_2)...\mathcal{R}_{F_n}(x_n)F$ is equivalent to connecting every point in the diagrams to F_1. F_1 must remain in the bottom of the connected point. Let us calculate explicitly the example discussed previously once again to make the idea clear. Since we have explicitly calculated $\mathcal{R}_{F_1}\mathcal{R}_{F_2}F$ we can check our "rules" for order $n=2$ and construct the diagrams for order $n=3$ (we will not write the whole expression). For $n=1$, $\mathcal{R}_{F_3}(x_3)F$ is given simply by $-\int dy_3\frac{\delta F_3}{\delta(x_3)}\Delta^{\text{ret}}(y_3-x_3)\frac{\delta F}{\delta\phi(y_3)}$. Diagrammatically:



Following the rule, $\mathcal{R}_{F_2}(x_2)\mathcal{R}_{F_3}(x_3)F$ basically connects the points in the previously diagram to F_2 while keeping F_2 below the connected point:



Last but not least, we can compute $\mathcal{R}_{F_1}(x_1)\mathcal{R}_{F_2}(x_2)\mathcal{R}_{F_3}(x_3)F$:



We end the discussion in this section using the diagrams above to recover the result obtained in example c) in third order in perturbation expansion, namely, we will use the diagrams above to compute $R_{3,1}\left((\int dx\, g(x)\phi^{k+1})^{\otimes 3},\phi(x)\right)$.

Since $\frac{\delta^2\phi}{\delta\phi(x)\delta\phi(y)}=0$, all the diagrams with more than 2 lines connecting F to the other points are zero. Hence, only the last two diagrams contribute. The first one reads:

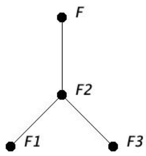


Figure 7.4: let us call the contribution to the first diagram FD just to name it.

$$
FD = \int_{x_1^0 \leq x_2^0 \leq x_3^0} dX_3 \int dY_3 \Big\{
\frac{\delta F_1}{\delta\phi(x_1)} \Delta^{\text{ret}}(y_1 - x_1) \frac{\delta^3 F_2}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(x_3)} \Delta^{\text{ret}}(y_2 - x_2) \frac{\delta F_3}{\delta\phi(x_2)} \Delta^{\text{ret}}(y_3 - x_3) \frac{\delta F(x)}{\delta\phi(y_3)} \Big\}. \tag{7.43}
$$

Using

$$
\frac{\delta F(x)}{\delta\phi(y_3)} = \frac{\delta\phi(x)}{\delta\phi(y_3)} = \delta(x - y_3) \tag{7.44}
$$

$$
\frac{\delta F_i(x)}{\delta\phi(a)} = \frac{\delta}{\phi(a)} \int dx\, g(x)\phi^{k+1}(x) = (k+1)g(a)\phi^k(a) \tag{7.45}
$$

$$
\frac{\delta^2 F_i(x)}{\delta\phi(b)\delta\phi(a)} = (k+1)kg(a)\phi^{k-1}(a)\delta(a - b) \tag{7.46}
$$

$$
\frac{\delta^3 F_i(x)}{\delta\phi(c)\delta\phi(b)\delta\phi(a)} = (k+1)k(k-1)g(a)\phi^{k-2}(a)\delta(a - b)\delta(a - c). \tag{7.47}
$$

We get

$$
\begin{aligned}
FD = &\int_{x_1^0 \leq x_2^0 \leq x_3^0} dX_3 \int dY_3 \Big\{ \\
&\Big((k+1)g(x_1)\phi^k(x_1)\Big)\, \Delta^{\text{ret}}(y_1 - x_1) \\
&\times \Big((k+1)k(k-1)g(x_3)\phi^{k-2}(x_3)\delta(y_1 - x_3)\delta(y_2 - x_3)\Big) \\
&\times \Delta^{\text{ret}}(y_2 - x_2) \Big(g(x_2)\phi^k(x_2)\Big)\, \Delta^{\text{ret}}(y_3 - x_3)\delta(x - y_3)\Big\}.
\end{aligned} \tag{7.48}
$$

Integrating in dY_3 we obtain:

$$
\begin{aligned}
FD = &\int_{x_1^0 \leq x_2^0 \leq x_3^0} dX_3 \Big\{ g(x_3)\Delta^{\text{ret}}(x - x_3) \Big((k+1)g(x_1)\phi^k(x_1)\Delta^{\text{ret}}(x_3 - x_1)\Big) \\
&\Big((k+1)g(x_2)\phi^k(x_2)\Delta^{\text{ret}}(x_3 - x_2)\Big) \Big((k+1)k(k-1)\phi^{k-2}(x_3)\Delta^{\text{ret}}(x_3 - x_1)\Big) \Big\} \\
\equiv\ & 3!(k+1) \int dx_3\, \Delta^{\text{ret}}(x - x_3)g(x_3) \frac{k(k-1)}{2} R_{0,1}^{k-2}(x_1) R_{1,1}^2(x_1).
\end{aligned} \tag{7.49}
$$

That is exactly the expression obtained in (7.36)!
the second diagram is

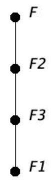


Figure 7.5: let us call the contribution to the first diagram SD just to name it.

The translation reads:

$$SD = \int_{x_1^0 \leq x_2^0 \leq x_3^0} dX_3 \int dY_3 \Bigg\{ \frac{\delta F_1}{\delta\phi(x_1)} \Delta^{\text{ret}}(y_1 - x_1) \frac{\delta^2 F_3}{\delta\phi(y_1)\delta\phi(x_2)} \Delta^{\text{ret}}(y_2 - x_2) \frac{\delta^2 F_2}{\delta\phi(y_2)\delta\phi(x_3)} \Delta^{\text{ret}}(y_3 - x_3) \frac{\delta F}{\delta\phi(y_3)} \Bigg\}. \tag{7.50}$$

Substituting the derivatives:

$$\begin{aligned} SD = \int_{x_1^0 \leq x_2^0 \leq x_3^0} & dX_3 \int dY_3 \Bigg\{ \\ & (k+1) g(x_1) \phi^k(x_1) \Delta^{\text{ret}}(y_1 - x_1)(k+1) k g(x_2) \phi^{k-1}(x_2) \delta(x_2 - y_1) \\ & \times \Delta^{\text{ret}}(y_2 - x_2)(k+1) k g(x_3) \phi^{k-1}(x_3) \delta(y_2 - x_3) \Delta^{\text{ret}}(y_3 - x_3) \delta(y_3 - x) \Bigg\}. \end{aligned} \tag{7.51}$$

Integrating in y:

$$\begin{aligned} SD = & \int_{x_1^0 \leq x_2^0 \leq x_3^0} dX_3 \Bigg\{ \\ & \Big((k+1) g(x_1) \phi^k(x_1) \Big) \Delta^{\text{ret}}(x_2 - x_1) \Big((k+1) k g(x_2) \phi^{k-1}(x_2) \Big) \Delta^{\text{ret}}(x_3 - x_2) \\ & \Big((k+1) k g(x_3) \phi^{k-1}(x_3) \Big) \Delta^{\text{ret}}(x - x_3) \Bigg\} \\ \equiv & -3!(k+1) \int dx_3 \, \Delta^{\text{ret}}(x - x_3) g(x_3) \frac{k}{2} R_{2,1}(x_1) R_{0,1}^{k-1}(x_1). \end{aligned} \tag{7.52}$$

Hence, to construct the product $R_{n,1}(F_1, ..., F_n; F)$ we have to construct the diagrams as mentioned above considering all the possible permutations of F_n.

7.6 Properties of the classical retarded product

This section is based on Section 1.10 of [24].

1. **Causality:** If $(\text{supp}\, F + V_-) \cap \text{supp}\, H = \emptyset$, then: $R(e_0^{S+\lambda H}, F) = R(e_0^S, F)$.
2. **Field independence:** The property above is basically the "Leibniz rule":

$$\frac{\delta R(e_{\otimes}^{S}, F)}{\delta \phi(z)} = R\left(e_{\otimes}^{S} \otimes \frac{\delta S}{\delta \phi}, F\right) + R\left(e_{\otimes}^{S}, \frac{\delta F}{\delta \phi}\right). \tag{7.53}$$

3. **The GLZ relation:** So called after Glaser, Lehmann and Zimmermann[36].

   Let $F, H, S \in \mathcal{F}_{\rm loc}$. Then the Poisson bracket of two interacting fields satisfies the relation [26]:

$$\begin{aligned}\left\{R(e_{\otimes}^{S}, F), R(e_{\otimes}^{S}, H)\right\} = \left.\frac{d}{d\lambda}\right|_{\lambda=0} &\left(R(e_{\otimes}^{S+\lambda F}, H) - R(e_{\otimes}^{S+\lambda H}, F)\right) \\ &= R(e_{\otimes}^{S} \otimes F, H) - R(e_{\otimes}^{S} \otimes H, F).\end{aligned} \tag{7.54}$$

4. Retarded product in terms of Poisson brackets The classical retarded product can be directly written in terms of the classical Poisson bracket of the theory:

$$\begin{aligned}R_{n,1}(S^{\otimes n}, F) = n! \int dX_n dy\, g(x_1)...g(x_n) f(y) \theta(y^0 - x_n^0) \\ \theta(x_n^0 - x_{n-1}^0)....\theta(x_2^0 - x_1^0)\{L(x_1), \{L(x_2), ...\{L(x_n), P(y)\}...\}\}.\end{aligned} \tag{7.55}$$

   This formula will be useful when constructing the quantum version of the retarded product. The proof of the statements above can be found in Section 1.10 of [24].

5. Parity and $*$ transformations

   From the general expression of retarded product it is easy to show ([24] page 42) that:

$$\begin{aligned}\left(R(e_{\otimes}^{S}, F)\right)^{*} &= R(e_{\otimes}^{S^{*}}, F^{*}) \\ \alpha(R(e_{\otimes}^{S}, F)) &= R(e^{\alpha(S)}, \alpha(F)).\end{aligned} \tag{7.56}$$

Chapter 8

Quantum retarded product

8.1 Introduction

Once the quantization procedure and the classical theory are done, we can move further to the real goal of this work, the construction of the retarded (quantum) product and the scattering matrix S. Although most courses in quantum field theory focus on the scattering matrix, there are some that use the retarded expansion instead [10, 79, 71]. The construction is done via axioms based on the classical theory. Those axioms are, only for simplicity, divided into two groups. The first, called the "basic axioms" allows us to construct the retarded product using induction on the coupling constant in a restricted domain. The second group, called "renormalization conditions", expands the domain of construction done using the basic axioms and incorporates the quantum nature of the theory. The construction will be done as follows: first the retarded product in a narrower domain is constructed using only the first group of axioms; then the second group will be introduced.

8.2 Construction part 1: Basic axioms

As one could expect, the first feature we have to adapt to pass from the classical word to the quantum one is the classical product. By these we mean that given $A, B \in \mathcal{F}$:

$$A^{\mathrm{ret}}(x) \cdot B^{\mathrm{ret}}(x) \neq (AB)^{\mathrm{ret}}(x). \tag{8.1}$$

Hence, the inductive construction presented in the preceding chapter does not hold anymore. To recover the retarded product

$$\begin{aligned} &\phi^{\mathrm{ret}} : \mathcal{C} \to \mathbb{C} \\ &\phi^{\mathrm{ret}}(x) \equiv R(e_{\otimes}^{\lambda \frac{\tilde{S}}{\hbar}}; \phi(x)) \end{aligned} \tag{8.2}$$

of a given interaction $\frac{S}{\hbar} := \frac{\lambda\tilde{S}}{\hbar}$ with coupling constant λ we introduce four axioms.

8.2.1 Linearity

We impose the retarded product is linear in its arguments. This axiom allows us to write:

$$R(e_{\otimes}^{\lambda\frac{\tilde{S}}{\hbar}}, \phi(x)) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\hbar^n n!} R_{n,1}(\tilde{S}^{\otimes n}; \phi(x))$$
$$R_{n,1} : \mathcal{F}_{\mathrm{loc}}^{\otimes(n+1)} \to \mathcal{F} \text{ is linear.} \tag{8.3}$$

Just as the classical retarded product. We emphasize: The sum above is to be understood as a formal power series, the convergence of it is not under control. Note that R is a formal power series both in λ and in $\hbar$. We denote the space of such a formal power series as $\mathcal{F}[[\lambda, \hbar]]$. For fields containing derivatives $A_1, ..., A_n, A \in \mathcal{P}, A_i = \sum_{a_i} \partial^{a_i} B_{a_i}$, we define:

$$\begin{aligned} &R_{n,1}(A_1(x_1), ..., A_n(x_n); A(x)) \\ &:= \sum_{a_1,...,a_n,a} \partial_{x_1}^{a_1} ... \partial_{x_n}^{a_n} \partial_x^a R_{n,1}(B_{a_1}(x_1), ..., B_{a_n}(x_n); B_a(x)). \end{aligned} \tag{8.4}$$

The above definition implies the “Action ward identity”:

$$\partial_{x_l} R_{n,1}(...A(x_l)...) = R_{n,1}(...\partial_{x_l} A(x_l)...). \tag{8.5}$$

8.2.2 Symmetry and initial condition

Just as in the classical case, we impose $R_{n,1}$ to be symmetric in the first n arguments:

$$R_{n,1}(F_{\pi(1)} \otimes ... \otimes F_{\pi(n)}, F) = R_{n,1}(F_1 \otimes ... \otimes F_n, F), \quad \pi \in S^n. \tag{8.6}$$

The condition above allows us to write a similar formula (7.41):

$$R_{n,1}(F_1, ..., F_n; F) = \frac{\hbar^n}{\partial\lambda_1 ... \partial\lambda_n}\bigg|_{\lambda_i = 0 \forall i} R(e_{\otimes}^{F_\lambda}, F) \tag{8.7}$$

where $F_\lambda := \sum_{k=1}^{n} \lambda_k F_k$. Another axiom we impose is the initial condition:

$$R_{0,1}(F) = F. \tag{8.8}$$

8.2.3 Causality

Last but not least, we want that only events that occur in the causal past of F can interfere with its dynamics. This feature can be mathematically stated as:

$$\begin{aligned}&\operatorname{supp} R_{n,1}(A(x_1) \otimes ... \otimes A(x_n); F(x)) \\ &\subseteq \left\{(x_1, ..., x_n, x) \in \mathbb{M}^{n+1} | x_j \in (x + \overline{V}_-) \forall j = 1, ..., n\right\}.\end{aligned} \tag{8.9}$$

The condition above can be equivalently formulated as follows:

$$R(e_\otimes^{(S+H)/\hbar}; F) = R(e_\otimes^{S/\hbar}; F) \text{ if } (\operatorname{supp} F + \overline{V}_-) \cap \operatorname{supp} H = \emptyset. \tag{8.10}$$

The proof of equivalence can be found in [24] page 77.

8.2.4 GLZ relation

In classical field theory, we can prove the GLZ relation, but in the quantum case one has to impose it. The reason for imposing it as an axiom is because it is a fundamental tool to construct the retarded product. The GLZ relation is the last of the basic axioms.

The GLZ relation provides a split of the commutator $[A(x), B(y)]_\star$ into advanced ($(x-y) \in \overline{V}_-$) and retarded ($(x-y) \in \overline{V}_+$)part. It reads:

$$\frac{1}{i\hbar}\left[R(e^{S/\hbar}; F), R(e^{S/\hbar}; H)\right]_\star = R(e^{S/\hbar} \otimes F/\hbar; H) - R(e^{S/\hbar} \otimes H/\hbar; F). \tag{8.11}$$

Or equivalently

$$\begin{aligned}&R_{n-1,1}(G_1 \otimes ... \otimes G_{n-2} \otimes F; H) - R_{n-1,1}(G_1 \otimes ... \otimes G_{n-2} \otimes H; F) \\ &= \hbar J_{n-2,2}(G_1 \otimes ... \otimes G_{n-2}; F \otimes H).\end{aligned} \tag{8.12}$$

Where:

$$\begin{aligned}&J_{n-2,2}(G_1 \otimes ... \otimes G_{n-2}, F \otimes H) \\ &:= \frac{1}{i\hbar} \sum_{I \subseteq \{1,...,n-2\}} \left[R_{|I|,1}(G_I, F), R_{|I^c|,1}(G_I^c, H)\right]_{\star_\hbar}.\end{aligned} \tag{8.13}$$

The proof is given in [24] page 78.

Some properties that can be shown using this formula are the Jacobi identity and support condition [24] page 82:

$$J_{n-2,2}(F_1, ..., F_{n-3}, F_{n-2}(x_{n-2}); F(y), G(z)) + \text{cyclic}(H, F, G) = 0 \tag{8.14}$$

$$\begin{aligned}&\text{supp}\, J_{n-2,2}(F_1, ..., F_{n-3}, F_{n-2}(x_{n-2}); F(y), G(z)) \\ &\subset \left\{(x_1, ..., x_{n-2}, y, z) \middle| \{x_1, ..., x_{n-2}, y\} \subset (z + \overline{V}_-) \text{ or } \{x_1, ..., x_{n-2}, z\} \subset (y + \overline{V}_-)\right\}.\end{aligned} \tag{8.15}$$

8.3 Construction part 1: Algorithm to construct the retarded product

We will follow [24], more specifically Sections 3.1 and 3.2. Unlike it, here we will try to give a more intuitive picture of how to construct $R_{n,1}$ using induction, emphasizing some details that are present but not explicitly written in [24]. For that reason, although the construction is essentially the same, the final result might look a bit different. The strategy is as follows: We first show how to construct the retarded product, and then we show that our construction respects the basic axioms. Later, when we introduce the second package of axioms, the renormalization conditions, we return to the construction and show that it is also coherent with it.

The example we must keep in mind is to calculate $R_{n,1}(\tilde{S}^{\otimes n}, \phi(x))$ with $\tilde{S} = \int dx\, g(x)L(x) \in \mathcal{F}_{\text{loc}}$. Using the linearity of $R_{n,1}$, we can write this product as:

$$R_{n,1}(\tilde{S}^{\otimes n}, \phi(x)) = \int dX_n\, g(x_1)...g(x_n) R_{n,1}(L(x_1)\otimes, ..., \otimes L(x_n), \phi(x)). \tag{8.16}$$

Hence, if we manage to calculate $R_{n,1}(G_1(x_1), ..., G_n(x_n), F(x_{n+1}))$ where $G_1(x_1), ..., G_n(x_n), F(x_{n+1}) \in \mathcal{P}$, we solve the problem. That is going to be the goal of this section.

Let $G_1(x_1), ..., G_n(x_n), F(x_{n+1}) \in \mathcal{P}$. The construction is done by induction in n. For $n = 0$, the axiom of the initial condition imposes $R_{0,1}(F) = F$. The case $n = 1$ is obtained using the GLZ-relation:

$$\begin{aligned}&R_{1,1}(G_1(x_1), F(x_2)) - R_{1,1}(F(x_2), G_1(x_1)) = \hbar J_{0,2}(G_1(x_1) \otimes F(x_2)) \\ &= -i \left[R_{0,1}(G_1(x_1)), R_{0,1}(F(x_2))\right]_{\star_\hbar} = -i \left[G_1(x_1), F(x_2)\right]_{\star_\hbar}.\end{aligned} \tag{8.17}$$

For now on we drop the subindex $\star_\hbar$ in the commutator. The problem in the above formula is that we have to subtract $R_{1,1}(F(x_2), G_1(x_1))$ from the term we want $R_{1,1}(G_1(x_1), F(x_2))$. We can bypass the problem by restricting the domain of $x_1, x_2 \in \mathbb{M}^2$. We have to analyze three cases.

$\mathbf{x_1, x_2}$ **space-like separated:**

In this particular scenario, the causality axiom and the properties of the commutator guarantee that

$$R_{1,1}(G(_1x_1), F(x_2)) = R_{1,1}(F(x_2), G_1(x_1)) = [G_1(x_1), F(x_2)] = 0. \tag{8.18}$$

$\mathbf{x_1} \notin (\mathbf{x_2} + \overline{\mathbf{V}}_-)$:

In this case again, due to the causality axiom:

$$R_{1,1}(G_1(x_1), F(x_2)) = 0. \tag{8.19}$$

$\mathbf{x_2} \notin (\mathbf{x_1} + \overline{\mathbf{V}}_+)$:

That is the only case we are really interested in. Due to causality:

$$R_{1,1}(F(x_2), G_1(x_1)) = 0 \Rightarrow R_{1,1}(G_1(x_1), F(x_2)) = -i\,[G_1(x_1), F(x_2)]\,. \tag{8.20}$$

Note that the union of the domains cited above is equal to $\mathbb{M}^2 \setminus \{x_1 = x_2\}$. Hence, we were able to construct $R_{1,1}(G_1(x_1), F(x_2))$ in this domain:

$$R_{1,1}(G_1(x_1), F(x_2)) = \begin{cases} -i\,[G_1(x_1), F(x_2)] & \text{if } x_2 \notin (x_1 + \overline{V}^+) \text{ and } x_1 \neq x_2 \\ 0 & (x_1 - x_2)^2 < 0 \text{ or } x_2 \in (x_1 + \overline{V}_-) \end{cases}$$

.Using the fact that the commutator is zero for space-like points, we can summarize the result above as:

$$R_{1,1}(G_1(x_1), F(x_2)) = -i\,[G_1(x_1), F(x_2)]\,\theta(x_2^0 - x_1^0). \tag{8.21}$$

Remark: Note that the product above is well defined as a product of distributions, since it respects the Hörmander criterion. To see it, we write $\Delta(x) = -i(\Delta^+(x) - \Delta^+(-x))$ and use $\mathrm{WF}(\theta(x^0)) = \{(0, (k^0, \vec{0})), k^0 \neq 0\}$ and $\mathrm{WF}(\Delta^+(x)) = \{(0, (|\vec{p}|, \vec{p})\}$. Hence, $(|\vec{p}|, \vec{p}) + (k^0, 0) \neq 0$.

We will explicitly construct the case $n = 2$.

We start the construction with some definitions to split the domain as in the previous case.

$$\begin{aligned}
\mathcal{M}_0 &:= \big\{(x_1, x_2, x_3) \in \mathbb{M}^3 | \exists j \in \{1,2\} : x_j \notin (x_3 + \overline{V}_-)\big\} \\
\mathcal{M}_1 &:= \big\{(x_1, x_2, x_3) \in \mathbb{M}^3 | x_1 \notin (x_3 + \overline{V}_+)\big\} \\
\mathcal{M}_2 &:= \big\{(x_1, x_2, x_3) \in \mathbb{M}^3 | x_2 \notin (x_3 + \overline{V}_+)\big\} .
\end{aligned} \tag{8.22}$$

First, we check that $\{\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2\}$ is an open cover (that is, a cover by open sets) of $\mathbb{M}^3 \setminus \Delta_3$ (recall: $\Delta_3 : \{x_1 = x_2 = x_3\}$).

$$\bigcup_{k=0}^{2} \mathcal{M}_k = \mathbb{M}^3 \setminus \Delta_3.. \tag{8.23}$$

The relation $\bigcup_k \mathcal{M}_k \subseteq \mathbb{M}^3 \setminus \Delta_3$ is obvious. To prove '$\supseteq$", let $x = (x_1, x_2, x_3) \notin \Delta_3$. Then there exists $j \in \{1,2,3\}$ with $x_j \neq x_3$. If $x_j \notin (x_3 + \overline{V_-})$, we have $x_j \in \mathcal{M}_0$; and if $x_j \notin (x_3 + \overline{V_+})$, we have $x_j \in \mathcal{M}_j$.

In the next step, we restrict ourselves to $\mathcal{M}_0$ in that case, due to the causality axiom:

$$R_{2,1}(G_1(x_1), G_2(x_2); F(x_3)) = 0. \tag{8.24}$$

If $(x_1, x_2, x_3) \in \mathcal{M}_1$, then either x_1 and x_3 are space-like or $x_1 \in (x_3 + \overline{V}_-)$.Using causality: $R_{2,1}(G(x_2), F(x_3); G(x_1)) = 0$. Now we use the GLZ-relation:

$$\begin{aligned}
&R_{2,1}(G_1(x_1), G_2(x_2); F(x_3)) - \underbrace{R_{2,1}(G_2(x_2), F(x_3); G_1(x_1))}_{=0} \\
&= \hbar J_{1,2}(G(x_2); G(x_1), F(x_3)) \\
&= -i \left[R_{1,1}(G_2(x_2); G_1(x_1)), R_{0,1}(F(x_3))\right] - i[R_{0,1}(G_1(x_1)), R_{1,1}(G_2(x_2); F(x_3))].
\end{aligned} \tag{8.25}$$

We can take the expression for $R_{1,1}(G(x_2); G(x_1)), R_{1,1}(G(x_2); F(x_3))$ from the first example:

$$\begin{aligned}
R_{1,1}(G_2(x_2); G_1(x_1)) &= -i \left[G_2(x_2), G_1(x_1)\right] \theta(x_1^0 - x_2^0) \\
R_{1,1}(G_2(x_2); F(x_3)) &= -i \left[G_2(x_2), F(x_3)\right] \theta(x_3^0 - x_2^0).
\end{aligned} \tag{8.26}$$

As explained above, the restriction of $R_{2,1}$ to $\mathcal{M}_1$ can be translated to $x_1 \in (x_3 + \overline{V}_-), x_1 \neq x_3$. It is also true that the commutator vanishes when the points are space-like. Hence, we can write the final expression as:

$$
\begin{aligned}
&R_{2,1}(G_1(x_1), G_2(x_2); F(x_3)) \\
&= (-i)^2\theta(x_3^0 - x_1^0)\Big([[G_2(x_2), G_1(x_1)]\,, F(x_3)]\theta(x_1^0 - x_2^0) \\
&+ [G_1(x_1), [G_2(x_2), F(x_3)]]\theta(x_3^0 - x_2^0)\Big). \qquad (8.27)
\end{aligned}
$$

Last but not least, in $\mathcal{M}_2$, we just have to change $2 \leftrightarrow 1$ in the formula above:

$$
\begin{aligned}
&R_{2,1}(G_1(x_1), G_2(x_2); F(x_3)) \\
&= (-i)^2\theta(x_3^0 - x_2^0)\Big([[G_1(x_1), G_2(x_2)]\,, F(x_3)]\theta(x_2^0 - x_1^0) + [G_2(x_2), [G_1(x_1), F(x_3)]]\theta(x_3^0 - x_1^0)\Big). \qquad (8.28)
\end{aligned}
$$

This last formula seems complicated, but we can write it in a cleaner way if we impose some causal order in x_1, x_2. If $x_1^0 < x_2^0$, $R_{2,1}$ in $\mathcal{M}_1$ can be written as:

$$
R_{2,1}(G_1(x_1), G_2(x_2); F(x_3)) = (-i)^2[G_1(x_1), [G_2(x_2), F(x_3)]]\theta(x_3^0 - x_2^0)\theta(x_2^0 - x_1^0). \qquad (8.29)
$$

For completeness, we check that the results agree in $\mathcal{M}_2 \cap \mathcal{M}_1$. To do it, just change $\theta(x_3^0 - x_1^0) \to \theta(x_2^0 - x_1^0)$ as in the equation above:

$$
\begin{aligned}
&R_{2,1}(G_1(x_1), G_2(x_2); F(x_3)) = \\
&(-i)^2\Big([[G_1(x_1), G_2(x_2)]\,, F(x_3)] + [G_2(x_2), [G_1(x_1), F(x_3)]]\Big)\theta(x_3^0 - x_2^0)\theta(x_2^0 - x_1^0) \\
&= (-i)^2[G_1(x_1), [G_2(x_2), F(x_3)]]\theta(x_3^0 - x_2^0)\theta(x_2^0 - x_1^0). \qquad (8.30)
\end{aligned}
$$

In the equation above, we have used the Jacobi identity:

$$
[[A, B], C] + [[B, C], A] + [[C, A], B] = 0. \qquad (8.31)
$$

On the other hand, if $x_2^0 < x_1^0$, $R_{2,1}$ in $\mathcal{M}_2$ can be written as:

$$
R_{2,1}(G_1(x_1), G_2(x_2); F(x_3)) = (-i)^2[G_2(x_2), [G_1(x_1), F(x_3)]]\theta(x_3^0 - x_1^0)\theta(x_1^0 - x_2^0). \qquad (8.32)
$$

Using the Jacobi identity, one can check that the result agrees in $\mathcal{M}_1$. Therefore, if $x_1 \neq x_2 \neq x_3$, we can summarize the result as:

$$R_{2,1}(G_1(x_1),G_2(x_2);F(x_3))$$
$$= (-i)^2 \sum_{\pi\in S^2} \theta(x_3^0 - x_{\pi_1}^0)\theta(x_{\pi_1}^0 - x_{\pi_2}^0)[G_{\pi_2}(x_{\pi_2}),[G_{\pi_1}(x_{\pi_1}),F(x_3)]]. \tag{8.33}$$

The above result is exactly the classical one:

$$R_{n,1}(S^{\otimes n},F) = n! \int dX_n dy\, g(x_1)...g(x_n)f(y)\theta(y^0 - x_n^0)$$
$$\theta(x_n^0 - x_{n-1}^0)....\theta(x_2^0 - x_1^0)\{L(x_1),\{L(x_2),...\{L(x_n),P(y)\}...\}\}. \tag{8.34}$$

with the change $\{.,.\} \to (-i)[.,.]$. As one can expect, that is not a mere coincidence. The above result will be imposed as a renormalization condition later.

8.3.1 General construction

The construction of $R_{n-1,1}(G_1(x_1),...,G_n(x_{n-1});F(x_n))$ is a generalization of the construction shown above. As stated above, the construction is done by induction. Suppose that we have constructed $R_{k,1}(G_1(x_1),...,G_k(x_k);F(x_{k+1}))$ for all $k\in\{1,...,n-2\}$ satisfying the basic axioms outside the diagonal $\Delta_k := (x,...x)$, k-times. We consider the following sets:

$$\mathcal{M}_0 := \left\{(x_1,\ldots,x_n) \mid \exists j\in\{1,\ldots,n-1\} \text{ with } x_j \notin (x_n+\overline{V_-})\right\},$$
$$\mathcal{M}_k := \left\{(x_1,\ldots,x_n) \mid x_k \notin (x_n+\overline{V_+})\right\}, \quad \text{for } k=1,\ldots,n-1.. \tag{8.35}$$

Just as in the previous, we check that $\{\mathcal{M}_0,\mathcal{M}_1,\ldots,\mathcal{M}_{n-1}\}$ is an open cover (i.e. a cover by open sets) of $\mathbb{M}^n\setminus\Delta_n$, that is:

$$\bigcup_{k=0}^{n-1}\mathcal{M}_k = \mathbb{M}^n\setminus\Delta_n.. \tag{8.36}$$

The relation $\bigcup_k \mathcal{M}_k \subseteq \mathbb{M}^n\setminus\Delta_n$ is obvious. To prove '$\supseteq$", let $x=(x_1,\ldots,x_n)\notin\Delta_n$. Then there exists $j\in\{1,\ldots,n-1\}$ with $x_j\neq x_n$. If $x_j\notin(x_n+\overline{V_-})$ we have $x\in\mathcal{M}_0$; and if $x_j\notin(x_n+\overline{V_+})$ we have $x\in\mathcal{M}_j$.

We impose $R_{n-1,1}(G_1(x_1),...,G_n(x_{n-1});F(x_n))=0$ on $\mathcal{M}_0$ due to causality.

Using the GLZ relation, we can construct $R_{n-1,1}(G_1(x_1),...,G_n(x_{n-1});F(x_n))$ in every subset $\mathcal{M}_k$:

$$
\begin{aligned}
&R_{n-1,1}(G_1(x_1),\ldots,G_{n-1}(x_{n-1});F(x_n)) - \underbrace{R_{n-1,1}(G_1(x_1),\ldots,F(x_n);G_k(x_k))}_{=0}\\
&= \hbar J_{n-2,2}(G_1(x_1),\ldots,\widehat{G_k(x_k)},\ldots,G_{n-1}(x_{n-1});G_k(x_k),F(x_n))\\
&= -i \sum_{I\subseteq\{1,\ldots,n-2\}} \left[R_{|I|,1}(G_I;G_k),R_{|I^c|,1}(G_{I^c},F)\right], (x_1,...,x_n)\in\mathcal{M}_k.
\end{aligned}
\tag{8.37}
$$

where $G_I := \otimes_{l\in I} G_l$ and $I^c := \{1,...,n-2\}\setminus I$. Remark: $I\setminus I = \{0\}$. Note that $R_{n-1,1}$ was constructed using only $R_{k,1}$, $k\in\{1,...,n-2\}$ which are known by hypotheses. If we can show that the object above is well defined, then, by construction, it is unique. To show that it is well defined we have to show that the formula above coincides in $\mathcal{M}_k\cap\mathcal{M}_j, \forall j,k\in\{0,...,n-1\}$.page 109 [24].

$(x_1,...,x_n)\in\mathcal{M}_0\cap\mathcal{M}_k$,$k\neq 0$:

Due to causality, we have:

$$
R_{n-1,1}(G_1(x_1),\ldots,G_{n-1}(x_{n-1});F(x_n)) = R_{n-1,1}(G_1(x_1),\ldots F)(x_n);G_k(x_k)) = 0. \tag{8.38}
$$

Thus, we need to prove that:

$$
J_{n-2,2}(G_1(x_1),...,G_{n-2}(x_{n-2});G_k(x_k),F(x_n)) = 0. \tag{8.39}
$$

It can be easily done using the support condition (8.15). A necessary condition for $J_{n-2,2}\neq 0$ is $x_k\in(x_n+\overline{V}_-)$ or $x_n\in(x_k+\overline{V}_-)$. If $x_k\in(x_n+\overline{V}_-)$, then by definition $(x_1,...,x_n)\notin\mathcal{M}_0$. On the other hand, if $x_n\in(x_k+\overline{V}_-)$,then $x_k\in(x_n+\overline{V}_+)\Rightarrow(x_1,...,x_n)\notin\mathcal{M}_k$. Hence, if

$$
(x_1,...,x_n)\in\mathcal{M}_0\cap\mathcal{M}_k \Rightarrow J_{n-2,2}(G_1(x_1),...,G_{n-2}(x_{n-2});G_k(x_k),F(x_n)) = 0. \tag{8.40}
$$

$(x_1,...,x_n)\in\mathcal{M}_j\cap\mathcal{M}_k$, $j\neq k\neq 0$:

In this case, we have:

$$
\begin{aligned}
&R_{n-1,1}(G_1(x_1),...,G_{n-1}(x_{n-1});F(x_n))\\
&= \hbar J_{n-2,2}(G_1(x_1),...,G_k(x_k);G_j(x_j),F(x_n))\\
&\overset{!}{=} \hbar J_{n-2,2}(G_1(x_1),...,G_j(x_j);G_k(x_k),F(x_n)).
\end{aligned}
\tag{8.41}
$$

Hence, we need to show that both "J's" are the same. We can do it using the Jacobi(8.15) identity and $(x_1, ..., x_n) \notin \mathcal{M}_0$ (this case was already discussed above). To simplify the notation, we will abbreviate $G_l(x_l) \equiv x_l$ in the argument of the function.

The Jacobi identity reads:

$$\begin{aligned} &J_{n-2,2}(..., x_k; x_j, x_n) + J_{n-2,2}(...., x_j; x_n, x_k) + \underbrace{J_{n-2,2}(..., x_n; x_j, x_k))}_{=0, \quad (x_1,...,x_n)\notin\mathcal{M}_0} \overset{!}{=} 0 \\ \Rightarrow &J_{n-2,2}(..., x_k; x_j, x_n) = -J_{n-2,2}(...., x_j; x_n, x_k) = J_{n-2,2}(...., x_j; x_k, x_n). \end{aligned} \tag{8.42}$$

In the last equation, we have used the skew symmetry of $J_{n-2,2}$ with respect to the last two arguments.

The above results confirm that the product exists and is unique per construction.

We claim that the construction above satisfies all basic axioms. Causality is imposed when restricting the domain to $\mathcal{M}_0$, and is therefore satisfied. The initial condition is also imposed on the construction of $R_{k,1}$ and is therefore satisfied. Symmetry can be proven using a similar argument that is used to show the existence. If $(x_1, ..., x_n) \in \mathcal{M}_0$, then $R_{n-1,1}(x_1, ..., x_{n-1}; x_n) = 0$ for every permutation in the first n arguments. If $(x_1, ..., x_n) \in \mathcal{M}_k$ and $(x_1, ..., x_n) \notin \mathcal{M}_0$., then using the GLZ equation:

$$R_{n-1,1}(x_1, ..., x_{n-1}, x_k; x_n) = \hbar J_{n-2,2}(x_1, ...; x_k, x_n). \tag{8.43}$$

The permutation of $x_1, ..., x_{n-1}$ except for x_k is immediate from the definition of $J_{n-2,2}(x_1, ..., ; x_k, x_n)$. The only tricky part is what happens if we change $x_j \leftrightarrow x_k$. If $(x_1, ..., x_n) \in \mathcal{M}_j$, the calculation was done in the last section, and the change is allowed. If $(x_1, ..., x_n) \notin \mathcal{M}_j$, then $x_j \in (x_n + \overline{V}_+)$ and $R_{n-1,1}(x_1, ..., x_{n-1}; x_n) = 0$ outside the thin diagonal to every permutation. The GLZ relation outside the thin diagonal holds by construction. Further commentaries on the GLZ relation can be found at [24] page 110.

8.4 Why we need more?

One could ask the following question "since we have constructed the retarded product using this set of axioms, why do we need the second group? Why can not I just assume the formula and perform the calculation?". That is a very good question, whose answer is also very subtle. To motivate the answer, we consider the distribution $t = \frac{1}{\|x\|^2} \in \mathcal{J}'(\mathbb{R}^3 \setminus \{0\})$. When integrated with a test or a Schwarz function:

$$\langle t, g\rangle = \int_{\mathbb{R}\setminus\{0\}} d^3x \frac{g(x)}{\|x\|^2} = \lim_{\epsilon\to 0}\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \int_\epsilon^\infty dr\, \frac{r^2 g(\vec{x})}{r^2} < \infty. \qquad (8.44)$$

In the above equation, we could extend $t \in \mathcal{J}'(\mathbb{R}^3 \setminus \{0\}) \to \mathcal{J}'(\mathbb{R}^3)$ using the same formula, and the product is still well defined.

On the other hand, the distribution $\frac{1}{\|x\|^4}$ is a legitimate distribution in $\mathcal{J}'(\mathbb{R}^3 \setminus \{0\})$, can be seen as the "square" of $\frac{1}{\|x\|^2}$ and if we simply "expand the domain":

$$\langle t^2, g\rangle = \int_{\mathbb{R}\setminus\{0\}} d^3x \frac{g(x)}{\|x\|^4} = \lim_{\epsilon\to 0}\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \int_\epsilon^\infty dr\, \frac{r^2 g(\vec{x})}{r^4} \to \infty. \qquad (8.45)$$

Hence, we need a better way to calculate the distribution passing through the singularity. This problem is not new; it is actually well known (see the appendix).

Similar problems arise in the star-product of fields. We will explicitly work out an example further in the text.

Technical remark: It is easy to prove that $\frac{1}{r^4}$ is a well-defined distribution. To do it, note that $\frac{1}{r^4}$ is proportional to $\frac{\partial^2}{\partial r^2}\frac{1}{r^2}$ and that the derivatives of a distribution are well defined as a distribution.

8.5 Construction part 2: Renormalization conditions

The second group of axioms is called "Renormalization conditions". They control the behavior of $R_{n,1}$ on the diagonal. The name "Renormalization conditions" is due to the singular behavior of those distributions on the thin diagonal. In QFT jargon, expanding the retarded product to the thin diagonal is equivalent to "subtracting infinity" in such a way that the final result remains physically relevant (see [9] for a historical review). We again appeal to our classical intuition when imposing the axioms. Essentially, the "quantum part" of the theory is manifested in the procedure of regularizing the theory.

As explained in the beginning of the section " Construction part 1: Basic Axioms", the construction is first going to be explicitly done in the lowest orders, and then we give the general recipe and show the construction respecting the desired set of axioms (basic axioms + renormalization conditions).

The renormalization conditions are:

Field independence

This axiom is inherited from classical theory and is simply:

$$\frac{\delta}{\delta\phi(x)}R_{n-1,n}(F_1\otimes...\otimes F_n)=\sum_{l=1}^{n}R_{n-1,n}(F_1\otimes...\otimes\frac{\delta F_l}{\delta\phi(x)}\otimes...\otimes F_n). \tag{8.46}$$

A consequence of the axiom is that we can relate the retarded expansion of local fields with the derivatives of the field and its derivatives. More specifically, for

$$F=f_0+\sum_{n=0}^{N}\int dX_n\, f_n(x_1,...,x_n)\phi(x_1)...\phi(x_n). \tag{8.47}$$

It is true that

$$f_n(x_1,...,x_n)=n!\omega_0(\frac{\delta^n F}{\delta\phi(x_1)...\delta\phi(x_n)}). \tag{8.48}$$

Hence:

$$\begin{aligned}
&R_{n-1,1}(F_1\otimes\cdots\otimes F_n)\\
&=\sum_{l=0}^{N}\frac{1}{l!}\int dx_1\cdots dx_l\,\omega_0\left(\frac{\delta^l R_{n-1,1}(F_1\otimes\cdots\otimes F_n)}{\delta\varphi(x_1)\cdots\delta\varphi(x_l)}\right)\varphi(x_1)\cdots\varphi(x_l)\\
&=\sum_{l_1,\dots,l_n}\frac{1}{l_1!\cdots l_n!}\int dx_{11}\cdots dx_{1l_1}\cdots dx_{n1}\cdots dx_{nl_n}\\
&\times\omega_0\left(R_{n-1,1}\left(\frac{\delta^{l_1}F_1}{\delta\varphi(x_{11})\cdots\delta\varphi(x_{1l_1})}\otimes\cdots\otimes\frac{\delta^{l_n}F_n}{\delta\varphi(x_{n1})\cdots\delta\varphi(x_{nl_n})}\right)\right)\\
&\times\varphi(x_{11})\cdots\varphi(x_{1l_1})\cdots\varphi(x_{n1})\cdots\varphi(x_{nl_n}).
\end{aligned}$$

Poicaré-covariance, $*$-structure and Field parity

These axioms are rather simple. $*$-structure is simply:

$$R_{n-1,1}(F_1\otimes...\otimes F_n)^*=R_{n-1,1}(F_1^*\otimes...\otimes F_n^*). \tag{8.49}$$

Field parity:

$$\alpha\circ R_{n-1,n}=R_{n-1,n}\circ\alpha^{\otimes n}. \tag{8.50}$$

Poicaré covariance:

$$\beta_{\Lambda,a}R_{n-1,1}(F_1\otimes...\otimes F_n)=R_{n-1,1}(\beta_{\Lambda,a}F_1\otimes...\otimes\beta_{\Lambda,a}F_n),\quad (\Lambda,a)\in\mathcal{P}_+^{\uparrow}. \tag{8.51}$$

This implies that the expansions depend only on the relative coordinates.

Off-shell field equation

One of the most important axioms is that we still want our expansion to solve the field equation:

$$\phi^{\mathrm{ret}}(x) = \phi(x) - \int dy\, \Delta^{\mathrm{ret}}(x-y)(\frac{\delta S}{\delta\phi(y)})^{\mathrm{ret}}. \tag{8.52}$$

The equivalent equation for the retarded product can be derived by substitution:

$$\begin{aligned}&R_{n,1}(F_1 \otimes ... \otimes F_n, \phi(x))\\ &= -\hbar \int dy \Delta_m^{\mathrm{ret}}(x-y) \sum_{l=1}^{n} R_{n-1,1}(F_1 \otimes ... \otimes \hat{F}_l \otimes ... \otimes F_n, \frac{\delta F}{\delta\phi(y)}).\end{aligned} \tag{8.53}$$

Scaling and mass expansion

This axiom cannot be so easily introduced as the previous ones. To properly introduce it and its motivation, we need some definitions.

(Almost) homogeneous scaling: We say that a distribution in $\mathcal{D}'(\mathbb{R}^k)$ or in $\mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ scales almost homogeneously with degree D and power $N \in \mathbb{N}$ if and only if:

$$\begin{aligned}&(\mathbb{E}_k + D)^{N+1} t(z_1, ..., z_n) = 0\\ &\iff (\rho\partial_\rho)^{N+1}(\rho^D t(\rho z_1, ..., \rho z_n)) = \frac{\partial^{N+1}}{\partial(\ln(\rho))^{N+1}}\rho^D t(\rho z_1, ..., \rho z_n)) = 0\\ &\text{and}\\ &(\mathbb{E}_k + D)^N t(z_1, ..., z_n) \neq 0.\end{aligned} \tag{8.54}$$

where $\mathbb{E}_k = \sum_{r=1}^{k} z_r \frac{\partial}{\partial z_r} = \sum_{r=1}^{k} \frac{\partial}{\partial \ln(z_r)}$. When $N = 0$, we say that the distribution scales homogeneously with degree D.

The **mass dimension** of $\partial^a\phi \in \mathcal{P}$ is defined by:

$$\dim(\partial^a\phi) := \frac{d-2}{2} + |a|, \quad a \in \mathbb{N}^d. \tag{8.55}$$

In practice, one uses the usual rule to determine the mass dimension: Each term in the Lagrangian (in d dimensions) must have dimensions d and $\dim m = 1$.

The final definition before the last axiom is:

Scaling and Mass expansion (Sm-Expansion) page 98 [24]:

A distribution $f^{(m)} \in \mathcal{D}'(\mathbb{R}^k)$ or $f^{(m)} \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$, depending on $m \geq 0$, fulfills the Sm-expansion with degree $D \in \mathbb{R}$ if and only if, for all $l, L \in \mathbb{N}$, there exist distributions $u_l^{(m)}, t_{L+1}^{(m)} \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ such that

$$f^{(m)}(X) = \sum_{l=0}^{L} m^l u_l^{(m)}(X) + \tau_{L+1}^{(m)}(X) \quad \forall L \in \mathbb{N},\, m > 0,. \tag{8.56}$$

where $X := (x_1, \ldots, x_k)$, and the following properties hold true:

(a) The leading term $u_0 \equiv u_0^{(m)}$ is independent of m and it agrees with $f^{(m=0)}$. $u_0 = f^{(m=0)}$.

(b) For $l \geq 1$ the m-dependence of $u_l^{(m)}(X)$ is a polynomial in $\ln \frac{m}{M}$, where $M > 0$ is a fixed mass scale. Explicitly, there exist m-independent distributions $u_{l,p} \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ such that

$$u_l^{(m)}(X) = \sum_{p=0}^{P_l} \ln^p \left(\frac{m}{M}\right) u_{l,p}(X), \quad P_l < \infty, \quad \forall m > 0.. \tag{8.57}$$

(Of course, the distributions $u_{l,p}$ may depend on M.)

(c) $u_l^{(m)}(X)$ scales almost homogeneously in X with degree $D - l$ and, hence, this holds also for all $u_{l,p}$, $p = 0, 1, \ldots, P_l$.

(d) $\tau_{L+1}^{(m)}(X)$ is almost homogeneous with degree D under the scaling $(X, m) \mapsto (\rho X, m/\rho)$.

(e) $\tau_{L+1}^{(m)}$ is smooth in m for $m > 0$ and

$$\lim_{m \to 0} \left(\frac{m}{M}\right)^{-(L+1)+\epsilon} \tau_{L+1}^{(m)} = 0 \quad \forall \epsilon > 0. \tag{8.58}$$

All properties are meant in the weak sense; e.g., property (e) holds for $\langle t_{L+1}^{(m)}, h\rangle$ for all $h \in \mathcal{D}(\mathbb{R}^k \setminus \{0\})$.

Finally, the last axiom states that for monomials $A_1, ..., A_n \in \mathcal{P}$ that do not depend on m, the numerical distribution:

$$r_{n-1,1}^{(m)}(A_1(x_1), ...; A_n(x_n)) := \omega_0(R_{n-1,1}^{(m)}(A_1(x_1) \otimes ... \otimes A_n(x_n))). \tag{8.59}$$

Fulfill the Sm-expansion with degree $D := \sum_{l=1}^{n} \dim(A_l)$. One could use a weaker version of this axiom and impose that $r^{(m)}$ fulfills the axioms with $\mathrm{sd}(r^{(m)}) \leq D$.

Strong Sm-expansion axiom

There is another axiom that can be imposed on the retarded product called " a stronger version of the Sm expansion". It states that in each inductive step,when expanding the distribution from

$\mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ to $\mathcal{D}'(\mathbb{R}^k)$, one needs to map the zeros of u_0, u_{lp} to the zeros of the new distributions.

Renormalization has to be done in each order of $\hbar$

This condition requires $r_{n,1}(A_1(x_1), ...; A_{n+1}(x_{n+1})) \sim \hbar^{\sum_{j=1}^{n+1} |A_j|/2}$ for $|A_j| \sim \hbar^0 \,\forall_j$ (we use the notation $0 \sim \hbar^r \,\forall r$).

Using this axiom we guarantee that $0 \in \mathcal{D}'(\mathbb{R}^n \setminus \Delta_n)$ is extended to $0 \in \mathcal{D}'(\mathbb{R}^n)$ (the distribution "0" is trivially extended and we do not sum any terms, since it would mean that we are computing renormalization of something that is not of the same order in $\hbar$) and that the limit $\hbar \to 0$[1] is equal to the classical retarded product. The proof of this statement uses the $T-$ product that will be constructed later. It can be found at [25]. The proof that the retarded product constructed outside the thin diagonal satisfies the renormalization condition can be found in [24] Chapter 3.2.

8.6 Extension of distributions

The construction process shown in the previous section solves the problem to the entire Minkowski space-time except for a point. The next step is to close that hole. There are more than one way to do it. For example, we could define a distribution $R_{n,1}(x_1, ..., x_n; x)$ following the aforementioned construction outside the thin diagonal and $R_{n,1} = 0$ if $x_1 = x_2 = ... = x$. Although it defines a distribution in the entire space, it is very artificial and it certainly does not have the properties we would expect for the retarded product (for example, the field equation). Hence, when constructing the retarded product in the points not included in the construction above, we have to impose some rules to guarantee that they will be "well behaved" as a distribution[70] page 57. To do it, we introduce the **scaling degree of a distribution** t: $\mathrm{sd}(t) \in \mathbb{R}$. This number basically controls the divergency of the singularity of a distribution. For $t \in \mathcal{D}'(\mathbb{R}^k)$ or $\mathcal{D}'(\mathbb{R}^k \setminus \{0\})$. It is defined as:

$$\begin{aligned} \mathrm{sd}(t) &:= \inf\{r \in \mathbb{R} | \lim_{\rho \to 0} \rho^r t(\rho x) = 0\} \\ \inf \emptyset &:= -\infty, \quad \inf \mathbb{R} := 0. \end{aligned} \tag{8.60}$$

The expansion needs to be done in such a way that the scaling degree is the same for the unrenormalized distribution and the extended distribution. One can prove that

$$\mathrm{sd}(\Delta^+(x)) = d - 2. \tag{8.61}$$

Since:

[1] remember: we do not have any kind of topology, thus the limit is simply plugging $\hbar = 0$

$$\Delta^{+}(\rho x) = \int \frac{d^d p}{(2\pi)^{d-1}} \theta(p_0)\delta(p^2 - m^2) e^{-ip\rho x}$$

$$\rho p \equiv u \Rightarrow \int \frac{d^d u}{\rho^d (2\pi)^{d-1}} \theta(\frac{u_0}{\rho}) \delta(\underbrace{\frac{u^2}{\rho^2} - m^2}_{\frac{1}{\rho^2}(u^2 - (\rho m)^2)}) e^{-ip\rho x} = \rho^{d-2} \Delta^{+}_{\rho m}(x). \tag{8.62}$$

And similarly

$$\mathrm{sd}(\Delta^{\mathrm{ret}}(x)) = \mathrm{sd}(\Delta(x)) = \mathrm{sd}(\Delta^{+}(x)) = d - 2. \tag{8.63}$$

One can also prove that

$$\mathrm{sd}(t \cdot u) = \mathrm{sd}(t) + \mathrm{sd}(u). \tag{8.64}$$

hence

$$\mathrm{sd}\left((\Delta^{+}(x))^n\right) = n(d-2). \tag{8.65}$$

The properties mentioned above will be important because given a distribution $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ with $\mathrm{sd}(t)$, then:

- if $\mathrm{sd}(r) < k$ the extension t of t^0 exist, preserve the scaling degree, i.e, $\mathrm{sd}(t) = \mathrm{sd}(t^0)$ and is given by **the same formula as** t^0
- if $k < \mathrm{sd}(r) < \infty$ there are more than one possible extension t of t^0 preserving the scaling degree but they are connected to each other by:

  $$t = t_0 + \sum_{|a| < \mathrm{sd}(t) - k} C_a \partial^a \delta(x), \quad C_a \in \mathbb{R}. \tag{8.66}$$

  i.e, they only differ by local terms. [24] page 117

The (informal) idea behind the formula above is that we only need to worry about distributions that are bad behaved in the origin, i.e., can not be treated as a function near the origin.

An important remark: This is only informal in the sense that we can define the distributions passing through a singularity [2]. The "divergence" only indicates the cases where we have some "liberty" to sum counter-terms. We will return to this example once the necessary tools are developed to extend a distribution.

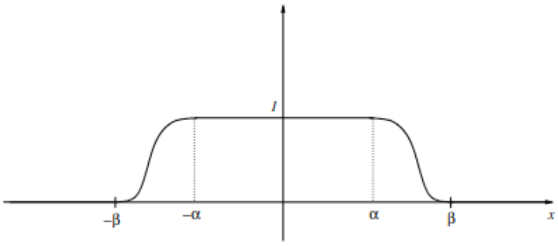


Figure 8.1: $v(x)$ with $\alpha = 1$ and $\beta = 2$. Available at [2] pg2143.

8.7 Existence and uniqueness of extension

In (8.6) we stated that there are extensions of a distribution $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\}) \to \mathcal{D}'(\mathbb{R}^k)$ that preserve the scaling degree. In this section, we will prove they indeed exist and show that they also preserve the other axioms. To do it, we use the W-expansion. Let us start with the easiest case first:

8.7.1 $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ and $\mathrm{sd}(t^0) < k$:

To prove that there is an extension t that satisfies $\mathrm{sd}(t) = \mathrm{sd}(t^0)$ and $t(x) = t^0(x) \forall \in \mathbb{R}^k \setminus \{0\}$ we will adapt the prove given in [12]. To do it, we need a preliminary result: The space of distributions sequentially is complete, that is, given $t_n(x) \in \mathcal{D}'(\mathbb{R}^k)$ a sequence of distributions such that $|\langle t_n - t_m, g\rangle| \to 0$ for sufficiently big $n, m \in \mathbb{N}$ for all $g \in \mathcal{D}(\mathbb{R}^k)$ then there is $t \in \mathcal{D}'(\mathbb{R}^k)$ such that $t_n \to t$. The proof of this statement can be found in [47] Chapter 2. Once that has been said, consider the smooth functions $v(x)$ given by:

$$v(x) := \begin{cases} 0 & \|x\| \geq 2 \\ 0 \leq g(x) \leq 1 & 1 \leq \|x\| \leq 2 \\ 1 & \|x\| \leq 1 \end{cases} \tag{8.67}$$

and $\chi(x) = 1 - v(x)$ given by

$$\chi(x) = \begin{cases} 0, & \|x\| < 1 \\ 0 \leq \chi(x) \leq 1 & 1 \leq \|x\| \leq 2 \\ 1 & \|x\| \geq 2 \end{cases} \tag{8.68}$$

.For $k = 1$ we can drawn such a function:

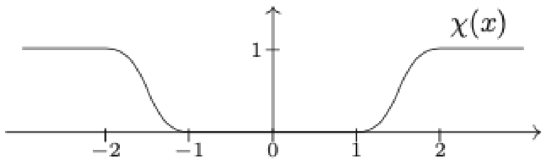


Figure 8.2: $\chi(x)$ Available at [24] pg. 117

For $n > 0, n \in \mathbb{N}$, define $\chi_n(x) := \chi(nx)$. Note that if $h \in \mathcal{D}(\mathbb{R}^k)$, then $\chi_n h \in \mathcal{D}\left(\mathbb{R}^k \setminus \{0\}\right) \forall n > 0$. Hence, $\langle t^0, \chi_n h\rangle$ exists. Due to the definition of χ, we have $\lim_{n\to 0} \chi_n(x) = 1$. Hence, if can prove that a distribution t defined by

$$t(x) = \lim_{n\to\infty} \chi_n(x)t^0(x) \tag{8.69}$$

is well defined, t is an extension of t^0. To prove that the limit above exists, we prove it is a Cauchy sequence. Let $n > m \in \mathbb{N}$, $v_n(x) := v(nx)$ and $g \in \mathcal{D}(\mathbb{R}^k)$, then:

$$\begin{aligned}
&\langle \chi_n(x)t^0(x), g(x)\rangle - \langle \chi_m(x)t^0(x), g(x)\rangle \\
&= \langle (1 - v_n(x))t^0(x), g(x)\rangle - \langle (1 - v_m(x))t^0(x), g(x)\rangle \\
&= \langle v_m(x)t^0(x), g(x)\rangle - \langle v_n(x)t^0(x), g(x)\rangle \\
&= m^{-k}\langle v(x)t^0(m^{-1}x), g(m^{-1}x)\rangle - n^{-k}\langle v(x)t^0(n^{-1}x), g(n^{-1}x)\rangle.
\end{aligned} \tag{8.70}$$

In the last line, we have made a change in variables: $nx = u \Rightarrow dx = n^{-k}du$. We kept the variable x for simplicity. We claim that the expression above gets arbitrary small for n, m sufficiently big. To prove it note that defining $\epsilon := n^{-1}$ we have:

$$n^{-k}\langle v(x)t^0(n^{-k}x), g(n^{-d}x)\rangle = \langle \epsilon^{-k}t^0(\epsilon x), v(x)g(\epsilon x)\rangle \to 0. \tag{8.71}$$

Since $n \to \infty \iff \epsilon \to 0$, $\mathrm{sd}(t^0) < k$ and t^0 is acting in a smooth function [2]. Hence, $\chi_n(x)t^0(x)$ is a Cauchy sequence and converges to $t \in \mathcal{D}'(\mathbb{R}^d)$, which is an expansion of t^0.

To prove that it is unique, let t_1 and t_2 be extensions of t^0 with the same scaling degree. Then

$$\mathrm{supp}(t_1 - t_2) \subseteq \{0\},$$

so that

$$t_1 - t_2 = \sum_a C_a \partial^a \delta_{(k)}$$

[2] $v(x)g(\epsilon x)$ is smooth because v and g are smooth and $\mathrm{supp}(v(x)g(\epsilon x)) \subseteq \mathrm{supp}(v(x)) = B_2(0)$

for some coefficients C_a. However, since

$$\mathrm{sd}(t_1 - t_2) \leq \max\{\mathrm{sd}(t_1), \mathrm{sd}(t_2)\} = \mathrm{sd}(t^0) < k,$$

while

$$\mathrm{sd}(\partial^a \delta_{(k)}) = k + |a|,$$

we see that $C_a = 0$ for all a.

8.7.2 $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ and $\mathrm{sd}(t^0) \geq k$:

In this case, let $\omega = \mathrm{sd}(t^0) - k \in \mathbb{R}^+$, $\lfloor \omega \rfloor$ the integer part of ω and the set

$$\mathcal{D}_\omega \equiv \mathcal{D}_\omega(\mathbb{R}^k) := \left\{ h \in \mathcal{D}(\mathbb{R}^k) | \partial^a h(0) = 0 \text{ for } a \leq \lfloor \omega \rfloor \right\}. \tag{8.72}$$

Note that $\mathcal{D}(\mathbb{R}^k \setminus \{0\}) \subset \mathcal{D}_\omega$ (if $h(0) = 0$, then, by definition, $\mathrm{supp}(h) = \mathbb{R}^k \setminus \{0\}$ and $h \in \mathcal{D}'_\omega$). We claim that every $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ has a unique extension $t_\omega \in \mathcal{D}'_\omega$. To prove it, first note that if $h \in \mathcal{D}_\omega$ one can write ([1] chapter 38.2 and [2] chapter 39.2):

$$h(x) = \sum_{a=\lfloor \omega \rfloor + 1} x^a g_a(x),\; g_a(x) \in \mathcal{D}(\mathbb{R}^k). \tag{8.73}$$

Using this decomposition, we define:

$$\langle t_\omega, h \rangle := \sum_{|a|=\lfloor \omega \rfloor + 1} \langle x^a t^0, g_a \rangle. \tag{8.74}$$

The extension of $x^a t^0$,is guaranteed by the first part, since

$$\mathrm{sd}(x^a t^0) \leq \mathrm{sd}(t^0) - (\lfloor \omega \rfloor + 1) < k.. \tag{8.75}$$

Therefore, if we are able to construct a projector $W : \mathcal{D}(\mathbb{R}^k) \to \mathcal{D}_\omega$, it defines an extension t^W (W-expansion) by:

$$\langle t^W, h \rangle := \langle t_\omega, W h \rangle. \tag{8.76}$$

If $h \in \mathcal{D}(\mathbb{R}^k \setminus \{0\})$, then $Wh = h$ since $\mathcal{D}(\mathbb{R}^k \setminus \{0\}) \subset \mathcal{D}_\omega$. Thus:

$$\langle t^W, h \rangle := \langle t_\omega, W h \rangle = \langle t^0, h \rangle. \tag{8.77}$$

hence, it is indeed an extension. To prove that $\mathrm{sd}(t^W) = \mathrm{sd}(t^0)$ we refer to [12]. The W

projections are, in general, not unique. The most general projection that one can construct is given by:

$$Wh(x) := h(x) - \sum_{|a|\leq\lfloor\omega\rfloor} \partial^a h(0)\omega_a(x). \tag{8.78}$$

with $\omega \in \mathcal{D}(\mathbb{R}^k)$, $\partial^b\omega_a(0) = \delta^b_a$ and $b < \lfloor\omega\rfloor$.

As an example of historical importance, we can construct the "central solution" [29]:

$$Wh(x) := h(x) - \sum_{|a|\leq\lfloor\omega\rfloor} \frac{x^a}{a!}\partial^a h(0). \tag{8.79}$$

Remark: If t^0 does not decay fast enough at infinity, we have to change the projection above by

$$Wh(x) := h(x) - \sum_{|a|\leq\lfloor\omega\rfloor} \frac{x^a}{a!}\omega(x)\partial^a h(0). \tag{8.80}$$

with $\omega \in \mathcal{D}(\mathbb{R}^k)$,$\omega(0) = 1$ and $\partial^a\omega(0) = 0$. The above projector was used in the original paper by Henri Epstein and Vladimir Glaser [29]. Further examples of w's can be found in [60] and references therein.

Once the function ω is fixed, $\langle t^0, Wh\rangle$ is uniquely determined. But we still have some freedom to "sum" the terms we subtracted, namely: $\sum_{|a|\leq\omega} C_a\partial_a h(0)$, $C_a \in \mathbb{C}$. Those are the famous "counter terms" of quantum field theories. The constants C_a need to be fixed using physical data or some symmetry (see Chapter 4 [24]).

The great part of this formalism is that we can prove that all extensions of t^0 (especially those respecting the axioms) are W projections. The proof can be found on [24] page.122. The price of using these methods is that the scaling power N (8.54) may be lost but, in the worst-case scenario, it increases to $N+1$. An important remark we are not going to explore in this work is that to maintain a symmetry, we have to impose further constrains to the projection, and that is not always possible. When such a case occurs, we obtain the famous anomalies of quantum field theory. For more details, we refer again to Chapter 4 [24].

8.7.3 Example

To motivate the discussion, we use a similar example of (8.44) adapted to physics (this example was extracted from [24] page 122). Consider the electric potential in 3 dimensions with a point charge at the origin:

$$\nabla^2\Phi(x) = -\frac{\rho(x)}{\epsilon_0} = -4\pi q\delta(x). \tag{8.81}$$

The solution is well known:

$$\Phi(x) = \frac{q}{\|x\|}. \tag{8.82}$$

We could *a priori* say that this distribution exists only in $\mathcal{D}'(\mathbb{R}^3 \setminus \{0\})$.But, since $\mathrm{sd}(\Phi(x)) = 1 < 3$, we can compute the extension using the same formula as above. The intuition is that this distribution exists because given $g(x) \in \mathcal{D}(x)$:

$$\langle \Phi, g\rangle = \int d^3x\, \frac{q}{\|x\|} g(x) \sim \int_0^\infty dr\, r^2 \frac{q}{r} g(x) < \infty. \tag{8.83}$$

Hence, the formula is sufficiently "well behaved". On the other hand, the energy density is given by:

$$U = \frac{1}{4\pi^2}\|\vec{E}\|^2 = \frac{1}{4\pi^2}\frac{q^2}{\|x\|^4}. \tag{8.84}$$

Again, the formula above is defined only in $\mathcal{D}'(\mathbb{R}^3 \setminus \{0\})$. But differently from last time $\mathrm{sd}(U) = 4 > 3$. Hence, we have more than one possibility to extend this distribution and still maintain the scaling degree. Note that differently from the electric potential:

$$\langle U, g\rangle = \frac{1}{4\pi^2}\int d^3x \frac{q^2}{\|x\|^4} g(x) \sim \int_0^\infty \frac{r^2}{r^4} g(x) \sim \lim_{r\to 0}\frac{1}{r} \to \infty. \tag{8.85}$$

Let us calculate an example using the central solution for the distribution (8.84).In this case, $\omega = \mathrm{sd}(U) - 3 = 1$.The ω subspace is given by:

$$\mathcal{D}_\omega = \{h \in \mathcal{D}(\mathbb{R}^k) | \partial^a h(0) = 0, a = 1, 2, 3\}. \tag{8.86}$$

And the $W-$ projection reads:

$$Wh(x) = h(x) - \sum_{k=1}^{3} \omega_k(x)\partial^k h(0) - \omega_0(x)h(0). \tag{8.87}$$

The central solution:

$$h(x) - \omega(x)(h(0) - \vec{\nabla} \cdot \vec{h}(0)). \tag{8.88}$$

The U^W is now given by:

$$\langle U, Wh \rangle = \int d^3x \frac{q}{4\pi \|\vec{x}\|^4} \Big(h(x) - \omega(x)(h(0) - \vec{\nabla} \cdot \vec{h}(0)) \Big). \tag{8.89}$$

And the most general extension we can do is:

$$\langle U, h \rangle = \langle U^W, h \rangle + C_0 h(0) + \vec{C}_k \cdot \vec{\nabla} h(0), \quad \vec{C}_k = (c_x, c_y, c_z) \in \mathbb{R}^3. \tag{8.90}$$

The self-energy is calculated for $h(x) = 1$. The simplest choice we can make to perform this calculation is $\omega(x) = 1$. Using these parameters, we conclude

$$\langle U, 1 \rangle = C_0 \stackrel{!}{=} mc^2. \tag{8.91}$$

The self-energy of the particle.

From the example above, we learned how to extend distributions avoiding divergences and the "extra term" we include in the most general expansion must be fixed by its physical meaning. This procedure is analogous to "subtracting infinity" from the usual QFT. Unfortunately, the W expansion is not very good for practical computations. In the future, we will use other methods of regularization that are easier to manipulate.

8.8 Inductive construction of the retarded product in the entire space

Now we have all the tools to construct the retarded product in the entire space. The goal is to inductively construct $R_{n,1}$ in such a way that all the basic and renormalization axioms are fulfilled. Off-the-diagonal, the work is done. If we find an extension to the thin diagonal that preserves the scaling degree, we finish the construction of $R_{n,1}$. The "formal" way of doing it is very technical and goes in the opposite direction to what the thesis is intended to. The construction can be found in Chapter 3.2.4, page 130 [24]. Instead of repeating the steps of the book, we present a different approach. We take examples, show how to solve them, and show how the recipe generalizes. To solve the extension problem, we will use what is called "differential renormalization". This method is the easiest to be implemented in the configuration space. We postpone the topic because we will face similar problems in the construction of $T-$ product, studied mainly in the context of scattering theory. Since the way to solve the problem is the same, we present it and use it to solve both the $T-$ and $R-$ products. The solution can be interpreted as Feynman diagrams. For the $R-$ product, this representation is not very useful. On the other hand, for the $T-$ product is very useful to compute amplitudes.

Chapter 9

T-product

9.1 Introduction

Once the construction of the $R-$product is complete, we can focus on the construction of the time-order product, also known as the $T-$ product. The construction itself is very similar, and the $R-$ and $T-$ products are closely related to each other through the S matrix. Here we will not explain the importance of the T product and the S matrix to scattering theory since, presumably, those reading this work are initiated in quantum field theory. If for some reason the reader has not previously studied the subject, we can refer to a series of books and notes:The first two are closely related to these dissertation [24] and [67] chapter 3.3. The notes of Klaus Fredenhangen on QFT are always a good place to learn basic concepts [35]. Last but not least, the last two references are the "standard ones" in QFT [68, 75].

Once that has been said, we will axiomatically construct the $T-$ product, just as we have done with the $R-$product. The axioms are divided into the "basic axioms" and the "renormalization conditions". The axioms are partially copied without edition from [24];

9.1.1 Basic axioms:

(I)Linearity, (II) Symmetry in the arguments, (III) Initial condition $T_1(F) = F$ and (IV) causality:

$$\begin{aligned}&T_n(A_1(x_1), ..., A_n(x_n))\\=&T_k(A_{\pi(1)}(x_{\pi(1)}), ..., A_{\pi(k)}(x_{\pi(k)})) \star T_{n-k}(A_{\pi(k+1)}(x_{\pi(k+1)}), ..., A_{\pi(n)}(x_{\pi(n)}))\end{aligned} \tag{9.1}$$

whenever $\{x_{\pi(1)}, ..., x_{\pi(k)}\} \cap (\{x_{\pi(k)}, ..., x_{\pi(n)}\} + \overline{V}_-) = \emptyset$.

Probably, the most important formula for practical computation is that the unrenormalized time order product on $\check{\mathbb{M}} := \{(x_1, ..., x_n) \in \mathbb{M}^n | x_l \neq x_j \forall 1 \leq l < j \leq n\}$ [1] is given by:

$$T_n(A_1(x_1) \otimes .... \otimes A_n(x_n)) = A_1(x_1) \star_{\Delta^F} ... \star_{\Delta^F} A_n(x_n). \tag{9.2}$$

Where Δ^F is the Feynman propagator:

$$\Delta^F(x) = \Delta^+(x)\theta(x^0) + \Delta^+(-x)\theta(-x^0) \tag{9.3}$$

and $\star_F \equiv \star_{\Delta^F}$ indicates that we have changed the Wightman two-point function by the Feynman propagator in the formula of the star product.

9.1.2 Renormalization conditions:

These additional axioms for the T-product read:

(v) **Field independence**:

$$\delta T_n / \delta\varphi = 0,$$

or more explicitly,

$$\frac{\delta T_n(F^{\otimes n})}{\delta\varphi(x)} = n T_n\left(\frac{\delta F}{\delta\varphi(x)} \otimes F^{\otimes(n-1)}\right). \tag{v}$$

Similarly to the R-product, this axiom is equivalent to the requirement that T_n satisfies the causal Wick expansion.

(vi) **$*$-structure and field parity**: Field parity is the condition

$$\alpha \circ T_n = T_n \circ \alpha^{\otimes n}. \tag{vi}$$

The formulation of the $*$-structure condition is more complicated. As discussed in the historical introduction, one of the most important features of the S-matrix is that it is unitary, i.e, $S^* = S^{-1}$. We can prove that for every $S \in \mathcal{V}[\![\lambda]\!]$ of the form $S = 1 + \sum_{n=1}^{\infty} a_n \lambda^n$, exist $S^{-1} \in \mathcal{V}[\![\lambda]\!]$. The construction is done term by term in λ. [2]. Hence, the existence of S^{-1} is guaranteed. The $*$- axiom is chosen in such a way $S^* = S^{-1}$. For the unrenormalized $T_n(A_1(x_1), ..., A_n(x_n)) \in \mathcal{D}(\check{\mathbb{M}}^n)$ we can find a explicitly formula [24] page 175:

$$(T_n(A_1(x_1), ..., A_n(x_n)))^* = \overline{T}_n(A_1^*(x_1), ..., A_n^*(x_n)). \tag{9.4}$$

[1] the restriction of the domain is just a fancy way to write $x_j \neq x_k \forall j, k \in \{1, ..., n\}$

[2] For example,in first order in λ $S^{-1} = 1 - a_n\lambda$ because $(1 - a_n\lambda)(1 + a_n\lambda) = 1 + O(\lambda^2)$

Where $\overline{T}$ is the "antichronological $T-$ product" (the expansion of $S^{-1}(F) := \overline{T}(e_{\otimes}^{-\frac{iF}{\hbar}})$. More details can be found in [24] chapter 3.3.

(vii) **Poincaré covariance**:

$$\beta_{\Lambda,a} \circ T_n = T_n \circ \beta_{\Lambda,a}^{\otimes n} \quad \forall (\Lambda, a) \in \mathcal{P}_{\uparrow}^{+}.$$

(viii) **Off-shell field equation**:

$$\begin{aligned} T_n(\varphi(g) \otimes F_1 \otimes \cdots \otimes F_{n-1}) &= \varphi(g) T_{n-1}(F_1 \otimes \cdots \otimes F_{n-1}) \\ &+ \frac{i}{\hbar} \int dx\, dy\, g(x) \Delta_F(x-y) \frac{\delta}{\delta\varphi(y)} T_{n-1}(F_1 \otimes \cdots \otimes F_{n-1}). \end{aligned} \tag{viii}$$

where $g \in \mathcal{D}(\mathbb{M})$.

(ix) **Sm-expansion**: For all monomials $A_1, \ldots, A_n \in \mathcal{P}$, the distributions

$$t^{(m)}(A_1, \ldots, A_n)(x_1 - x_n, \ldots, x_{n-1} - x_n) := \omega_0 \left(T^{(m)}(A_1(x_1) \otimes \cdots \otimes A_n(x_n)) \right). \tag{ix}$$

fulfill the Sm-expansion with degree $D = \sum_{j=1}^{n} \dim A_j$.

- **Scaling degree**: Similarly to the R-product, the Scaling degree axiom,

$$\operatorname{sd} t(A_1, \ldots, A_n)(x_1 - x_n, \ldots, x_{n-1} - x_n) \le \sum_{j=1}^{n} \dim A_j \quad \forall A_1, \ldots, A_n \in \mathcal{P}_{\text{hom}}. \tag{Scaling Axiom}$$

is a less restrictive substitute for the Sm-expansion axiom.

(x) **$\hbar$-dependence**:

$$t(A_1, \ldots, A_n) \sim \hbar^{\sum_{j=1}^{n} \dim A_j - n|A_j|/2}. \tag{x}$$

for all monomials $A_1, \ldots, A_n$ which fulfill $A_j \sim h^0 \,\forall j$ where $|A_j|$ is the order of the field.

The scattering matrix is given by:

$$S(F) := 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!\hbar^n} T_n \left(F^{\otimes n} \right) \equiv T(e_{\otimes}^{\frac{iF}{\hbar}}). \tag{9.5}$$

9.2 Construction of T product

We will inductively construct the $T-$ product. Just as in the case of the retarded product, the basic axioms are enough to construct the T product everywhere except in the thin diagonal Δ_n. The renormalization conditions extend the distribution to all points in Minkowski space-time.

In this context we will first show (9.2), then following the steps of Epstein Glaser[29] show how to construct inductively T in $\mathbb{M}\setminus\Delta_n$. The extension is completely analogous to the extension of the $R-$ product, and therefore we will not repeat it here.

9.2.1 Proof of $T_n(A_1(x_1)\otimes....\otimes A_n(x_n)) = A_1(x_1)\star_{\Delta^F}...\star_{\Delta^F} A_n(x_n)$

Given a $T-$ product $T_n : \mathcal{F}_{\text{loc}}^{\otimes n} \to \mathcal{F}$ that meets the basic axioms and $(x_1,...,x_n)\in\mathbb{M}^n, x_i \neq x_j \forall i,j$. We will prove the statement using induction on n. $n=1$ is trivial. For pedagogical reasons, we will first show the case $n=2$, the generalization is immediate.

Since $x_1 \neq x_2$, we can divide our analyses into three cases, I) $x_1^0 < x_2^0$, II) $x_2^0 < x_1^0$ and III) $x_1^0 = x_2^0 \Rightarrow (x_1-x_2)^2 < 0$. In the first case, we can use causality to write:

$$T_2(A_1(x_1), A_2(x_2)) = T_1(A_1(x_1)) \star T_1(A_2(x_2)) = A_1(x_1)\star A_2(x_2). \tag{9.6}$$

The second case is exactly the same with the changed index $(1\leftrightarrow 2)$:

$$T_2(A_1(x_1), A_2(x_2)) = T_1(A_2(x_2)) \star T_1(A_1(x_1)) = A_2(x_2)\star A_1(x_1). \tag{9.7}$$

In the third case, the two formulas above agree since for space-like distant points $\Delta^+(x_1 - x_2) = \Delta^+(x_2-x_1)$. Explicitly:

$$\begin{aligned}
A_1(x_1)\star A_2(x_2) &= \sum_{l=0}^{\infty}\frac{\hbar^l}{l!}D_l(A_1(x_1))(\Delta^+(x_1-x_2))^l D_l(A_2(x_2))\\
&= \sum_{l=0}^{\infty}\frac{\hbar^l}{l!}D_l(A_1(x_1))(\Delta^+(x_2-x_1))^l D_l(A_2(x_2))\\
= \sum_{l=0}^{\infty}\frac{\hbar^l}{l!}D_l(A_2(x_2))&(\Delta^+(x_2-x_1))^l D_l(A_1(x_1)) = A_2(x_2)\star A_1(x_1).
\end{aligned} \tag{9.8}$$

where

$$D_l(A_1(x_1)) := \int dY_l\,\frac{\delta^l A_1(x_1)}{\delta\phi(y_1)...\delta\phi(y_l)}. \tag{9.9}$$

We can summarize the tree cases using the Feynman propagator defined as:

$$\Delta^F(x) := \theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^+(-x). \tag{9.10}$$

Thus, in case I) $\Delta^F(x_1-x_2) = \Delta^+(x_2-x_1)$, in case $II)\Delta^F(x_1-x_2) = \Delta^+(x_1-x_2)$, and in case III), $\Delta^F(x_1-x_2) = \Delta^+(x_1-x_2) = \Delta^+(x_2-x_1)$. Hence, changing the propagator $\Delta^+ \leftrightarrow \Delta^F$

in (9.7) contemplates all possible scenarios:

$$T_2(A_1(x_1), A_2(x_2)) = A_1(x_1) \star_F A_2(x_2). \tag{9.11}$$

Note that the product above is also symmetric since Δ^F is.

To finish the proof, assume that $T_{n-1}(A_1(x_1), ..., A_{n-1}(x_{n-1})) = A_1(x_1) \star_F ... \star_F A_{n-1}(x_{n-1})$ for $(x_1, ..., x_{n-1}) \in \mathbb{M}^{n-1}$ with $x_i \neq x_j \, \forall i, j$. We want to show that a similar formula holds for $T_n(A_1(x_1), ..., A_n(x_n))$, $(x_1, ..., x_n) \in \mathbb{M}^n$ with $x_i \neq x_j \, \forall i, j$. Let us consider without loss of generality[3] that $x_1^0 \leq x_j^0 \, \forall j \in \{2, ..., n\}$.

Since $x_1 \cap \{x_2, ..., x_n\} + \overline{V}_- = \emptyset$, we use causality to write:

$$\begin{aligned} T_n(A_1(x_1), ..., A_n(x_n)) &= A_1(x_1) \star T_{n-1}(A_2(x_2), ..., A_n(x_n)) \\ &= A_1(x_1) \star_F T_{n-1}(A_2(x_2), ..., A_n(x_n)). \end{aligned} \tag{9.12}$$

Using the induction hypothesis, we obtain:

$$T_n(A_1(x_1), ..., A_n(x_n)) = A_1(x_1) \star_F ... \star_F A_n(x_n). \tag{9.13}$$

Since the product above is symmetric and linear in each entry, it satisfies the basic axioms and the proof is complete.

The formula above is very useful for practical calculations (we will use it later) since it enables us to calculate very quickly the lowest order of the unrenormalized T-product. Usually, the same formula defines the T- product not only in $\check{\mathbb{M}}^n$ but also in $\mathbb{M}^n \setminus \Delta_n$. Unfortunately, the construction of the $T-$product needs more than a formula in $\check{\mathbb{M}}$, we need to construct it, just as the construction of the R-product, in the entire space except in the thin diagonal. One can construct it using the "splitting property" [67, 29, 61]. Another construction using partitions of unity is given in [12] and [55]. Here we only indicate how one can construct it using induction, but do not go through all the details of the calculation. For those, we refer to the references cited above.

The formula worked on in this section has a nice interpretation in terms of Feynman diagrams. To construct such diagrams, every point represents a field A_i and a line connecting A_i and A_j, a Feynman propagator. For example, consider $A^2(x) \star_F A^2(y) = A^2(x)A^2(y) + 4\hbar A(x)\Delta_F(x - y)A(y) + 2\hbar^2 \Delta_F^2(x - y)$. Diagrammatically, we can represent it by:

[3] If it is not the case, let us say $x_k^0 \leq x_j^0 \forall j \in \{1, ..., n\} j \neq k$, just relabel the indices with $x_k \equiv x_1$.

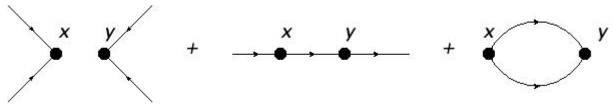


Figure 9.1: The external lines represent powers of fields and internal lines powers of Feynman propagator

9.2.2 Construction of the unrenormalized $T-$ product on $\mathbb{M}^n \setminus \Delta_n$

We divide the space into the following sets:

$$
\begin{aligned}
C_I :=& \{(x_1, ..., x_n) \in \mathbb{M}^n | x_i \notin (x_j + \overline{V}_-) \forall i \in I, j \in I^C\} \\
&\text{with } I \subset \{1, ..., n\}, I \neq \{1, ..., n\}, I \neq \emptyset.
\end{aligned} \tag{9.14}
$$

Note that any set of points $(x_1, ..., x_n)$ in $\mathbb{M}^n \setminus \Delta_n$ is within at least one of the above sets. Hence:

$$\bigcup_I C_I = \mathbb{M}^n \setminus \Delta_n. \tag{9.15}$$

And, for a given subset I causality implies:

$$
\begin{aligned}
&T_n^0(A_1(x_1), ..., A_n(x_n)) \\
=&T_{|I|}(A_{i_1}(x_{i_1}), ..., A_{i_{|I|}}(x_{i_{|I|}})) \star T_{|I|^c}(A_{i_1}(x_{i_1}), ..., A_{i_{|I|^c}}(x_{i_{|I|^c}})).
\end{aligned} \tag{9.16}
$$

The final step is to show that the formula above is the same when $C_I \cap C_J \neq \emptyset$. To simplify the notation, we divide the proof into three cases. First, we consider $\emptyset \neq I \subset J$ and enumerate the indices such that $I = \{1, .., i\}$, $J = \{1, ..., i, i+1, ..., j\} = I \cup (I^c \cap J)$ (which is always possible due to the symmetry of T). Hence: $I^c = \{i+1, ..., j, .., n\} = J \setminus (I \cup J^c)$ and $J^c = \{j+1, ..., n\}$. Using the formula described above:

$$
\begin{aligned}
T_n(x_1, ..., x_n) &= T_{|I|}(x_1, ..., x_i) \star T_{|I|^c}(x_{i+1}, ..., x_n) \\
=&T_{|I|}(x_1, ..., x_i) \star T_{|J \setminus I|}(x_{i+1}, ..., x_j) \star T_{|J|^c}(x_{j+1}, .., x_n).
\end{aligned} \tag{9.17}
$$

In the last equation, we have used causality and $(x_1, ..., x_n) \in C_J$. On the other hand:

$$
\begin{aligned}
&T_{|J|}(x_1, ..., x_j) \star T_{|J^c|}(x_{j+1}, ..., x_n) \\
=&T_{|I|}(x_1, ..., x_i) \star T_{|I^c \cap J|}(x_{i+1}, ..., x_j) \star T_{|J^c|}(x_{j+1}, ..., x_n).
\end{aligned} \tag{9.18}
$$

In the last equation, we have used causality and $(x_1, ..., x_n) \in C_I$.

The second case is $I = \{1, ..., i\}$, $J = \{i+1, ..., j\}$ and $J \cap I = \emptyset$. Again, we organize the indices in a convenient form for simplicity. In that case $I^c = J \cup (I^c \cap J^c)$ and $J^c = I \cup (J^c \cap I^c)$. Hence, using causality, we can write:

$$T_I(x_1, .., x_i) \star T_{|I^c|}(x_{i+1}, ..., x_n) = T_{|I|}(x_1, ..., x_i) \star T_{|J|}(x_{i+1}, ..., x_j) \star T_{|(I\cup J)^c|}(x_{j+1}, ..., x_n). \tag{9.19}$$

On the other hand:

$$\begin{aligned} &T_{|J|}(x_{i+1}, ..., x_j) \star T_{|J^c|}(x_1, ..., x_i, x_{j+1}, ..., x_n) \\ =&T_{|J|}(x_{i+1}, ..., x_j) \star T_{|I|}(x_1, ..., x_i) \star T_{|(J\cup I)^c|}(x_{j+1}, .., x_n). \end{aligned} \tag{9.20}$$

Last but not least, note that $C_I \cap C_J = \emptyset$ implies $(x_i - x_j)^2 < 0 \forall i \in I, j \in J$ (basically the intersection means that x_i is not in the past cone of x_j and x_j is not in the past cone of x_i, i.e., x_i is not in the future cone of x_j). Thus,

$$T_{|I|}(x_1, ..., x_i) \star T_{|J|}(x_{i+1}, ..., x_j) = T_{|J|}(x_{i+1}, ..., x_j) \star T_{|I|}(x_1, ..., x_i). \tag{9.21}$$

The last scenario is the mixture of the two above, when $I \cap J \neq \emptyset$ but $I \not\subseteq J$. In this case, we define $I = \{1, ..., k, ..., i\}$ and $J = \{k, ..., i, ..., j\}$. Using the notation $\{x_1, .., x_n\} \notin \{y_1, ..., y_m\} + \overline{V}_-$ as synonym for $x_i \notin (y_j + \overline{V}_-) \, \forall i \in \{1, ..., n\} \, j \in \{1, ..., m\}$ we can prove that

i $I \cap J = \{k, ..., i\}$.

ii $\{x_1, ..., x_{k-1}\}$ and $\{x_{i+1}, ..., x_j\}$ are space-like.[4]

iii $\{x_k, ...x_i\} \notin \{x_{i+1}, ..., x_n\} + \overline{V}_-$.

iv $\{x_k, ..., x_i\} \notin \{x_1, ..., x_{k-1}, x_{j+1}, ..., x_n\} + \overline{V}_-$

The justification to the sentences above are:

i Trivial by the enumeration

ii It is consequence of the previously case, where $C_I \cap C_J = \emptyset$.

iii We use $(x_k, ..., x_i) \in C_I$ and $(x_{k+1}, ..., x_n)$ are in the complement.

iv We use $(x_k, ..., x_i) \in C_J$ and $(x_1, ..., x_{k-1}, x_{j+1}, ..., x_n)$ are in the complement.

[4]Here we have a similar notation: $(x-y)^2 < 0 \, \forall \, x \in \{x_1, ..., x_{k-1}\}$ and $y \in \{x_{i+1}, ..., x_j\}$

Now we just have to write the $T-$ product both in C_I and in C_J. In C_I:

$$\begin{aligned} &T_n(x_1,...,x_n) = T_{|I|}(x_1,...,x_k,...,x_i) \star T_{|I^C|}(x_{i+1},...,x_n) \\ =&T_{|I\cap J|}(x_k,...,x_i) \star T_{|I\setminus J|}(x_1,...,x_k) \star T_{|I^C|}(x_{i+1},...,x_j,...,x_n) \\ =&T_{|I\cap J|}(x_k,...,x_i) \star T_{|I\setminus J|}(x_1,...,x_k) \star T_{|J\cap I^C|}(x_{i+1},...,x_j) \star T_{|I^C\setminus J|}(x_{j+1},...,x_n). \end{aligned} \tag{9.22}$$

On the other hand, in C_J:

$$\begin{aligned} &T_n(x_1,...,x_n) = T_{|J|}(x_k,...x_j) \star T_{|J^C|}(x_1,...,x_{k-1},x_{j+1},...,x_n) \\ =&T_{|J\cap I|}(x_k,...,x_i) \star T_{|J\setminus I|}(x_{i+1},...,x_j) \star T_{|J^C|}(x_1,...,x_{k-1},x_{j+1},...,x_n) \\ =&T_{|J\cap I|}(x_k,...,x_i) \star T_{|J\setminus I|}(x_{i+1},...,x_j) \star T_{|I\cap J^C|}(x_1,...,x_k) \star T_{|J^C\setminus I|}(x_{j+1},...,x_n). \end{aligned} \tag{9.23}$$

Since

$$T_{|I\setminus J|}(x_1,...,x_k) \star T_{|J\cap I^C|}(x_{i+1},...,x_j) = T_{|J\setminus I|}(x_{i+1},...,x_j) \star T_{|I\cap J^C|}(x_1,...,x_k). \tag{9.24}$$

Both expression are the same.

The proof that the aforementioned construction of the T product satisfies the axioms can be found in [24] Chapter 3.3. We also recommend the original work of Epstein Glaser [29] and the discussion related to causal perturbation theory in [67]. The method used in these last two references is different, since they do not work with star products, but the construction follows the same philosophy of dividing the domain.

This construction is different from the construction cited in the beginning of the section. Naturally, one could ask if the construction of the $T-$ product on $\mathbb{M}^n \setminus \Delta_n$ is unique? The answer is yes and the proof can be found with minor adaptations at [47], Theorem 2.4 page 42. Once again, we have to extend the domain to the thin diagonal. We will do it now with some examples.

9.3 Example 1: R- product considering $L_{int} = -\kappa \int dx\, g(x)\phi(x)$ up to order 2

In the classical case, the example was the simplest one. In the quantum case, it will be as well. We start with the definition of retarded field:

$$\phi^{\text{ret}}(x) \equiv R(e_{\otimes}^{S/\hbar}, \phi(x)) = R_{0,1}(\phi(x)) + R_{1,1}(S/\hbar, \phi(x)) + R_{2,1}(\frac{S\otimes S}{2\hbar^2}, \phi(x)) + ... \tag{9.25}$$

The axiom of initial condition gives us:

$$R_{0,1}(x) = \phi(x). \tag{9.26}$$

In Section (8.3), we found a formula for $R_{1,1}(x)$ (8.21):

$$R_{1,1}(-\frac{\kappa}{\hbar}\int dy\, g(y)\phi(y), \phi(x)) = \frac{i\kappa}{\hbar}\int dy\, g(y)[\phi(y)\,\phi(x)]_\star \theta(x^0 - y^0). \tag{9.27}$$

The commutator is very simple to calculate:

$$\begin{aligned}
[\phi(y)\,\phi(x)]_\star &= \phi(y) \star \phi(x) - \phi(x) \star \phi(y) \\
&= \phi(y)\phi(x) + \hbar\Delta^+(y-x) - (\phi(x)\phi(y) + \hbar\Delta^+(x-y)) \\
&= \hbar\left(\Delta^+(y-x) + \Delta^+(x-y)\right) = -i\hbar\Delta(x-y).
\end{aligned} \tag{9.28}$$

In the above equation, we have used:

$$\Delta^+(y-x) - \Delta^+(x-y) = i\Delta(y-x) = -i\Delta(x-y). \tag{9.29}$$

Last but not least, we use one more relation involving the propagators:

$$\Delta(x)\theta(x^0) = \Delta^{\mathrm{ret}}(x). \tag{9.30}$$

and conclude:

$$R_{1,1}(x) = \frac{i\kappa}{\hbar}\int dy\, g(y)(-i\hbar\Delta(x-y))\theta(x^0 - y^0) = \kappa\int dy\, g(y)\Delta^{\mathrm{ret}}(x-y). \tag{9.31}$$

The computation of higher orders is even simpler! As discussed in section (8.3), the inductive construction of $R_{2,1}$, consists in calculating commutations of the form $[R_{0,1}(\phi(a)), R_{1,1}(\phi(b)), \phi(c))$. But $R_{1,1} \in \mathbb{C}$. Therefore, all the commutators are zero and we do not have any higher orders. Thus:

$$\phi_S(x) = R(e^{\frac{S}{\hbar}}, \phi(x)) = \phi(x) + \kappa\int dy\, g(y)\Delta^{\mathrm{ret}}(x-y). \tag{9.32}$$

That is exactly the classical result (7.16)!

9.4 Example 2: R- product considering $L_{int} = -\kappa \int dx\, g(x)\phi^2(x)$ up to order 2

Again, we start with

$$R(e_{\otimes}^{S/\hbar}, \phi(x)) = R_{0,1}(\phi(x)) + R_{1,1}(S/\hbar, \phi(x)) + R_{2,1}(\frac{S \otimes S}{2\hbar^2}, \phi(x)). \tag{9.33}$$

To organize the calculation, we will compute each term individually starting by $R_{0,1}(\phi(x)) = \phi(x)$.

$R_{1,1}(S/\hbar, \phi(x))$

By definition:

$$R_{1,1}(S/\hbar, \phi(x)) = R_{1,1}(-\frac{\kappa}{\hbar}\int dy\, g(y)\phi^2(y), \phi(x)) = -\frac{\kappa}{\hbar}\int dy\, g(y) R_{1,1}(\phi^2(y), \phi(x)). \tag{9.34}$$

using (8.21):

$$R_{1,1}(S/\hbar, \phi(x)) = -\frac{i\kappa}{\hbar}\int dy\, g(y)[\phi^2(y), \phi(x)]_{\star}\theta(x^0 - y^0). \tag{9.35}$$

To calculate the commutator we use:

$$\phi^2(y) \star \phi(x) = \phi^2(y)\phi(x) + \hbar\frac{\delta\phi^2(y)}{\delta\phi}\Delta^+(y-x)\frac{\delta\phi(x)}{\delta\phi} + ... \tag{9.36}$$

Where the ... indicates higher derivatives terms that are zero and $\frac{\delta A(\phi(x))}{\delta\phi}$ is a short notation for:

$$\frac{\delta A(\phi(x))}{\delta\phi} := \int dy \frac{\delta A(\phi(y))}{\delta\phi(x)}. \tag{9.37}$$

Hence:

$$\begin{aligned}
&\phi^2(y) \star \phi(x) = \phi^2(y)\phi(x) + 2\hbar\phi(y)\Delta^+(y-x)\\
&\phi(x) \star \phi^2(y) = \phi(x)\phi^2(y) + 2\hbar\phi(y)\Delta^+(x-y)\\
\Rightarrow &\left[\phi^2(y), \phi(x)\right]_{\star} = 2\hbar\phi(y)\left(\Delta^+(y-x) - \Delta^+(x-y)\right) = 2\hbar\phi(y)(-i\Delta(x-y)).
\end{aligned} \tag{9.38}$$

In the above equation we have once again used:

$$\Delta^+(y-x) - \Delta^+(x-y) = i\Delta(y-x) = -i\Delta(x-y). \tag{9.39}$$

Hence:

$$R_{1,1}(\phi^2(y), \phi(x)) = 2\kappa \int dy\, g(y)\phi(y)\Delta^{\mathrm{ret}}(x-y) = R_{\mathrm{cl}}(S, \phi(x)). \tag{9.40}$$

To derive the previously result we have used:

$$\Delta(x)\theta(x^0) = \Delta^{\mathrm{ret}}(x). \tag{9.41}$$

In first order we recover the classical product and do not have any quantum corrections.

Now we proceed to the next term.

$R_{2,1}(\frac{S\otimes S}{2\hbar^2}, \phi(x))$

Once again we start by a simple definition:

$$R_{2,1}(\frac{S\otimes S}{2\hbar^2}, \phi(x)) = \frac{\kappa^2}{2\hbar^2} \int dx_1 dx_2\, g(x_1)g(x_2) R_{2,1}(\phi^2(x_1), \phi^2(x_2); \phi(x)). \tag{9.42}$$

Remark: We will omit the $\star$ when writing $[]_\star \equiv []$ when there is no risk of confusion.

The first step in the inductive construction is to restrain the domain. On $\mathcal{M}_1$ we have the following relation (8.27):

$$\begin{aligned} &R_{2,1}(\phi^2(x_1), \phi^2(x_2); \phi(x)) \\ &= -\theta(x^0 - x_1^0)\Big(\left[\left[\phi^2(x_2), \phi^2(x_1)\right], \phi(x)\right] \theta(x_1^0 - x_2^0) + \\ &\left[\phi^2(x_1), \left[\phi^2(x_2), \phi(x)\right]\right] \theta(x^0 - x_2^0)\Big). \end{aligned} \tag{9.43}$$

Before continuing, let us calculate $[\phi^2(x_2), \phi^2(x_1)]$:

$$
\begin{aligned}
&\phi^2(x_2)\star\phi^2(x_1)=\phi^2(x_2)\phi^2(x_1)+\hbar\frac{\delta\phi^2(x_2)}{\delta\phi}\Delta^+(x_2-x_1)\frac{\delta\phi^2(x_1)}{\delta\phi}\\
&+\frac{\hbar^2}{2}\frac{\delta^2\phi^2(x_2)}{\delta^2\phi}(\Delta^+(x_2-x_1))^2\frac{\delta^2\phi^2(x_1)}{\delta^2\phi}\\
&=\phi^2(x_2)\phi^2(x_1)+4\hbar\phi(x_2)\phi(x_1)\Delta^+(x_2-x_1)+2\hbar^2(\Delta^+(x_2-x_1))^2.
\end{aligned}\tag{9.44}
$$

Hence:

$$
\begin{aligned}
&[\phi^2(x_2),\phi^2(x_1)]=4\hbar\phi(x_2)\phi(x_1)(\Delta^+(x_2-x_1)-\Delta^+(x_1-x_2))\\
&+2\hbar^2\left\{(\Delta^+(x_2-x_1))^2-(\Delta^+(x_1-x_2))^2\right\}\\
&=4\hbar i\phi(x_2)\phi(x_1)\Delta(x_2-x_1)+\omega_0(R_{1,1}(\phi^2(x_2);\phi^2(x_1))).
\end{aligned}\tag{9.45}
$$

In the last equation, we have used

$$
\omega_0(R_{1,1}(\phi^2(y);\phi^2(z)))\equiv\frac{2\hbar^2}{i}\left\{(\Delta^+(y-z))^2-(\Delta^+(z-y))^2\right\}
$$

Using this result we can calculate the next term,$[[\phi^2(x_2);\phi^2(x_1)],\phi(x)]$. To do it, we use:

$$
\phi(x_1)\phi(x_2)\star\phi(x)=\phi(x)\phi(x_1)\phi(x_2)+\hbar\Delta^+(x_1-x)\phi(x_2)+\hbar\Delta^+(x_2-x)\phi(x_1).\tag{9.46}
$$

Hence:

$$
[[\phi^2(x_2),\phi^2(x_1)],\phi(x)]=-4\hbar^2\Delta(x_2-x_1)\left(\Delta(x_1-x)\phi(x_2)+\Delta(x_2-x)\phi(x_1)\right).\tag{9.47}
$$

The calculation of the other term is analogous:

$$
[\phi^2(x_1),[\phi^2(x_2),\phi(x)]]=-4\hbar^2\Delta(x_2-x)\Delta(x_1-x_2)\phi(x_1).\tag{9.48}
$$

Combining the results above, on $\mathcal{M}_1$, the expression for the retarded product is:

$$
\begin{aligned}
&R_{2,1}(x)=-\frac{\kappa^2}{2\hbar^2}\int dx_1dx_2\,g(x_1)g(x_2)\theta(x^0-x_1^0)\\
&\Big((-4\hbar^2\Delta(x_2-x_1)(\Delta(x_1-x)\phi(x_2)+\Delta(x_2-x)\phi(x_1))\theta(x_1^0-x_2^0))\\
&-4\hbar^2\Delta(x_2-x)\Delta(x_1-x_2)\phi(x_1)\theta(x^0-x_2^0)\Big).
\end{aligned}\tag{9.49}
$$

We can simplify the expression above using $\Delta(x)\theta(x^0)=\Delta^{\rm ret}(x)$:

$$R_{2,1}(x) = 2\kappa^2 \int dx_1 dx_2\, g(x_1) g(x_2) \Big(\Delta^{\text{ret}}(x_1 - x_2) \Delta^{\text{ret}}(x - x_1) \phi(x_2)$$
$$+ \Delta^{\text{ret}}(x_1 - x_2) \Delta^{\text{ret}}(x - x_2) \phi(x_1) - \Delta^{\text{ret}}(x - x_2) \Delta(x_1 - x_2) \theta(x^0 - x_1^0) \phi(x_1) \Big). \qquad (9.50)$$

The expression is still not as we want. We factorize $\Delta(x_1 - x_2)$ using the identity:

$$\Delta(x_1 - x_2) = \Delta(x_1 - x_2)(\theta(x_1^0 - x_2^0) + \theta(x_2^0 - x_1^0)) = \Delta^{\text{ret}}(x_1 - x_2) - \Delta^{\text{ret}}(x_2 - x_1). \qquad (9.51)$$

To finally get the final result:

$$R_{2,1}(x) = 2\kappa^2 \int dx_1 dx_2\, g(x_1) g(x_2) \Delta^{\text{ret}}(x_1 - x_2) \Delta^{\text{ret}}(x - x_1) \phi(x_2) + x_1 \leftrightarrow x_2. \qquad (9.52)$$

We have used the notation $x_1 \leftrightarrow x_2$ to indicate that there are more terms identical to the written ones except that we need to change x to y.

Changing variables:

$$R_{2,1}(x) = 4\kappa^2 \int dx_1 dx_2\, g(x_1) g(x_2) \Delta^{\text{ret}}(x - x_1) \Delta^{\text{ret}}(x_1 - x_2) \phi(x_2). \qquad (9.53)$$

The expression above is clearly symmetric under exchange $x_1 \leftrightarrow x_2$. To complete the construction, we need to calculate $R_{2,1}(x)$ on $\mathcal{M}_2$. Due to the symmetry of the problem, that is equivalent to changing $x_1 \to x_2$. Since both expressions are equal, the final result is simply:

$$R_{2,1}(x) = 8\kappa^2 \int dx_1 dx_2\, g(x_1) g(x_2) \Delta^{\text{ret}}(x - x_1) \Delta^{\text{ret}}(x_1 - x_2) \phi(x_2). \qquad (9.54)$$

Once again, we recover the classical product. Just as in the case of $L_{int} = -\kappa \int dx\, g(x)\phi(x)$, we do not have any quantum corrections. We can, actually, claim a stronger sentence about quantum corrections of this interaction. We claim that there is no quantum correction in any order of perturbative expansion. We can prove this using an induction argument. Essentially, what we are going to show is that $\frac{1}{\hbar^n} R_{n,1}(S^{\otimes n}, \phi(x))$ is independent of $\hbar$. Since the construction is carried out in such a way that we recover the classical expression when $\hbar = 0$, the quantum and the classical retarded product are the same.

Since we have proved that $R_{1,1}(\frac{S}{\hbar}; \phi(x))$ and $R_{2,1}(\frac{S^{\otimes 2}}{\hbar^2}; \phi(x))$ are independent of $\hbar$, the beginning of the induction is already complete. Suppose that the argument holds for $R_{n,1}(\frac{S^{\otimes n}}{\hbar^n}, \phi(x))$. Now, let us consider the field equation axioms for $R_{n+1,1}(\frac{S^{\otimes n+1}}{\hbar^{n+1}}, \phi(x))$:

$$R_{n+1,1}(\frac{S^{\otimes n+1}}{\hbar^{n+1}},\phi(x)) = -\hbar\int dy\Delta^{\text{ret}}(x-y)\sum_{l=1}^{n} R_{n,1}\left(\frac{S_1}{\hbar}\otimes...\frac{\widehat{S_l}}{\hbar}\otimes...\otimes\frac{S_{n-1}}{\hbar};\frac{1}{\hbar}\frac{\delta S_l}{\delta\phi(y)}\right). \tag{9.55}$$

Using $\frac{\delta S}{\delta\phi(y)} = 2g(y)\phi(y)$ and that all interactions are equal, we can rewrite the expression above as:

$$\begin{aligned}R_{n+1,1}(\frac{S^{\otimes n+1}}{\hbar^{n+1}},\phi(x)) &= -(n-1)\hbar\int dy\,\Delta^{\text{ret}}(x-y)R_{n,1}(\frac{S^{\otimes n}}{\hbar^n},\frac{2g(y)\phi(y)}{\hbar})\\ &= -2(n-1)\hbar\int dy\,g(y)\Delta^{\text{ret}}(x-y)R_{n,1}(\frac{S^{\otimes n}}{\hbar^n},\phi(y)).\end{aligned} \tag{9.56}$$

Using the inductive hypothesis, $R_{n,1}(\frac{S^{\otimes n}}{\hbar^n},\phi(y))$ is independent of $\hbar$. Thus, $R_{n+1,1}$ is also independent of $\hbar$ and that completes the proof.

9.5 Example 3: R- product considering $L_{int} = -(\frac{\lambda}{4!})\int dx\,g(x)\phi^4(x$ up to order 2

Just as in the case of the classical retarded product, the interaction described in this problem leads to a much more sophisticated expression, compared to the case of $L_{int} = \int dx\,g(x)\phi^2$ or $L_{int} = \int dx\,g(x)\phi$. We will derive the expression, but we will not work through the renormalization.

We star with the definition:

$$\begin{aligned}\phi_S(x) \equiv R(e_{\otimes}^{-\frac{(\frac{\lambda}{4!})}{\hbar}\int dx_1\,g(x_1)\phi^4(x_1)};\phi(x)) &= R_{0,1}(\phi(x))\\ &+ R_{1,1}(-\frac{(\frac{\lambda}{4!})}{\hbar}\int dx_1\,g(x_1)\phi^4(x_1);\phi(x))\\ &+ R_{2,1}(\frac{(\frac{\lambda}{4!})^2}{2\hbar^2}\int dx_1\,g(x_1)\phi^4(x_1),\int dx_2\,g(x_2)\phi^4(x_2);\phi(x)) + \mathcal{O}((\frac{\lambda}{4!})^3).\end{aligned} \tag{9.57}$$

Using linearity, we can write:

$$\begin{aligned}&R_{1,1}(-\frac{(\frac{\lambda}{4!})}{\hbar}\int dy\,g(x_1)\phi^4(x_1);\phi(x)) = -(\frac{\lambda}{4!})\int dx_1\,g(x_1)R_{1,1}\left(\phi^4(x_1);\phi(x)\right)\\ &R_{2,1}(\frac{(\frac{\lambda}{4!})^2}{2\hbar^2}\int dx_1\,g(x_1)\phi^4(x_1),\int dx_2\,g(x_2)\phi^4(x_2);\phi(x))\\ &= \frac{(\frac{\lambda}{4!})^2}{2\hbar^2}\int dX_2\,g(x_1)g(x_2)R_{2,1}(\phi^4(x_1),\phi^4(x_2);\phi(x)).\end{aligned} \tag{9.58}$$

The initial condition reads:

$$R_{0,1}(\phi(x)) = \phi(x). \tag{9.59}$$

Now, let us calculate $R_{1,1}$. To do it,we use (8.21):

$$\begin{aligned}
&R_{1,1}(-\frac{(\frac{\lambda}{4!})}{\hbar}\int dx_1\, g(x_1)\phi^4(x_1), \phi(x)) \\
&= -\frac{(\frac{\lambda}{4!})}{\hbar}\int dx_1\, g(x_1)(-i)\left[\phi^4(x_1), \phi(x)\right]\theta(x^0 - x_1^0) \\
&= i\frac{(\frac{\lambda}{4!})}{\hbar}\int dx_1\, g(x_1)(\phi^4(x_1) \star \phi(x) - \phi(x) \star \phi^4(x_1))\theta(x^0 - x_1^0) \\
&= i\frac{(\frac{\lambda}{4!})}{\hbar}\int dx_1\, g(x_1)\left(4\hbar\phi^3(x_1)(\Delta^+(x_1 - x) - \Delta^+(x - x_1)\right)\theta(x^0 - x_1^0) \\
&= i\frac{(\frac{\lambda}{4!})}{\hbar}\int dx_1\, g(x_1)\left(4i\hbar\phi^3(x_1)\Delta(x_1 - x)\theta(x^0 - x_1^0)\right) \\
&= -\frac{\lambda}{6}\int dx_1, g(x_1)\Delta^{\mathrm{ret}}(x - x^1)\phi^3(x_1).
\end{aligned} \tag{9.60}$$

In the equation above we have used:

$$\begin{aligned}
&\phi^4(x_1) \star \phi(x) = \phi(x)\phi^4(x_1) + 4\hbar\phi^3(x_1)\Delta^+(x_1 - x) \\
&\phi(x) \star \phi^4(x_1) = \phi(x)\phi^4(x_1) + 4\hbar\phi^3(x_1)\Delta^+(x - x_1) \\
&\Delta^+(z) - \Delta^+(-z) = i\Delta(z) \\
&\Delta(z)\theta(-z) = -\Delta(-z)\Delta(-z) = \Delta^{\mathrm{ret}}(z).
\end{aligned} \tag{9.61}$$

Note that, once again, the result is the same as in the classical case.

The formula for $R_{2,1}$ is more complicated since we have to divide the domain. We start in $\mathcal{M}_1 := \left\{(x_1, x_2, x_3) \in \mathbb{M}^3 | x_1 \notin (x_3 + \overline{V}_+)\right\}$. The corresponding expression is given in (8.27). Hence:

$$\begin{aligned}
&R_{2,1}(\phi^4(x_1), \phi^4(x_2); \phi(x)) = \\
&\theta(x^0 - x_2^0)\Big(\left[\left[\phi^4(x_1), \phi^4(x_2)\right], \phi(x)\right]\theta(x_2^0 - x_1^0) + \left[\phi^4(x_2), \left[\phi^4(x_1), \phi(x)\right]\right]\theta(x^0 - x_1^0)\Big).
\end{aligned} \tag{9.62}$$

The calculation is long, but straightforward. To simplify the notation, we follow the notation from [24] section 3.2.8 and denote

$$j_k(x) := -i((\Delta^+(x))^k - (\Delta^+(-x))^k). \tag{9.63}$$

We start by calculating $\left[\phi^4(x_1), \phi^4(x_2)\right]$:

$$\begin{aligned}\left[\left[\phi^4(x_1), \phi^4(x_2)\right], \phi(x)\right] &= 4^2 i\hbar\phi^3(x_1)\Delta(x_1 - x_2)\phi^3(x_2) \\ &+ 4^2 3^2 \frac{\hbar^2}{2!}\phi^2(x_1)\frac{j_2(x_1 - x_2)}{-i}\phi^2(x_2) \\ &+ 4^2 3^2 2^2 \frac{\hbar^3}{3!}\phi(x_1)\frac{j_3(x_1 - x_2)}{-i}\phi(x_2).\end{aligned} \tag{9.64}$$

Now we commute the term above with $\phi(x)$:

$$\begin{aligned}&[[\phi^2(x_1), \phi^4(x_2)], \phi(x)] \\ &= 48(i\hbar)^2(\phi^2(x_1)\Delta(x_1 - x)\Delta(x_1 - x_2)\phi^3(x_2) + \phi^3(x_1)\Delta(x_2 - x)\Delta(x_1 - x_2)\phi^2(x_2)) \\ &- 144\hbar^3 j_2(x_1 - x_2)(\phi(x_1)\Delta(x_1 - x)\phi^2(x_2) + \phi^2(x_1)\Delta(x_2 - x)\phi(x_2)) \\ &- 96\hbar^4 j_3(x_1 - x_2)(\Delta(x_1 - x)\phi(x_2) + \phi(x_1)\Delta(x_2 - x)).\end{aligned} \tag{9.65}$$

The second commutator is equal to:

$$\begin{aligned}&[\phi^4(x_2), [\phi^4(x_1), \phi(x)]] = [\phi^4(x_2), 4i\hbar\Delta(x_1 - x)\phi^3(x_1)] = 4i\hbar\Delta(x_1 - x) \\ &\times \left(4 \cdot 3i\hbar\phi^3(x_2)\Delta(x_2 - x_1)\phi^2(x_1) + 4 \cdot 3^2\hbar^2 \frac{j_2}{-i}(x_2 - x_1)\phi(x_1)\right) \\ &= -48\hbar^2\phi^2(x_1)\Delta(x_1 - x)\Delta(x_2 - x_1)\phi^3(x_2) - 144\hbar^3\phi(x_1)j_2(x_2 - x_1)\phi^2(x_2) \\ &- 96\hbar^4 j_3(x_2 - x_1)\phi(x_2).\end{aligned} \tag{9.66}$$

Hence:

Next we separate the terms proportional to $\hbar^2$. We are going to factorize then to obtain the classical retarded product. The rest of the terms are the quantum corrections.

Note that:

$$
\begin{aligned}
&\phi^3(x_2)\Big(\Delta(x_1-x)\Delta(x_2-x_1)\theta(x^0-x_1^0)\\
&+\Delta(x_1-x)\Delta(x_1-x_2)\theta(x_2^0-x_1^0)\Big)\theta(x^0-x_2^0)\phi^2(x_1)\\
&+\phi^3(x_1)\Delta(x_2-x)\Delta(x_1-x_2)\theta(x^0-x_2^0)\theta(x_2^0-x_1^0)\phi^2(x_2)\\
&=-\phi^3(x_2)\left(\Delta^{\mathrm{ret}}(x-x_1)\Delta(x_2-x_1)+\Delta(x_1-x)\Delta^{\mathrm{ret}}(x_1-x_2)\right)\theta(x^0-x_2^0)\phi^2(x_1)\\
&+\phi^3(x_1)\Delta^{\mathrm{ret}}(x-x_2)\Delta^{\mathrm{ret}}(x_2-x_1)\phi^2(x_2).
\end{aligned}
\tag{9.67}
$$

We can manipulate the first term above:

$$
\begin{aligned}
&\Delta^{\mathrm{ret}}(x-x_1)\Delta(x_2-x_1)+\Delta(x_1-x)\Delta^{\mathrm{ret}}(x_1-x_2)\\
=&\Delta^{\mathrm{ret}}(x-x_1)\Big(\Delta^{\mathrm{ret}}(x_2-x_1)-\Delta^{\mathrm{ret}}(x_1-x_2)\Big)\\
+&\Big(\Delta^{\mathrm{ret}}(x_1-x)-\Delta^{\mathrm{ret}}(x_1-x)\Big)\Delta^{\mathrm{ret}}(x_1-x_2)\\
=&-\Delta^{\mathrm{ret}}(x-x_1)\Delta^{\mathrm{ret}}(x_1-x_2)+\Delta^{\mathrm{ret}}(x_2-x_1)\Delta^{\mathrm{ret}}(x_1-x).
\end{aligned}
\tag{9.68}
$$

The original expression is multiplied by $\theta(x^0-x_2^0)$:

$$
\begin{aligned}
&-\Delta^{\mathrm{ret}}(x-x_1)\Delta^{\mathrm{ret}}(x_1-x_2)\theta(x^0-x_2^0)+\Delta^{\mathrm{ret}}(x_2-x_1)\Delta^{\mathrm{ret}}(x_1-x)\theta(x^0-x_2^0)\\
&=-\Delta^{\mathrm{ret}}(x-x_1)\Delta^{\mathrm{ret}}(x_1-x_2).
\end{aligned}
\tag{9.69}
$$

In the above equation, we use the fact $\operatorname{supp}\Delta(x)\subseteq\{x^2\leq 0, x^0>0\}$ and the definition of $\theta(x)$ to cancel the second term and simplify the first one. Taking all the results, the term proportional to $\hbar^2$ is equal to:

$$
\begin{aligned}
&-48\hbar^2(\phi^3(x_2)\phi^2(x_1)\Delta^{\mathrm{ret}}(x-x_1)\Delta^{\mathrm{ret}}(x_1-x_2)\\
&+\phi^3(x_1)\phi^2(x_2)\Delta^{\mathrm{ret}}(x_2-x_1)\Delta^{\mathrm{ret}}(x-x_2))\\
&=-48\hbar^2\phi^3(x_2)\phi^2(x_1)\Delta^{\mathrm{ret}}(x-x_1)\Delta^{\mathrm{ret}}(x_1-x_2)+x_1\leftrightarrow x_2\\
&\equiv\hbar^2 r^{\mathrm{cl}}_{2,1}(x_1,x_2;x).
\end{aligned}
\tag{9.70}
$$

The quantum corrections is:

$$\hbar^3 r^q_{2,1}(x_1,x_2;x) = -144\hbar^3 j_2(x_1-x_2)(\phi(x_1)\Delta(x_1-x)\phi^2(x_2)$$
$$+\phi^2(x_1)\Delta(x_2-x)\phi(x_2))\theta(x^0-x^0_2)\theta(x^0_2-x^0_1)$$
$$-144\hbar^3\phi(x_1)j_2(x_2-x_1)\phi^2(x_2)\theta(x^0-x^0_2)\theta(x^0-x^0_1)$$
$$-96\hbar^4 j_3(x_1-x_2)\Bigg(\Delta(x_1-x)\phi(x_2)$$
$$+\phi(x_1)\Delta(x_2-x)\Bigg)\theta(x^0-x^0_2)\theta(x^0_2-x^0_1)$$
$$-96\hbar^4 j_3(x_2-x_1)\phi(x_2)\theta(x^0-x^0_2)\theta(x^0-x^0_1). \tag{9.71}$$

Due to the symmetry of the retarded product, $R_{2,1}(x_1,x_2;x)$ in $\mathcal{M}_2$ basically changes $x_1 \leftrightarrow x_2$. Hence:

$$R_{2,1}(\frac{S^{\otimes 2}}{\hbar^2},\phi(x)) = \frac{\lambda^2}{2\cdot 4!^2}\int dx_1 dx_2\, g(x_1)g(x_2)$$
$$(r^{\mathrm{cl}}_{2,1}(x_1,x_2;x)+\hbar r^q_{2,1}(x_1,x_2;x)) + x_1 \leftrightarrow x_2. \tag{9.72}$$

We emphasize that if $d \geq 4$, the terms $j_2(z), j_3(z)$ need renormalization. We will not do it here (instead we are going to focus on the renormalization of the T matrix), but it can be found on page 154 [24].

9.6 Example 4:Scattering amplitude considering $L_{int} = -(\frac{\lambda}{4!})\int dx\, g(x)\phi^4(x)$ up to order 2

The $T-$ product is much simpler than the $R-$ product. Just as in the case of the $R-$ product, we want to calculate:

$$S(L_{int}) \equiv T(e^{\frac{iL_{int}}{\hbar}}) = \frac{i}{\hbar}T_1(L_{int}) + \frac{i^2}{2!\hbar^2}T_2(L_{int},L_{iny}) + \mathcal{O}((\frac{\lambda}{4!})^3). \tag{9.73}$$

In first order:

$$\frac{i}{\hbar}T_1\left(-\frac{\lambda}{4!}\int dx_1\, g(x_1)\phi^4(x_1)\right) = \frac{-i(\frac{\lambda}{4!})}{\hbar}\int dx_1\, g(x_1)\phi^4(x_1). \tag{9.74}$$

Second order:

$$
\begin{aligned}
&\frac{i^2}{2\hbar^2}T_2(\frac{-\lambda}{4!}\int dx_1\, g(x_1)\phi^4(x_1), \frac{-\lambda}{4!}\int dx_2\, g(x_2)\phi^4(x_2))\\
&= -\frac{\lambda^2}{2(4!\hbar)^2}\int dx_1 dx_2\, g(x_1)g(x_2)\phi^4(x_1)\star_F \phi^4(x_2) = -\frac{\lambda^2}{2(4!\hbar)^2}\int dX_2 g(x_1)g(x_2)\\
&\times \Bigg(\phi^4(x_1)\phi^4(x_2) + 16\hbar\phi^3(x_1)\phi^3(x_2)\Delta^F(x_1-x_2) + 72\hbar^2\phi^2(x_1)\phi^2(x_2)(\Delta^F(x_1-x_2))^2\\
&+ 32\hbar^3\phi(x_1)\phi(x_2)(\Delta^F(x_1-x_2))^3 + 8\hbar^4(\Delta^F(x_1-x_2))^4\Bigg). \qquad (9.75)
\end{aligned}
$$

Using the examples above, we can compute the scattering amplitude $\mathcal{T}$. In the formalism of QFT in Fock space, the fields that appear in the $T-$ product are normal ordered [67]. We will use this fact to invoke the theorem of Section (6.4). We will focus on the scattering involves 4 particles, two incoming and two outgoing particles. The first order in λ reads:

$$
\begin{aligned}
\mathcal{T}_1 &= \langle \Omega a^*_{\mathfrak{F}}(p_1)a^*_{\mathfrak{F}}(p_2)|\int dx\,\frac{-i\lambda}{4!\hbar}g(x):\phi^4(x):|a^*_{\mathfrak{F}}(p_3)a^*_{\mathfrak{F}}(p_4)\Omega\rangle\\
&= \left(\prod_{i=1}^{4}\frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\int d\vec{Y}_4 dx\\
&\times \omega_0\left(e^{ip_1y_1+ip_2y_2}\phi(y_1)\phi(y_2)\star\frac{-i\lambda}{4!\hbar}g(x)\phi^4(x)\star e^{-ip_3y_3-ip_4y_4}\phi(y_3)\phi(y_4)\right). \qquad (9.76)
\end{aligned}
$$

The calculation above is long, but not very hard. We start by attacking the star product. For simplicity, we first calculate $\phi(y_1)\phi(y_2)\star\phi^4(x)$ and then the result of the mentioned calculation star product with the rest of the fields.

$$
\begin{aligned}
\phi(y_1)\phi(y_2)\star\phi^4(x) = {}&\phi(y_1)\phi(y_2)\phi^4(x)\\
&+ 4\hbar\left(\phi(y_1)\Delta^+(y_2-x)\phi^3(x) + \phi(y_2)\Delta^+(y_1-x)\phi^3(x)\right)\\
&+ \frac{4\cdot 3\hbar^2}{2!}\Bigg(\Delta^+(y_1-x)\Delta^+(y_2-x)\phi^2(x)\\
&+ \Delta^+(y_2-x)\Delta^+(y_1-x)\phi^2(x)\Bigg). \qquad (9.77)
\end{aligned}
$$

Since we are going to compute the vacuum expectation value, we need to consider only the terms proportional to ϕ^2 of the above equation when computing $(\phi(y_1)\phi(y_2)\star\phi^4(x))\star\phi(y_3)\phi(y_4)$, that is, the term in the last line. Using:

$$
\begin{aligned}
\phi^2(x) \star \phi(y_3)\phi(y_4) &= \phi^2(x)\phi(y_3)\phi(y_4) \\
&+ 2\hbar \left(\phi(x)\Delta^+(x-y_3)\phi(y_4) + \phi(x)\Delta^+(x-y_4)\phi(y_3)\right) \\
&= \frac{2\hbar^2}{2}(\Delta^+(x-y_4)\Delta^+(x-y_3) + \Delta^+(x-y_4)\Delta^+(x-y_3)) \\
&= 2\hbar^2\Delta^+(x-y_4)\Delta^+(x-y_3).
\end{aligned} \tag{9.78}
$$

We conclude:

$$
\begin{aligned}
&\omega_0(\phi(y_1)\phi(y_2) \star \phi^4(x) \star \phi(y_3)\phi(y_4)) \\
&= 24\hbar^4\Delta^+(y_1-x)\Delta^+(y_2-x)\Delta^+(x-y_3)\Delta^+(x-y_4).
\end{aligned} \tag{9.79}
$$

Hence:

$$
\begin{aligned}
&\mathcal{T}_1 = -i\lambda\hbar^3 \left(\prod_{i=1}^{4} \frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right) \\
&\times \int d\vec{Y}_4 dx\, g(x)\Delta^+(y_1-x)\Delta^+(y_2-x)\Delta^+(x-y_3)\Delta^+(x-y_4)e^{ip_1y_1+ip_2y_2-ip_3x_3-ip_4x_4}.
\end{aligned} \tag{9.80}
$$

We can calculate the integral on $\vec{Y}_4$ above. The integral in x must be considered with caution since the test function g complicates the calculation. The goal is to take the limit $g(x) \to 1 \forall x$, but to do it we need to guarantee that the integral above is well defined. The integrals that need to be calculated are:

$$
\begin{aligned}
&\int d\vec{y}_1 \Delta^+(y_1-x)e^{ip_1y_1} = \int d\vec{y}_1 \frac{1}{(2\pi)^{d-1}} \int \frac{d\vec{k}_1}{2\omega_{k_1}} e^{-ik_1(y_1-x)}e^{ip_iy_1} \\
&= \frac{e^{-i(k_1^0-p_1^0)y_1^0}}{(2\pi)^{d-1}} \int \frac{d\vec{k}_1}{2\omega_{k_1}} (2\pi)^{d-1}\delta(\vec{k}_1-\vec{p_1})e^{ik_1x} = \frac{e^{ip_1x}}{2\omega_{p_1}}
\end{aligned} \tag{9.81}
$$

$$
\int d\vec{y}_3 \Delta^+(x-y_3)e^{-ip_3y_3} = \frac{1}{(2\pi)^{d-1}} \int d\vec{y}_3 \frac{d\vec{k}_3}{2\omega_{k_3}} e^{-ik_3(x-y_3)}e^{-ip_3y_3} = \frac{e^{-ip_3x}}{2\omega_{p_3}}. \tag{9.82}
$$

Collecting all the results, we obtain:

$$
\begin{aligned}
\mathcal{T}_1 = -i\lambda\hbar^3 \left(\prod_{i=1}^{4} \frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\left(\prod_{i=1}^{4} \frac{1}{2\omega_{p_i}}\right) \int dx\, g(x)e^{i(p_1+p_2-p_3-p_4)x} \\
\overset{g\to 1}{=} -i\frac{\lambda}{(2\pi)^{d-2}}\hbar^3\delta(p_1+p_2-p_3-p_4).
\end{aligned} \tag{9.83}
$$

Note that in this case, the adiabatic limit is well defined. The above result is known as the Feynman rule (in momentum space) for the vertex in $\lambda\phi^4$ theories. Diagrammatically:

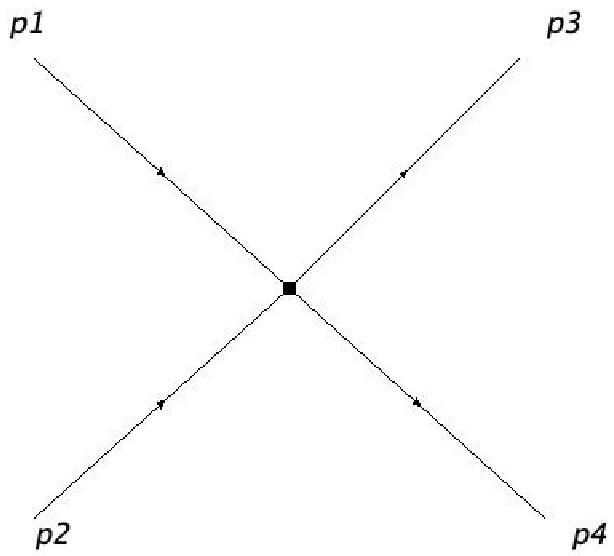


Figure 9.2: Every vertex correspond to a factor of $\frac{-i\lambda\hbar^3}{(2\pi)^{d-2}}$. The $\delta(p_1+p_2-p_3-p_4)$ just tell us we have momentum conservation.

Remark: The difference between the usual text book result $\mathcal{T}_1 = -i\lambda\delta(p_1+p_2-p_3-p_4)$ is due to different conventions of Fourier transformations.

To compute the scattering amplitude in second order of lambda we first need $\frac{i^2}{2!\hbar^2}T_2(L_{int}, L_{int})$. It is given by:

$$\begin{aligned}
\frac{i^2}{2\hbar^2}T_2(L_{int}, L_{int}) &= -\frac{1}{2\hbar^2}L_{int} \star_F L_{int} \\
&= -\frac{\lambda^2}{2(4!)^2\hbar^2}\int dxy\, g(x)g(y)\phi^4(x) \star_F \phi^4(y) \\
&= -\frac{\lambda^2}{2(4!)^2\hbar^2}\int dxy\, g(x)g(y)\Big(\phi^4(x)\phi^4(y) + 16\hbar\phi^3(x)\Delta^F(x-y)\phi^3(y) \\
&+ \frac{\hbar^2}{2}144\phi^2(x)(\Delta^F(x-y))^2\phi^2(y) + \frac{\hbar^3}{6}576\phi(x)(\Delta^F(x-y))^3\phi(y) \\
&+ \frac{\hbar^4}{24}576(\Delta^F(x-y))^4\Big).
\end{aligned} \tag{9.84}$$

Only terms with four powers of ϕ contribute to the amplitude. Hence:

$$\omega_0\left(\overline{\phi(p_1)\phi(p_2)} \star \left(\frac{-36\lambda^2\hbar^2}{(4!\hbar)^2}\int dxdy\, g(x)g(y)\phi^2(x)\phi^2(y)(\Delta^F(x-y))^2\right) \star \phi(p_3)\phi(p_4)\right). \tag{9.85}$$

Where we have used the short notation:

$$
\begin{aligned}
\overline{\phi(p_1)\phi(p_2)} &\equiv \frac{4\omega_{p_1}\omega_{p_2}}{(2\pi)^{d-1}}\int d\vec{x}_1 d\vec{x}_2\, e^{ip_1x_1+ip_2x_2}\phi(x_1)\phi(x_2) \\
\phi(p_3)\phi(p_4) &\equiv \frac{4\omega_{p_3}\omega_{p_4}}{(2\pi)^{d-1}}\int d\vec{x}_3 d\vec{x}_4\, e^{-ip_3x_3-ip_4x_4}\phi(x_3)\phi(x_4).
\end{aligned}
\tag{9.86}
$$

The calculation of the star product is long but straightforward. As usual, we first compute $\phi(x_1)\phi(x_2)\star\phi^2(x)\phi^2(y)$:.

$$
\begin{aligned}
&\phi(x_1)\phi(x_2)\star\phi^2(x)\phi^2(y) = \phi(x_1)\phi(x_2)\phi^2(x)\phi^2(y) \\
&+ 2\hbar\bigg(\phi(x_2)(\Delta^+(x_1-x)\phi(x)\phi^2(y) + \Delta^+(x_1-y)\phi^2(x)\phi(y)) \\
&+ \phi(x_1)(\Delta^+(x_2-x)\phi(x)\phi^2(y) + \Delta^+(x_2-y)\phi^2(x)\phi(y))\bigg) \\
&+ \hbar^2(\Delta^+(x_1-x)\Delta^+(x_2-x)\phi^2(y) + 2\Delta^+(x_1-x)\Delta^+(x_2-y)\phi(x)\phi(y)) \\
&+ \hbar^2(2\Delta^+(x_1-y)\Delta^+(x_2-x)\phi(x)\phi(y) + \Delta^+(x_1-y)\Delta^+(x_2-y)\phi^2(x)) \\
&+ \hbar^2(\Delta^+(x_2-x)\Delta^+(x_1-x)\phi^2(y) + 2\Delta^+(x_2-x)\Delta^+(x_1-y)\phi(x)\phi(y)) \\
&+ \hbar^2(2\Delta^+(x_2-y)\Delta^+(x_1-x)\phi(x)\phi(y) + \Delta^+(x_2-y)\Delta^+(x_1-y)\phi^2(x)) \\
&\equiv 2\phi^2(x)\Delta^+(x_1-y)\Delta^+(x_2-y) \\
&+ 4\phi(x)\phi(y)\Delta^+(x_1-x)\Delta^+(x_2-y) + x\leftrightarrow y.
\end{aligned}
\tag{9.87}
$$

The relevant term is the one with two powers of the field:

$$
\begin{aligned}
&2\hbar^2\Delta^+(x_1-x)\Delta^+(x_2-x)\phi^2(y) + 2\hbar^2\Delta^+(x_1-y)\Delta^+(x_2-y)\phi^2(x) \\
&+4\hbar^2(\Delta^+(x_1-x)\Delta^+(x_2-y) + \Delta^+(x_1-y)\Delta^+(x_2-x))\phi(x)\phi(y).
\end{aligned}
\tag{9.88}
$$

Now we need to compute the term above "star" $\phi(x_3)\phi(x_4)$. The term that is not proportional to any power of the field is

$$
\begin{aligned}
&4\hbar^4\Delta^+(x_1-x)\Delta^+(x_2-x)\Delta^+(y-x_3)\Delta^+(y-x_4) \\
&4\hbar^4\Delta^+(x_1-x)\Delta^+(x_2-y)\Delta^+(x-x_3)\Delta^+(y-x_4) \\
&4\hbar^4\Delta^+(x_1-x)\Delta^+(x_2-y)\Delta^+(y-x_3)\Delta^+(x-x_4) \\
&+x\leftrightarrow y.
\end{aligned}
\tag{9.89}
$$

The corresponding Feynman diagrams are:

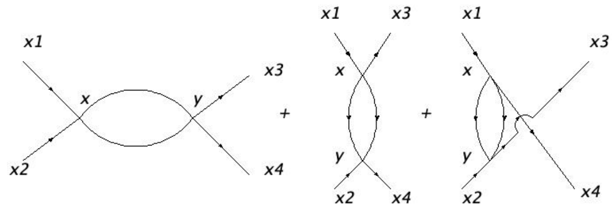


Figure 9.3: There are other three diagrams corresponding to $x \leftrightarrow y$

Using (9.82), we can perform the integrals in $\vec{X}_4$:

$$
\begin{aligned}
&\left(\prod_{i=1}^{4} \frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right) \int d\vec{X}_4 \, e^{i(p_1 x_1 + p_2 x_2 - p_3 x_y - p_4 x_4)} \\
&\omega_0 \left(\phi(x_1)\phi(x_2) \star \phi^2(x)\phi^2(y) \star \phi(x_3)\phi(x_4)\right) \\
&= \frac{4\hbar^4}{(2\pi)^{2(d-1)}} e^{i(p_1+p_2)x - i(p_3+p_4)y} \\
&+ \frac{4\hbar^4}{(2\pi)^{2(d-1)}} e^{i(p_1-p_3)x + i(p_2-p_4)y} \\
&+ \frac{4\hbar^4}{(2\pi)^{2(d-1)}} e^{i(p_1-p_4)x + i(p_2-p_3)y} \\
&+ x \leftrightarrow y.
\end{aligned} \tag{9.90}
$$

The terms above correspond to de diagrams drawn before them. Finally, the corresponding expression to each diagram reads.

$$
-\frac{144\lambda^2\hbar^2}{(2\pi)^{2(d-1)}(4!)^2} \int dx dy \, g(x)g(y) e^{i(p_1+p_2)x - i(p_3+p_4)y} (\Delta^F(x-y))^2 \tag{9.91}
$$

$$
-\frac{144\lambda^2\hbar^2}{(2\pi)^{2(d-1)}(4!)^2} \int dx dy \, g(x)g(y) e^{i(p_1-p_3)x + i(p_2-p_4)y} (\Delta^F(x-y))^2 \tag{9.92}
$$

$$
-\frac{144\lambda^2\hbar^2}{(2\pi)^{2(d-1)}(4!)^2} \int dx dy \, g(x)g(y) e^{i(p_1-p_4)x + i(p_2-p_3)y} (\Delta^F(x-y))^2. \tag{9.93}
$$

Until now, all calculations have been done independently of the number of dimensions. The problem of renormalization becomes apparent in the expressions above if $d \leq 4$. If we naively try to compute the integrals above, we will find a divergence in the limit $x \to y$. The reason is,as discussed in the renormalization section, due to:

$$
\mathrm{sd}(\Delta^F) = 2 \Rightarrow \mathrm{sd}((\Delta^F)^2) = 4. \tag{9.94}
$$

In this case, the W expansion is quite simple. Let us calculate, for example,

$$
\int dx dy \, g(x)g(y) e^{i(p_1+p_2)x - i(p_3+p_4)y} (\Delta^F(x-y))^2. \tag{9.95}
$$

First, we change the variables $z := x - y, \quad x = z + y$:

$$\int dxdy\, g(z+y)g(y)e^{i(p_1+p_2)(z+y)-i(p_3+p_4)y}(\Delta^F(z))^2$$
$$\equiv \int dy\, g(y)e^{i(p_1+p_2-p_3-p_4)y}\langle(\Delta^F)^2, e^{i(p_1+p_2)z}g(y+z)\rangle_z. \tag{9.96}$$

Since $\mathrm{sd}((\Delta^F)^2) = 4$, following the notation from section (8.7.2), we have $\omega = 0$. Hence, the $W-$ projection is simply:

$$\langle(\Delta^F)^2, We^{i(p_1+p_2)z}g(y+z)\rangle_z = \int dz\,(\Delta^F(z))^2(e^{i(p_1+p_2)z}g(y+z) - \omega(z)g(y))$$
$$+ Ce^{i(p_1+p_2)0}g(y+0). \tag{9.97}$$

Where $Cg(y)$ is the counter-term. Using:

$$(\Delta^F(z))^2 \overset{d=4}{=} \frac{1}{(2\pi)^8}\int dpdq\,\frac{e^{-ipz}}{p^2-m^2+i0}\frac{e^{-iqz}}{q^2-m^2+i0}. \tag{9.98}$$

we conclude:

$$\langle(\Delta^F)^2, We^{i(p_1+p_2)z}g(y+z)\rangle_z$$
$$= \frac{1}{(2\pi)^8}\int dz\int dpdq$$
$$\left\{\frac{e^{-ipz}}{p^2-m^2+i0}\frac{e^{-iqz}}{q^2-m^2+i0}(e^{i(p_1+p_2)z}g(y+z) - \omega(z)g(y))\right\} + Cg(y). \tag{9.99}$$

The expression above is sufficiently "well behaved" to choose $\omega(z) = 1 \forall z$ and take the adiabatic limit $g \to 1$. Once the parameters was chosen as mentioned, the corresponding expression is:

$$
\begin{aligned}
&\langle (\Delta^F)^2, We^{i(p_1+p_2)z}\rangle_z - C \\
&= \frac{1}{(2\pi)^8}\int dz \int dpdq\,\frac{e^{-ipz}}{p^2-m^2+i0}\frac{e^{-iqz}}{q^2-m^2+i0}(e^{i(p_1+p_2)z}-1) \\
&= \frac{1}{(2\pi)^4}\int dpdq\,\frac{\delta(p_1+p_2-p-q)-\delta(p+q)}{(p^2-m^2+i0)(q^2-m^2+i0)} \\
&= \frac{1}{(2\pi)^4}\int dp\,\frac{1}{(p^2-m^2+i0)((p-p_1-p_2)^2-m^2+i0)} - \frac{1}{(p^2-m^2+i0)^2} \\
&= \frac{1}{(2\pi)^4}\int dp\,\frac{p^2-m^2+i0-((p-p_1-p_2)^2-m^2+i0)}{(p^2-m^2+i0)^2((p-p_1-p_2)^2-m^2+i0)} \\
&= \frac{1}{(2\pi)^4}\int dp\frac{2p(p_1+p_2)-(p_1+p_2)^2}{(p^2-m^2+i0)^2((p-p_1-p_2)^2-m^2+i0)}.
\end{aligned}
\tag{9.100}
$$

The integral above is not simple to calculate, but its result is well known in the literature [19] eq.22.8 and eq22.42 or [20] page 46 eq.2.74 and page 49 eq.2.87. The result is:

$$
\langle (\Delta^F)^2, We^{i(p_1+p_2)z}\rangle_z = C_s + \begin{cases}
\frac{i}{4(2\pi)^2}\left(2-2\sqrt{\frac{4m^2-s}{s}}\arctan\left[\sqrt{\frac{s}{4m^2-s}}\right]\right), & 0<s<4m^2 \\
\frac{i}{4(2\pi)^2}\left(2+\sqrt{\frac{s-4m^2}{s}}\ln\left[\frac{\sqrt{s}-\sqrt{s-4m^2}}{\sqrt{s}+\sqrt{s-4m^2}}\right]+i\pi\right), & s>4m^2 \\
\frac{i}{4(2\pi)^2}\left(2+\sqrt{\frac{4m^2-s}{|s|}}\ln\left[\frac{\sqrt{4m^2-s}-\sqrt{|s|}}{\sqrt{4m^2-s}+\sqrt{|s|}}\right]\right), & s<0
\end{cases}
\tag{9.101}
$$

where $s=(p_1+p_2)^2$. We are left with the integral in the y coordinate. In the adiabatic limit it is simply:

$$
\begin{aligned}
&\int dy\, g(y)e^{i(p_1+p_2-p_3-p_4)y}\langle (\Delta^F)^2, e^{i(p_1+p_2)z}g(y+z)\rangle_z \\
&\quad = \delta(p_1+p_2-p_3-p_4)\langle (\Delta^F)^2, e^{i(p_1+p_2)z}g(y+z)\rangle_z.
\end{aligned}
\tag{9.102}
$$

Note that all integrals (9.93) have the same format, the only difference being the combination of momentum in the exponential. Therefore, we can summarize the expression of the amplitude by writing:

$$
\mathcal{T} = (\tilde{\Gamma}(s)+\tilde{\Gamma}(u)+\tilde{\Gamma(t)}+C_s+C_u+C_t)\delta(p_1+p_2-p_3-p_4). \tag{9.103}
$$

Where

$$\tilde{\Gamma}(s) = -\lambda^2\hbar^2 \begin{cases} \frac{i}{8(2\pi)^2}\left(2 - 2\sqrt{\frac{4m^2-s}{s}}\arctan\left[\sqrt{\frac{s}{4m^2-s}}\right]\right), & 0 < s < 4m^2 \\ \frac{i}{8(2\pi)^2}\left(2 + \sqrt{\frac{s-4m^2}{s}}\ln\left[\frac{\sqrt{s}-\sqrt{s-4m^2}}{\sqrt{s}+\sqrt{s-4m^2}}\right] + i\pi\right), & s > 4m^2 \\ \frac{i}{8(2\pi)^2}\left(2 + \sqrt{\frac{4m^2-s}{|s|}}\ln\left[\frac{\sqrt{4m^2-s}-\sqrt{|s|}}{\sqrt{4m^2-s}+\sqrt{|s|}}\right]\right), & s < 0 \end{cases} \tag{9.104}$$

.In the above expression, we have already considered the change $x \leftrightarrow y$. The constants C_s, C_u, C_t need to be fixed by physical arguments. A suggestion is done in [19]. We fixed the scale at $s_0 = 4m^2, t_0 = u_0 = 0$ imposing

$$\begin{cases} \tilde{\Gamma}(4m^2) + C_s = 0 \\ \tilde{\Gamma}(u_0 = 0) + C_u = 0 \\ \tilde{\Gamma}(t_0 = 0) + C_t = 0 \end{cases} . \tag{9.105}$$

Since the extension in $\mathcal{D}_\omega$ is unique, the above solution is unique and the problem is completely solved.

9.7 Example 5: Amplitude of massless $L_{int} = -\lambda \int dx\, g(x)(\phi\partial_\mu\phi)$ up to order 1 in the $\phi\phi \to \phi\phi$ process.

The last example we want to show involves a calculation with a derivative.

$$\mathcal{T}_1 = -\frac{i\lambda}{\hbar}\left(\prod_{i=i}^{4}\frac{2\omega_{p_i}}{(2\pi)^{\frac{d}{2}}}\right)\int dx d\vec{X}_4\, g(x) e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4}$$
$$\times\, \omega_0(\phi(x_1)\phi(x_2) \star \phi^2\partial_\mu\phi\partial^\mu\phi(x) \star \phi(x_3)\phi(x_4)). \tag{9.106}$$

We can calculate the star product $\phi^2\partial_\mu\phi\partial^\mu\phi(x) \star \phi(x_3)\phi(x_4)$:

$$
\begin{aligned}
&\phi^2(x)\partial_\mu\phi(x)\partial^\mu\phi(x)\star\phi(x_3)\phi(x_4)=\phi^2(x)\partial_\mu\phi(x)\partial^\mu\phi(x)\phi(x_3)\phi(x_4)+\\
&\hbar\bigg(2\phi(x)(\partial^\mu\phi(x))^2\Delta^+(x-x_3)\phi(x_4)+2\phi(x)(\partial^\mu\phi(x))^2\Delta^+(x-x_4)\phi(x_3)\\
&-2\phi^2(x)\partial_\mu\phi(x)\partial^\mu\Delta^+(x-x_3)\phi(x_4)-2\phi^2(x)\partial_\mu\phi(x)\partial^\mu\Delta^+(x-x_4)\phi(x_3)\bigg)\\
&+\hbar^2\bigg((\partial^\mu\phi(x))^2\Delta^+(x-x_3)\Delta^+(x-x_4)-2\phi(x)\partial_\mu\phi(x)\Delta^+(x-x_3)\partial_x^\mu\Delta^+(x-x_4)\\
&+(\partial^\mu\phi(x))^2\Delta^+(x-x_3)\Delta^+(x-x_4)-2\phi(x)\partial_\mu\phi(x)\Delta^+(x-x_4)\partial_x^\mu\Delta^+(x-x_3)\\
&-2\phi(x)\partial_\mu\phi(x)\partial_x^\mu\Delta^+(x-x_3)\Delta^+(x-x_4)+\phi^2(x)\partial_{x,\mu}\Delta^+(x-x_3)\partial_x^\mu\Delta^+(x-x_4)\\
&-2\phi(x)\partial_\mu\phi(x)\Delta^+(x-x_3)\partial_x^\mu\Delta^+(x-x_4)+\phi^2(x)\partial_{x,\mu}\Delta^+(x-x_3)\partial_x^\mu\Delta^+(x-x_4)\bigg).
\end{aligned}
\tag{9.107}
$$

It is meaningful to explain why some terms changed sign. The sign comes from the derivative of the delta function:

$$
\begin{aligned}
\partial^\mu\phi(x)\star\phi(x_3)&=\int dy_1dy_2\,\frac{\delta\partial^\mu\phi(x)}{\partial\phi(y_1)}\Delta^+(y_1-y_2)\frac{\delta\phi(x_3)}{\partial\phi(y_2)}\\
&=\int d_1dy_2\,\partial_x^\mu\delta(x-y_1)\Delta^+(y_1-y_2)\delta(x_3-y_2)\\
&=-\int d_1dy_2\,\delta(x-y_1)\partial_x^\mu(\Delta^+(y_1-y_2)\delta(x_3-y_2))\\
&=-\partial_x^\mu\Delta^+(x-x_3).
\end{aligned}
\tag{9.108}
$$

The term important to the amplitude is the one that contains two powers of the fields. We can summarize it as:

$$
\begin{aligned}
&\hbar^2\bigg(\phi^2(x)\partial_{x,\mu}\Delta^+(x-x_3)\partial_x^\mu\Delta^+(x-x_4)+(\partial^\mu\phi(x))^2\Delta^+(x-x_3)\Delta^+(x-x_4)\\
&-4\phi(x)\partial_\mu\phi(x)\Delta^+(x-x_3)\partial_x^\mu\Delta^+(x-x_4)+x_3\leftrightarrow x_4\bigg).
\end{aligned}
\tag{9.109}
$$

The term that does not contain any power of fields of the star product of the expression above with $\phi(x_1)\phi(x_2)$ is:

$$
\begin{aligned}
&\frac{\hbar^4}{2}(4\Delta^+(x_1-x)\Delta^+(x_2-x)\partial_{x,\mu}\Delta^+(x-x_3)\partial^\mu_x\Delta^+(x-x_4)\\
&+4\partial_{\mu,x}\Delta^+(x_1-x)\partial^\mu_x\Delta^+(x_2-x)\Delta^+(x-x_3)\Delta^+(x-x_4)\\
&+4\Delta^+(x_1-x)\partial_{\mu,x}\Delta^+(x_2-x)\partial^\mu_x\Delta^+(x-x_3)\Delta^+(x-x_4)\\
&+4\partial_{x,\mu}\Delta^+(x_1-x)\Delta^+(x_2-x)\partial^\mu_x\Delta^+(x-x_3)\Delta^+(x-x_4)\\
&\left.+x_3\leftrightarrow x_4\right)\\
&.
\end{aligned}
\tag{9.110}
$$

The expression above can be "simplified" if we disconsider the permutation of the terms. To fix the notation, consider:

$$
\begin{aligned}
P(x_1,x_2,x_3,x_4;x):=\hbar^4\Big(&\Delta^+(x_1-x)\Delta^+(x_2-x)\partial_{x,\mu}\Delta^+(x-x_3)\partial^\mu_x\Delta^+(x-x_4)\\
&+\partial_{\mu,x}\Delta^+(x_1-x)\partial^\mu_x\Delta^+(x_2-x)\Delta^+(x-x_3)\Delta^+(x-x_4)\\
&+2\Delta^+(x_1-x)\partial_{\mu,x}\Delta^+(x_2-x)\partial^\mu_x\Delta^+(x-x_3)\Delta^+(x-x_4)\Big).
\end{aligned}
\tag{9.111}
$$

Then, the full expression for the vacuum of the interaction is given by:

$$
\begin{aligned}
&\omega_0\left(\phi(x_1)\phi(x_2)\star(\phi(x)\partial^\mu\phi(x))\star\phi(x_3)\phi(x_4)\right)=\\
&P(x_1,x_2,x_3,x_4;x)+P(x_1,x_2,x_4,x_3;x)+P(x_2,x_1,x_3,x_4;x)+P(x_2,x_1,x_4,x_3;x).
\end{aligned}
\tag{9.112}
$$

Now, we plug $P(x_1,x_2,x_3,x_4;x)$ in the expression for the amplitude:

$$
\begin{aligned}
\mathcal{T}_1=&-\frac{i\lambda}{\hbar}\left(\prod_{i=i}^{4}\frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\int dxd\vec{X}_4\, g(x)e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4}\\
&\times P(x_1,x_2,x_3,x_4;x).
\end{aligned}
\tag{9.113}
$$

$P(x_1,x_2,x_3,x_4;x)$ can be written as:

$$
\begin{aligned}
&P(x_1,x_2,x_3,x_4;x)=\frac{\hbar^4}{(2\pi)^{4(d-1)}}\int\left(\prod_{i=1}^{4}\frac{d\vec{k}_i}{2\omega_{k_i}}\right)\\
&(k_3\cdot k_4+k_1\cdot k_2+2k_1\cdot k_3)e^{-ik_1(x_1-x)-ik_2(x_2-x)-ik_3(x-x_3)-ik_4(x-x_4)}\\
&.
\end{aligned}
\tag{9.114}
$$

The calculation of the integrals in the adiabatic limit is essentially the same. Once the integral over all the P's is done, the final result, i.e, the Feynman rule is:

$$
\begin{aligned}
\mathcal{T}_1 = &-i\frac{\lambda\hbar^3}{(2\pi)^2}\Big((p_3\cdot p_4 + p_1\cdot p_2 + 2p_1\cdot p_3) + (p_4\cdot p_3 + p_1\cdot p_2 + 2p_1\cdot p_4) \\
&+ (p_3\cdot p_4 + p_2\cdot p_1 + 2p_2\cdot p_3) + (p_4\cdot p_3 + p_2\cdot p_1 + 2p_2\cdot p_4)\Big)\delta(p_1+p_2-p_3-p_4) \\
&= -4i\frac{\lambda\hbar^3}{(2\pi)^2}\left(p_1\cdot p_2 + p_3\cdot p_4 + \frac{1}{2}(p_1+p_2)\cdot(p_3+p_4)\right)\delta(p_1+p_2-p_3-p_4).
\end{aligned} \tag{9.115}
$$

Note that since we have the momentum of the particles appearing in the formula, it is important to distinguish incoming from outgoing particles when drawing the diagram!!

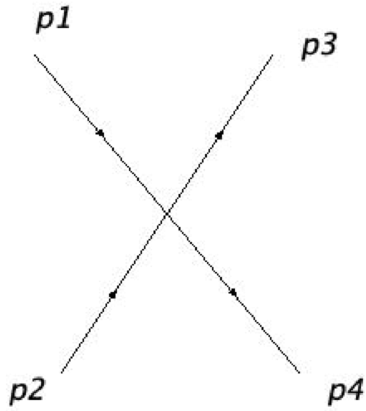


Figure 9.4: The diagram of the scattering $\phi\phi \to \phi\phi$

Chapter 10

Renormalization

10.1 Introduction: The problem in simple words

Before describing the abstract theory, let us give a taste of the problem we are about to attack. Suppose that we want to calculate the S-matrix considering an interaction of the form $L_{\text{int}} = -\lambda \int g(x)\phi^4(x)dx$ in 4 dimensions. As we have seen, we need to fix the counter-terms using physical inputs. Since these inputs are physically relevant, what does that mean? In the above example, we were able to perform the adiabatic limit, but fixing the parameters of the theory should not depend on how the interaction is turned on/off by the test function $g(x)$. Is it possible to find a formulation regardless of g?

We will answer all the aforementioned by studying the so-called "Stückelberg-Petermann" renormalization group.

10.2 Stückelberg-Petermann renormalization group

Here we change the discussion presented in [24] Chapter 3.6 to a most "intuitive one". The price one pays by doing it is limiting the exposure to the lowest-order terms. However, the most general discussion is not of central importance in this work.

Given two $T-$ products, $T(e_{\otimes}^{\frac{i\lambda}{\hbar}F}), \widetilde{T}(e_{\otimes}^{\frac{i\lambda}{\hbar}F})$ where $F \in \mathcal{F}_{\text{loc}}$. We want to construct a map $Z : \mathcal{F}_{\text{loc}}[\![\hbar,\lambda]\!] \to \mathcal{F}_{\text{loc}}[\![\hbar,\lambda]\!]$ such that

$$T\left(e_{\otimes}^{\frac{i}{\hbar}\lambda F}\right) = \widetilde{T}\left(e_{\otimes}^{\frac{i}{\hbar}Z(\lambda F)}\right) \tag{10.1}$$

and Z can be written as a series:

$$Z(F) := \sum_{n=0}^{\infty} \frac{1}{n!} Z_n(F^{\otimes n}). \tag{10.2}$$

We simplify the notation using $F_n \equiv F(x_n)$. We also impose, for simplicity, symmetry in the arguments $Z(F_{\pi(1)}, ..., F_{\pi(n)}) = Z(F_1, ..., F_n) \forall \pi \in S^n$. As usual, let us construct the map Z order by order in perturbation theory.

$[\lambda] = \mathbf{1}$:

In first order in perturbation theory we obtain:

$$
\begin{aligned}
T_1(\frac{i\lambda}{\hbar}F) = \frac{i\lambda}{\hbar}F \stackrel{!}{=} \widetilde{T}_1(\frac{i\lambda}{\hbar}Z(F)) = \frac{i\lambda}{\hbar}Z_1(F) \\
\Rightarrow Z_1(F) = F.
\end{aligned}
\tag{10.3}
$$

$[\lambda] = \mathbf{2}$:

Now the formula gets a bit more engaging[1]:

$$
\begin{aligned}
e_{\otimes}^{\frac{i}{\hbar}Z(\lambda F)} &= 1 + \frac{i}{\hbar}Z(\lambda F) + \frac{i^2}{\hbar^2}(Z(\lambda F))^{\otimes 2} \\
&= \frac{i}{\hbar}\left(\lambda Z_1(F) + \frac{\lambda^2}{2}Z_2(F_1, F_2)\right) \\
&+ \frac{i^2}{\hbar^2}\lambda^2 Z_1(F_1) \otimes Z_1(F_2).
\end{aligned}
\tag{10.4}
$$

Thus:

$$
\begin{aligned}
\frac{i^2\lambda^2}{2\hbar^2}T_2(F_1, F_2) = \frac{i\lambda^2}{2\hbar}\tilde{T}_1(Z_2(F_1, F_2)) + \frac{i^2\lambda^2}{2\hbar^2}\tilde{T}_2\left(Z_1(F_1), Z_1(F_2)\right) \\
\Rightarrow Z_2(F_1, F_2) = \frac{i}{\hbar}\left(T_2(F_1, F_2) - \tilde{T}_2(F_1, F_2)\right).
\end{aligned}
\tag{10.5}
$$

In the above equation, we have used $Z_1(F) = F$ and $T_1(F) = F$. The general formula for the $n-$th order can be found using the same procedure. Just one last example to "get the feeling" of how higher orders are constructed:

$[\lambda] = \mathbf{3}$:

Now the combinatorial are more complicated. First, we write:

[1] Note that we have two tipes of terms proportional to λ^2 in the expansion from $e_{\otimes}^{\frac{i}{\hbar}Z(\lambda F)}$, one is Z_1^2 and the other Z_2

$$
\begin{aligned}
Z(\lambda F) &= \underbrace{Z_0(\lambda F)}_{=0} + \lambda Z_1(F_1) + \frac{\lambda^2}{2} Z_2(F_1, F_2) + \frac{\lambda^3}{3!} Z_3(F_1, F_2, F_3) \\
\Rightarrow \exp_{\otimes}(\frac{i}{\hbar} Z(F)) &\overset{\lambda^3}{=} \frac{i^3}{\hbar^3} \frac{\lambda^3}{3!} (Z_1(F_1))^{\otimes 3} \\
&+ \frac{1}{2} \frac{i^2}{\hbar^2} \left(\lambda Z_1(F_1) + \frac{\lambda^2}{2} Z_2(F_1, F_2) \right)^{\otimes 2} + \frac{i}{\hbar} \frac{\lambda^3}{3!} Z_3(F_1, F_2, F_3) \\
&\overset{\lambda^3}{=} \frac{\lambda^3}{3!} \left[\frac{i^3}{\hbar^3} Z_1(F_1) \otimes Z_1(F_2) \otimes Z_1(F_3) \right. \\
&+ \frac{i^2}{\hbar^2} (Z_2(F_1, F_2) \otimes Z_1(F_3) \\
&+ Z_2(F_1, F_3) \otimes Z_1(F_2) + Z_2(F_2, F_3) \otimes Z_1(F_1)) \\
&\left. + \frac{i}{\hbar} Z_3(F_1, F_2, F_3) \right].
\end{aligned} \tag{10.6}
$$

The term in the fourth line was symmetrized to maintain the symmetry of Z. Therefore, the $T-$ product is given by:

$$
\begin{aligned}
\frac{i^3}{\hbar^3} T_3(F_1, F_2, F_3) &= \frac{i^3}{\hbar^3} \widetilde{T}_3(F_1, F_2, F_3) \\
&+ \frac{i^2}{\hbar^2} \sum_{\pi \in S^3} \widetilde{T}_2(Z_2(F_{\pi_1}, F_{\pi_2}), Z_1(F_{\pi_3})) + \frac{i}{\hbar} \widetilde{T}_1(Z_3(F_1, F_2, F_3)) \\
\Rightarrow Z_3(F_1, F_2, F_3) &= \frac{i^2}{\hbar^2} T_3(F_1, F_2, F_3) - \frac{i^2}{\hbar^2} \widetilde{T}_3(F_1, F_2, F_3) \\
&- \frac{i}{\hbar} \sum_{\pi \in S^3} \widetilde{T}_2(Z_2(F_{\pi_1}, F_{\pi_2}), Z_1(F_{\pi_3})).
\end{aligned} \tag{10.7}
$$

Just a quick remark on the general formula: Note that by expanding $Z(\lambda F)$ to the $n-$th order and composing with T, the result is always of the form:

$$
T \left(\frac{i\lambda^n}{\hbar n!} Z_n(F_1, ..., F_n) \right) = \frac{i\lambda^n}{\hbar n!} T_1(Z_n(F_1, ..., F_n)) = \frac{i\lambda^n}{\hbar n!} Z_n(F_1, ..., F_n). \tag{10.8}
$$

That means that we can explicitly write Z_n as a combination of the preceding orders, namely:

$$
\frac{i\lambda^n}{\hbar n!} Z_n(F_1, ..., F_n) = \widetilde{T}_n(F_1, ..., F_n) - \text{Combinations of lowest orders from T's and Z's.} \tag{10.9}
$$

A proof that this construction is well defined can be found in [60] and on page 227 [24].

The set of maps $Z(n)$ defines the so-called Stückelberg-Petermann renormalization group [58]. There is a lot that can be talked about the group, but, for us, the importance of the group is the so called "**The main theory of perturbative renormalization**". It states [24] page

225:

a) *Given two renormalization prescriptions, i.e., two time-ordered products T and $\widetilde{T}$ both fulfilling the axioms, there exists a map $Z : \mathcal{F}_{\text{loc}}[\![\hbar, \lambda]\!] \to \mathcal{F}_{\text{loc}}[\![\hbar, \lambda]\!]$ of the form $Z(F) = \sum_{n=0}^{\infty} \frac{Z_n(F)}{n!}$, which is uniquely determined by*

$$\widetilde{T}(e^{\frac{i}{\hbar}F}) = T(e^{\frac{i}{\hbar}Z(F)}) \iff \tilde{S} = S \circ Z. \tag{10.10}$$

where Z is an element of the Stückelberg-Petermann group.

b) *Conversely, given an S-matrix S fulfilling the axioms for the time-ordered product and an arbitrary Z of the Stückelberg-Petermann group, the composition $\tilde{S} = S \circ Z$ also satisfies these axioms.*

The proof of the statement can be found on page 225 [24]. The theorem above basically gives the most general transformations between two normalization conditions and also justified why is called a group. There is also a similar construction for the $R-$ product [24] page 233.

The next question we want to answer is what the physical meaning of the renormalization group is. To do it, we first introduce a change in the scale $(x, m) \to (\rho x, \frac{m}{\rho})$page 93 [24]. We call a **scaling transformation** $\sigma_\rho : \mathcal{F} \to \mathcal{F}, \rho > 0$ a linear transformation that acts on a single field as:

$$\sigma_\rho(\phi(x)) := \rho^{-\dim(\phi)}\phi(\rho^{-1}x). \tag{10.11}$$

and in a generic field as:

$$\begin{aligned} &\sigma_\rho\left(\int dX_n f(x_1, ..., x_n)\phi(x_1)...\phi(x_n)\right) \\ &:= \rho^{-n\dim\phi} \int dX_n f(x_1, ..., x_n)\phi(\rho^{-1}x)...\phi(\rho^{-1}x_n). \end{aligned} \tag{10.12}$$

We emphasize that in the above formula σ_ρ only change the argument of the fields!!

The formula above justifies the change in the space, but what about the change in the mass? The dependence on the mass is restricted to the propagators $\Delta^+(x) \equiv \Delta^+_m(x)$ and they appear as argument in the star product $\star \equiv \star_m$. The is to compute $\sigma_\rho(F \star G)$ and see what happens to the mass.

In our analyses, we will consider $F = \int dX_n\, f(x)\phi^n(x)$, $G = \int dY_n\, g(y)\phi^m(y)$, $\infty > n \geq m$ for simplicity. The general case involves only a more sophisticated combinatorial, but the result is the same.

$$
\begin{aligned}
&\sigma_\rho(F \star_m G) = \sigma_\rho\Big(\int dxdy\, f(x)g(y)\phi^n(x)\phi^m(y) \\
&+ \sum_{k=1}^{m} \frac{(n-k)!(m-k)!\hbar^k}{k!} \int dxdy\, f(x)g(y)\phi^{n-k}(x)(\Delta^+(x-y))^k \phi^{m-k}(y)\Big) \\
&= \int dxdy\, f(x)g(y)\rho^{-(n+m)\dim\phi}\phi^n(\rho^{-1}x)\phi^m(\rho^{-1}y) \\
&+ \sum_{k=0}^{m} \frac{(n-k)!(m-k)!\hbar^k}{k!} \int dxdy\, f(x)g(y) \\
&\rho^{-(n+m+2k)\dim\phi}\phi^{n-k}(\rho^{-1}x)(\Delta^+(x-y))^k\phi^{m-k}(\rho^{-1}y).
\end{aligned} \tag{10.13}
$$

Now we write:

$$
\begin{aligned}
\rho^{-(n+m+2k)\dim\phi}(\Delta^+(x-y))^k &= \rho^{-(n+m)\dim\phi}(\rho^{-2\dim\phi}\Delta^+_m(x-y))^k \\
&= \rho^{-(n+m)\dim\phi}(\rho^{-2(\frac{d}{2}-1)}\Delta^+_m(x-y))^k \\
&= \rho^{-(n+m)\dim\phi}(\rho^{d-2}\Delta^+_m(x-y))^k \\
&= \rho^{-(n+m)\dim\phi}((\rho^{-1})^{2-d}\Delta^+_m(x-y))^k \\
&= \rho^{-(n+m)\dim\phi}(\Delta^+_{\rho m}(\rho^{-1}x-\rho^{-1}y))^k.
\end{aligned} \tag{10.14}
$$

In the last equality we have used the last identity of (3.5). Hence:

$$
\begin{aligned}
\sigma_\rho(F \star_m G) &= \int dxdy\, f(x)g(y)\rho^{-(n+m)\dim\phi}\phi^n(\rho^{-1}x)\phi^m(\rho^{-1}y) \\
&+ \sum_{k=0}^{m} \frac{(n-k)!(m-k)!\hbar^k}{k!} \int dxdy\, f(x)g(y) \\
&\times \rho^{-(n+m)\dim\phi}\phi^{n-k}(\rho^{-1}x)(\Delta^+_{\rho m}(\rho^{-1}(x-y)))^k\phi^{m-k}(\rho^{-1}y) \\
&= \rho^{-(n+m)\dim\phi}\Big(\int dxdy\, f(x)g(y)\phi^n(\rho^{-1}x)\phi^m(\rho^{-1}y) \\
&+ \sum_{k=0}^{m} \frac{(n-k)!(m-k)!\hbar^k}{k!} \int dxdy\, f(x)g(y) \\
&\times \phi^{n-k}(\rho^{-1}x)(\Delta^+_{\rho m}(\frac{x-y}{\rho}))^k\phi^{m-k}(\rho^{-1}y)\Big) \\
&= \sigma_\rho(F) \star_{m\rho} \sigma_\rho(G).
\end{aligned} \tag{10.15}
$$

From the formula above, we see that the scaling transformation indeed changes $(x,m) \to (\rho^{-1}x, \rho m)$.

It is not difficult to check that a new scaled $S-$ matrix defined by

$$\mathbf{S}^{m}_{\rho} := \sigma_{\rho} \circ \mathbf{S}^{\frac{m}{\rho}} \circ \sigma_{\rho^{-1}}. \tag{10.16}$$

is also a $S-$ matrix, that is, it fulfills the axioms of the $T-$ product. The main theorem of perturbative renormalization guarantees us that both descriptions are connected by a change of renormalization. Thus, the physical meaning of the renormalization group is a change in scale! Usually in QFT, this phenomenon is called "running of the constants" ([57] chapter 12). The meaning of the name becomes clearer once the adiabatic limit is taken. Fortunately, when working with renormalization, this limit always exists. It is far from obvious to show this feature, but it can be done using the formalism of **algebraic quantum field theory** (AQFT). Roughly speaking, AQFT studies the most fundamental aspects of quantum field theory using very abstract mathematics (net of algebras, category theory, and so on). A special case is the perturbative approach of QFT. One can show that the perturbative approach developed here can be formulated using AQFT, then it is shown that the crucial properties of the renormalization can be obtained if we restrict the space-time to a diamond-shaped region, and last but not least, one can then show that in this region the renormalization can be done independently from g. A deeper discussion can be found in Chapter 3.7 from [24]. For more statements about the adiabatic limit, we recommend [23] and references therein [2]. The last reference we recommend is about the general features and advances of AQFT [14].

For now on, we assume that the limit has been taken and use it to compute the renormalization of a scalar theory with $L_{int} = -\frac{\lambda}{4!}\int dx\, \phi^4(x)$.

10.3 Renormalization of $L_{int} = -\frac{\lambda}{4!}\int dx\, \phi^4(x)$ up to order 2

The strategy for computing the renormalization is the following: We know, due to the main theorem of renormalization, that two $T-$ products are connected by a scaling transformation. Hence, we renormalize all the terms in T_2 once and compute $Z_2 = \sigma_{\rho} \circ T_2^{\frac{m}{\rho}} \circ \sigma_{\rho^{-1}} - T_2^{m}$. Using (9.84), the terms that need a non-trivial extension, that is, have a scaling degree $d \geq 4$, are $(\Delta^F)^2, (\Delta^F)^3, (\Delta^F)^4$. Unlike (9.6), we will use a different method to extend distributions. The main reason to do so is that $W-$ expansion is easier to compute amplitudes that can be measured, but the computations for Z are not that clear (see [60]). The calculations become easier if we give up the W expansion and substitute it for a method called **differential renormalization** page193 [24].

[2]in this article it is proved that the adiabatic limit we simply calculated in the examples of last section always exists for massive theories! It also established boundaries in massless theories

10.3.1 Differential renormalization

Before attacking the problem, let us briefly explain the method. Let $t^0 \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$ such that $\mathrm{sd}(t^0) = k \geq d$. If we manage to write t^0 as

$$t^0 = \mathfrak{D} f, \quad \mathfrak{D} = \sum_{|a|=l} C_a \partial^a, C_a \in \mathbb{C}. \tag{10.17}$$

With $k - l < d$ and $(x\partial_x + k - l)^N f = 0$. Then, we have an extension of the distribution t^0. We can prove this by writing:

$$\langle t, h \rangle = \langle \mathfrak{D} f, h \rangle = (-1)^{|l|} \langle f, \mathfrak{D} h \rangle. \tag{10.18}$$

Since $\mathrm{sd}(f) < d$, the extension is given by the same formula and the expression above is well defined. The expression also scales almost homogeneously with degree D, since:

$$(x\partial_x + k)^N \mathfrak{D} f = (x\partial_x + k - l)^N f = 0. \tag{10.19}$$

The challenge is that there is no general method to find such a f. The existence of counter-terms reflect that difficulty. Given a f_0 with the desired properties, we can always sum $g^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ with degree $D - l$ and $\mathfrak{D} g^0 = 0$ in $\mathcal{D}(\mathbb{R}^k \setminus \{0\})$. Since $\mathfrak{D} g^0 = 0$ in $\mathcal{D}(\mathbb{R}^k \setminus \{0\})$, $\mathrm{supp}\, \mathfrak{D} g \subseteq \{0\} \Rightarrow \mathfrak{D} g = \sum_a C_a \partial^a \delta(x)$, that is, g contributes by adding counter-terms. Fortunately, we have enough tolls to fix the propagators discussed in this dissertation.

To perform the renormalization, we need an auxiliary formula.

$$\begin{aligned}
\partial^2 \frac{\ln^j(M^2x^2)}{(x^2)^k} &= \partial_\mu \left(2jx^\mu \frac{\ln^{j-1}(M^2x^2)}{(x^2)^{k+1}} - 2kx^\mu \frac{\ln^j(M^2x^2)}{(x^2)^{k+1}} \right) \\
&= 2jd \frac{\ln^{j-1}(M^2x^2)}{(x^2)^{k+1}} + 4j(j-1) \frac{\ln^{j-2}(M^2x^2)}{(x^2)^{k+1}} - 4j(k+1) \frac{\ln^{j-1}(M^2x^2)}{(x^2)^{k+1}} \\
&- 2kd \frac{\ln^j(M^2x^2)}{(x^2)^{k+1}} - 4kj \frac{\ln^{j-1}(M^2x^2)}{(x^2)^{k+1}} + 4k(k+1) \frac{\ln^j(M^2x^2)}{(x^2)^{k+1}} \\
&= (2jd - 4j(k+1) - 4jk) \frac{\ln^{j-1}(M^2x^2)}{(x^2)^{k+1}} \\
&+ (4k(k+1) - 2kd) \frac{\ln^j(M^2x^2)}{(x^2)^{k+1}} + 4j(j-1) \frac{\ln^{j-2}(M^2x^2)}{(x^2)^{k+1}}.
\end{aligned} \tag{10.20}$$

where M is a mass scale, d is the number of space-time dimensions, and $\ln^a(x^2) \equiv (\ln(x))^a$. The same formula holds for $x^2 \pm i0$ and $x^2 \pm ip^0 0$ instead of x^2.

To perform the calculation, we also need the Feynman propagator in position space. Although it can be computed with great generality for an arbitrary dimension [46], for our porpoises, it is easier to start from formula 2.2.3 from [24]:

$$\Delta^+_{d=4}(z) = \frac{-1}{4\pi^2(z^2 - iz^0 0)} + m^2 f(m^2 z^2) \ln\left(-\frac{m^2}{4}(z^2 - iz^0 0)\right) + m^2 g(m^2 z^2). \tag{10.21}$$

Where:

$$\begin{aligned} f(x) &:= \frac{1}{8\pi^2\sqrt{x}} J_1(\sqrt{x}) \equiv \sum_{k=0}^{\infty} a_k x^k \\ g(x) &:= -\frac{1}{16\pi^2} \sum_{k=0}^{\infty} \left(\frac{\Gamma'(k+1)}{\Gamma(k+1)} + \frac{\Gamma'(k+2)}{\Gamma(k+2)}\right) \frac{(-x/4)^k}{k!(k+1)!} \equiv \sum_{k=0}^{\infty} b_k x^k. \end{aligned} \tag{10.22}$$

Where $J_1(x)$ is the Bessel function of order 1:

$$J_1(\sqrt{x}) := \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+1} l!(l+1)!} x^l \sqrt{x}. \tag{10.23}$$

And $\Gamma(x)$ is the gamma function:

$$\Gamma(x+1) := \int_0^{\infty} dt\, e^{-t} t^x. \tag{10.24}$$

The Feynman propagator can be computed using:

$$\begin{aligned} &\Delta^F(z) = \theta(z^0)\Delta^+(z) + \theta(-z^0)\Delta^+(-z) \\ &\overset{d=4}{=} \theta(z^0)\left(\frac{1}{4\pi^2(z^2 - iz^0 0)} + m^2 f(m^2 z^2) \ln\left(-\frac{m^2}{4}(z^2 - iz^0 0)\right) + m^2 g(m^2 z^2)\right) \\ &+ \theta(-z^0)\left(\frac{1}{4\pi^2(z^2 + iz^0 0)} + m^2 f(m^2 z^2) \ln\left(-\frac{m^2}{4}(z^2 + iz^0 0)\right) + m^2 g(m^2 z^2)\right) \\ &= \frac{1}{4\pi^2(z^2 - i0)} + m^2 f(m^2 z^2) \ln\left(-\frac{m^2}{4}(z^2 - i0)\right) + m^2 g(m^2 z^2). \end{aligned} \tag{10.25}$$

Note that, as expected,

$$\rho^2 \Delta^F_{\frac{m}{\rho}}(\rho z) = \Delta^F(z). \tag{10.26}$$

Returning to the problem of renormalization, we want to compute $Z_2(L_{int}, L_{int})$. Using the main theorem of perturbative renormalization, it can be computed using:

$$Z_2 = \frac{i}{\hbar}\left(\sigma_\rho \circ T_2^{\frac{m}{\rho}}(\sigma_{\rho^{-1}}(L_{int}), \sigma_{\rho^{-1}}(L_{int})) - T_2(L_{int}, L_{int})\right). \tag{10.27}$$

Using (9.75), we can easily compute $\sigma_\rho \circ T_2^{\frac{m}{\rho}}(\sigma_{\rho^{-1}}(L_{int}), \sigma_{\rho^{-1}}(L_{int}))$:

$$\begin{aligned}
&\sigma_\rho \circ T_2^{\frac{m}{\rho}}(\sigma_{\rho^{-1}}(L_{int}), \sigma_{\rho^{-1}}(L_{int})) = \frac{\lambda^2}{(4!)^2}\int dX_2 \\
&\times \Big(\phi^4(x_1)\phi^4(x_2) + 16\hbar\phi^3(x_1)\phi^3(x_2)\rho^2\Delta^F_{\frac{m}{\rho}}(\rho(x_1-x_2)) \\
&+ 72\hbar^2\phi^2(x_1)\phi^2(x_2)(\rho^2\Delta^F_{\frac{m}{\rho}}(\rho(x_1-x_2)))^2 \\
&+ 96\hbar^3\phi(x_1)\phi(x_2)(\rho^2\Delta^F_{\frac{m}{\rho}}(\rho(x_1-x_2)))^3 \\
&+ 24\hbar^4(\rho^2\Delta^F_{\frac{m}{\rho}}(\rho(x_1-x_2)))^4\Big).
\end{aligned} \tag{10.28}$$

Let us compute term by term of Z_2. For simplicity, we denote $y \equiv x_1 - x_2$ and $Y \equiv -(y^2 - i0)$. The first term reads::

$$\frac{\lambda^2}{(4!)^2}16\hbar\int dX_2\,(\rho^2\Delta^F_{\frac{m}{\rho}}(\rho y) - \Delta^F(y))\phi^3(x_1)\phi^3(x_2). \tag{10.29}$$

Now, the renormalization strikes with full force. Since $\Delta^F(y) \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\})$ has a trivial extension to $y = 0$ and $\rho^2\Delta^F_{\frac{m}{\rho}}(\rho y) = \Delta^F(y)$ outside the diagonal, the contribution of the first term is zero. The second term reads:

$$\frac{\lambda^2}{(4!)^2}72\hbar^2\int dX^2\,((\rho^2\Delta^F_{\frac{m}{\rho}}(\rho y))^2 - (\Delta^F(y))^2)\phi^2(x_1)\phi^2(x_2). \tag{10.30}$$

$(\Delta^F(y))^2$ need a non-trivial renormalization. To do it, we write:

$$\begin{aligned}
(\Delta^F(y))^2 &= \left(\frac{1}{4\pi^2 Y} + m^2 f(m^2y^2)\ln\left(\frac{m^2}{4}Y\right) + m^2 g(m^2y^2)\right)^2 \\
&= \frac{1}{(4\pi^2)^2Y^2} + \frac{1}{2\pi Y}\left(m^2 f(m^2y^2)\ln\left(\frac{m^2}{4}Y\right) + m^2 g(m^2y^2)\right) \\
&+ \left(m^2 f(m^2y^2)\ln\left(\frac{m^2}{4}Y\right) + m^2 g(m^2y^2)\right)^2.
\end{aligned} \tag{10.31}$$

Note that the terms in the second line have scaling degree sd < 4 and therefore, it expansion to $y = 0$ is given by the same formula. The term $\frac{1}{(4\pi)^2Y^2}$ has scaling degree $d = 4$ and is the only one that needs a renormalization. That can be done substituting $d = 4, j = 1, k = 1$ in (10.20):

$$\frac{1}{Y^2} = \frac{1}{4}\partial_y^2 \frac{\ln\left(M^2 Y\right)}{Y}. \tag{10.32}$$

Note that $\mathrm{sd}(\frac{\ln\left(M^2 Y\right)}{Y}) = 2 < 4$. Hence, it is renormalizable. Thus:

$$\begin{aligned}
&\frac{\lambda^2}{(4!)^2} 72\hbar^2 \int dX^2 \, ((\rho^2 \Delta^F_{\frac{m}{\rho}}(\rho y))^2 - (\Delta^F(y))^2)\phi^2(x_1)\phi^2(x_2) \\
&= \frac{\lambda^2}{(4!)^2} 72\hbar^2 \int dX^2 \, \frac{1}{4(4\pi^2)^2} (\rho^4 \partial^2_{\rho y} \frac{\ln(M^2\rho^2 Y)}{\rho^2 Y} - \partial_y^2 \frac{\ln(M^2 Y)}{Y})\phi^2(x_1)\phi^2(x_2) \\
&= \frac{\lambda^2}{(4!)^2} 72\hbar^2 \int dX^2 \, \frac{1}{4(4\pi^2)^2} (2\ln(\rho)\partial_y^2 \frac{1}{Y})\phi^2(x_1)\phi^2(x_2).
\end{aligned} \tag{10.33}$$

Last but not least, we use:

$$\partial_y^2 \frac{1}{Y} = -4\pi^2 i \delta(y). \tag{10.34}$$

To write:

$$\begin{aligned}
&\frac{\lambda^2}{(4!)^2} 72\hbar^2 \int dX^2 \, ((\rho^2 \Delta^F_{\frac{m}{\rho}}(\rho y))^2 - (\Delta^F(y))^2)\phi^2(x_1)\phi^2(x_2) \\
&= \frac{\lambda^2}{(4!)^2} 72\hbar^2 \int dX^2 \, \frac{-i}{8\pi^2} \ln(\rho)\delta(y)\phi^2(x_1)\phi^2(x_2) = -i\frac{\lambda^2}{(4!)^2\pi^2} 9\hbar^2 \ln(\rho) \int dx_1 \, \phi^4(x_1) \\
&= \frac{\hbar}{i}\frac{1}{4!}\frac{-3\hbar\lambda^2}{8\pi^2} \ln(\rho) \int dx_1 \phi^4(x_1) \equiv \frac{\hbar}{i}\frac{1}{4!} \int dx_1 \tilde{C}\phi^4(x_1).
\end{aligned} \tag{10.35}$$

Where $\tilde{C} = \frac{3\hbar\lambda^2}{8\pi^2}\ln(\rho)$.The factor $\frac{\hbar}{4!i}$ was kept in the above formula because the original expression of Z_2 is proportional to $\frac{i}{\hbar}$ and the factor $\frac{1}{4!}$ will make the calculation of renormalized λ_ρ easier (10.3.1).

The next term is proportional to $(\Delta^F(y))^3$. Before computing it, let us do the renormalization of the term:

$$\begin{aligned}
&(\Delta^F(y))^3 = \left(\frac{1}{4\pi^2 Y} + m^2 f(m^2y^2)\ln\left(\frac{m^2}{4}Y\right) + m^2 g(m^2y^2)\right)^3 \\
&\left(\frac{1}{4\pi^2 Y}\right)^3 + 3\left(\frac{1}{4\pi^2 Y}\right)^2 \left(m^2 f(m^2y^2)\ln\left(\frac{m^2}{4}Y\right) + m^2 g(m^2y^2)\right) + R.
\end{aligned} \tag{10.36}$$

We denote all terms by R by $\mathrm{sd} < 4$. Now we expand:

$$
\begin{aligned}
&3\left(\frac{1}{4\pi^2 Y}\right)^2\left(m^2 f(m^2y^2)\ln\left(\frac{m^2}{4}Y\right)+m^2 g(m^2y^2)\right)\\
=&3\left(\frac{1}{4\pi^2 Y}\right)^2\left(m^2\frac{1}{16\pi^2}\ln\left(\frac{m^2}{4}Y\right)+m^2\frac{-1}{16\pi^2}\left(\frac{\Gamma'(2)}{\Gamma(2)}+\frac{\Gamma'(1)}{\Gamma(1)}\right)\right)+R\\
=&\frac{3m^2}{8\pi^2}\left(\frac{1}{4\pi^2 Y}\right)^2\left(\ln\left(\frac{m^2}{4}Y\right)+\frac{2\gamma-1}{2}\right)+R.
\end{aligned}
\tag{10.37}
$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

Hence:

$$
\begin{aligned}
(\Delta^F(y))^3 &= a\frac{1}{Y^3}+b\frac{m^2\ln(\frac{m^2}{4}Y)}{Y^2}+c\frac{m^2}{Y^2}+R\\
a\equiv \frac{1}{(4\pi^2)^3}\quad & b\equiv\frac{3}{16\pi^2(4\pi^2)^2}\quad c\equiv\frac{3(2\gamma-1)}{16\pi^2(4\pi^2)^2}.
\end{aligned}
\tag{10.38}
$$

Remark:

There is only a small technical problem with the formula above; it does not fulfill the Sm-expansion axiom. We can fix this by artificially introducing a mass scale $M>0$. We write:

$$
\ln(\frac{m^2}{4}Y)=\ln(\frac{m^2}{4}Y)+\ln(\frac{M^2}{4}Y)-\ln(\frac{M^2}{4}Y)=\ln(\frac{M^2}{4}Y)+2\ln\left(\frac{m}{M}\right).
\tag{10.39}
$$

Leading to:

$$
(\Delta^F(y))^3=a\frac{1}{Y^3}+\frac{m^2(b\ln(\frac{M^2}{4}Y)+c)}{Y^2}+2b\frac{m^2}{Y^2}\ln\left(\frac{m}{M}\right).
\tag{10.40}
$$

Remark: The “right” way to do the calculation is to star from the Sm-expansion and derive the formula above, not the other way around. We decide to take the opposite direction to keep the idea “as intuitive as possible”.

The term $\frac{1}{Y^3}$ in $d=4$. We fix $j=0$ and $k=2$ in the formula (10.20) which results in:

$$
\partial^2\frac{1}{y^2}=(4b(b+1)-2bd)\frac{1}{y^3}=\frac{8}{y^3}\Rightarrow\frac{1}{-(y^2-i0)^3}=\frac{-1}{8}\partial^2\frac{1}{Y^2}.
\tag{10.41}
$$

The term $\frac{1}{Y^2}$ can be written as

$$
\frac{1}{Y^2}=\frac{1}{4}\partial^2\frac{\ln(M^2Y)}{Y}.
\tag{10.42}
$$

Thus:

$$\frac{1}{Y^3} = -\frac{1}{8}\partial^2\frac{1}{Y^2} = -\frac{1}{32}\partial^2\partial^2\frac{\ln(M^2Y)}{Y}. \tag{10.43}$$

The term above can be expanded to $y = 0$ using integration by parts.

To tackle the term $\frac{\ln(M^2Y)}{Y^2}$, the simplest choice we can make in (10.20) is $j = 2$ and $k = 1$, leading to:

$$\partial^2\frac{\ln^2(M^2y^2)}{y^2} = -8\frac{\ln(M^2y^2)}{(y^2)^{1+1}} + 8\frac{1}{(y^2)^{1+1}} \Rightarrow \frac{\ln(M^2y^2)}{(y^2)^2} = \frac{1}{(y^2)^2} - \frac{1}{8}\partial^2\frac{\ln^2(M^2y^2)}{y^2}$$
$$\Rightarrow \frac{\ln(M^2Y)}{Y^2} = \frac{1}{4}\partial^2\frac{\ln(M^2Y)}{Y} + \frac{1}{8}\partial^2\frac{\ln^2(M^2Y)}{Y}. \tag{10.44}$$

Since

$$\mathrm{sd}(\frac{\ln(M^2Y)}{Y}) = \mathrm{sd}(\frac{\ln^2(M^2Y)}{Y}) = 2 < 4. \tag{10.45}$$

The extension to $y = 0$ is immediate.
Hence, we can write:

$$\begin{aligned}(\Delta^F(y))^3 &= a\frac{1}{Y^3} + \frac{m^2(b\ln(\frac{M^2}{4}Y) + c)}{Y^2} + 2b\frac{m^2}{Y^2}\ln\left(\frac{m}{M}\right)\\ &= -\frac{a}{32}\partial^2\partial^2\frac{\ln(M^2Y)}{Y} + \frac{bm^2}{4}\partial^2\frac{\ln(M^2Y)}{Y} + \frac{bm^2}{8}\partial^2\frac{\ln^2(M^2Y)}{Y}\\ &- \frac{bm^2}{4}\ln(4)\partial^2\frac{\ln(M^2Y)}{Y} + \frac{cm^2}{4}\partial^2\frac{\ln(M^2Y)}{Y}\\ &+ \frac{bm^2}{2}\ln\left(\frac{m}{M}\right)\partial^2\frac{\ln(M^2Y)}{Y} + R.\end{aligned} \tag{10.46}$$

Thus:

$$\begin{aligned}&\left(\rho^2\Delta^F_{\frac{m}{\rho}}(\rho y)\right)^3 - \left(\Delta^F_m(y)\right)^3 = -\frac{a}{16}\ln(\rho)\partial^2\partial^2\frac{1}{Y}\\ &+ \frac{bm^2}{2}\ln(\rho)\partial^2\frac{1}{Y} + \frac{bm^2}{2}\ln^2(\rho)\partial^2\frac{1}{Y} + \frac{bm^2}{2}\ln(\rho)\partial^2\frac{\ln(M^2Y)}{Y}\\ &- \frac{bm^2}{2}\ln(4)\ln(\rho)\partial^2\frac{1}{Y} + \frac{cm^2}{2}\ln(\rho)\partial^2\frac{1}{Y} - bm^2\ln^2(\rho)\partial^2\frac{1}{Y}\\ &- \frac{bm^2}{2}\ln(\rho)\partial^2\frac{\ln(M^2Y)}{Y} + bm^2\ln(\rho)\ln\left(\frac{m}{M}\right)\partial^2\frac{1}{Y}\\ &= -\frac{a}{16}\ln(\rho)\partial^2\partial^2\frac{1}{Y} + \frac{bm^2}{2}\ln(\rho)\partial^2\frac{1}{Y} - \frac{bm^2}{2}\ln^2(\rho)\partial^2\frac{1}{Y}\\ &- \frac{bm^2}{2}\ln(4)\ln(\rho)\partial^2\frac{1}{Y} + \frac{cm^2}{2}\ln(\rho)\partial^2\frac{1}{Y} + bm^2\ln(\rho)\ln\left(\frac{m}{M}\right)\partial^2\frac{1}{Y}.\end{aligned} \tag{10.47}$$

Note: the non-local terms $\sim \partial^2 \frac{\ln(M^2 Y)}{Y}$ cancel in the above equation.

Now we use $\partial^2 \frac{1}{Y} = -4\pi^2 i\delta(y)$ to write the equation above as:

$$\begin{aligned}
&\frac{a\pi^2 i}{4}\ln(\rho)\partial^2\delta(y) - 2\pi^2 ibm^2\ln(\rho)\delta(y) + 2\pi^2 ibm^2\ln^2(\rho)\delta(y)\\
&+ 2\pi^2 ibm^2\ln(4)\ln(\rho)\delta(y) - 2\pi^2 icm^2\ln(\rho)\delta(y)\\
&- 4\pi^2 ibm^2\ln(\rho)\ln\left(\frac{m}{M}\right)\delta(y)\\
&\equiv \frac{a\pi^2 i}{4}\ln(\rho)\partial^2\delta(y) + \frac{6\tilde{B}}{i\lambda^2\hbar^2}\delta(y).
\end{aligned} \tag{10.48}$$

Remark: The complete expression includes a constant C_1 page 196 [24] due to the most general expansion to the thin diagonal $t = t^0 + C_1\delta(y)$. For simplicity, we set the constant to 0 in our calculations.

Where:

$$\begin{aligned}
\frac{6\tilde{B}}{i\lambda^2\hbar^2} &= -2\pi^2 ibm^2\ln(\rho) + 2\pi^2 ibm^2\ln^2(\rho)\\
&+ 2\pi^2 ibm^2\ln(4)\ln(\rho) - 2\pi^2 icm^2\ln(\rho) - 4\pi^2 ibm^2\ln(\rho)\ln\left(\frac{m}{M}\right).
\end{aligned} \tag{10.49}$$

The constants multiplying $\tilde{B}$ will become clear in a second. To finish, we compute

$$\begin{aligned}
&\frac{i}{\hbar}\frac{\lambda^2}{(4!)^2}\int dx_1 dx_2\left(96\hbar^3\left(\rho^2\Delta^F_{\frac{m}{\rho}}(\rho y)\right)^3 - \left(\Delta^F_m(y)\right)^3\right)\phi(x_1)\phi(x_2)\\
&= \frac{i\lambda^2\hbar^2}{6}\int dx_1 dx_2\left(\frac{a\pi^2 i}{4}\ln(\rho)\partial^2\delta(y) + \frac{6\tilde{B}}{i\lambda^2\hbar^2}\delta(y)\right)\phi(x_1)\phi(x_2)\\
&= \frac{i\lambda^2\hbar^2}{6}\int dx_1\frac{a\pi^2 i\ln(\rho)}{2}(\phi(x_1)\partial^2\phi(x_1) - \partial^\mu\phi(x_1)\partial_\mu\phi(x_1)) + \frac{6\tilde{B}}{i\lambda^2\hbar^2}\phi^2(x_1)\\
&\equiv \int dx_1\tilde{A}(\phi(x_1)\partial^2\phi(x_1) - \partial^\mu\phi(x_1)\partial_\mu\phi(x_1)) + \tilde{B}\phi^2(x_1).
\end{aligned} \tag{10.50}$$

In the last equation we have used:

$$\begin{aligned}
&\int dx_1 dx_2\partial_y^2\delta(y)\phi(x_1)\phi(x_2) = \int dx_1 dx_2\delta(y)(\partial_{x_1} - \partial_{x_2})^2\phi(x_1)\phi(x_2)\\
&= \int dx_1 dx_2\,\delta(y)(\phi(x_1)\partial^2_{x_2}\phi(x_2) + (\partial^2_{x_1}\phi(x_1))\phi(x_2) - 2\phi^\mu_{x_1}\phi(x_1)\partial_{\mu,x_2}\phi(x_2)\\
&= 2\int dx_1((\partial^2\phi)\phi - \partial^\mu\phi\partial_\mu\phi)(x_1).
\end{aligned} \tag{10.51}$$

and

$$\tilde{A} = \frac{i\lambda^2\hbar^2}{6}\frac{a\pi^2 i\ln(\rho)}{2} = -\frac{\lambda^2\hbar^2\ln(\rho)}{48(2\pi)^4}. \tag{10.52}$$

The last term we have to renormalize is $(\Delta^F(y))^4$. Note that is not indeed necessary. To see it, let us consider the general form of the last term:

$$\int dx_1 dx_2\, (\rho^2\Delta^F_{\frac{m}{\rho}}(\rho y))^4 - (\Delta^F_m(y))^4 = \int dx_1 dx_2\, C\delta(y) + \sum_{n>0}^{N} C_n(\partial^2)^n\delta(y). \tag{10.53}$$

Where C, C_n are functions of ρ, m, M. The last term do not contribute, since:

$$\int dx_1 dx_2\, (\partial^2)^n\delta(y) + \int dx_1 dx_2\, (\partial^2)^n\delta(y)\cdot 1 = (-1)^{2n}\int dx_1 dx_2\, \delta(y)(\partial^2)^n 1 = 0. \tag{10.54}$$

The first term contributes with a constant term in the interaction lagrangian and can be ignored.

Therefore, what we obtained is:

$$\begin{aligned} Z^2(L_{int}, L_{int}) &= \int dx_1\, \tilde{A}(\phi(x_1)\partial^2\phi(x_1) - \partial_\mu\phi(x_1)\partial^\mu\phi(x_1)) + \tilde{B}\phi^2(x_1) + \tilde{C}\phi^4(x_1) \\ &= \int dx_1\, - 2\tilde{A}\partial^\mu\phi(x_1)\partial_\mu\phi(x_1) + \tilde{B}\phi^2(x_1) + \frac{1}{4!}\tilde{C}\phi^4(x_1). \end{aligned} \tag{10.55}$$

Note that the result has the same formula as the original lagrangian! Actually, we can define a new field $\phi_\rho(x) := f(\rho)\phi(x)$, a new mass m_ρ and a new coupling constant λ_ρ such that $L_0 - Z(L_{int})$ is the new Lagrangian. Explicitly:

$$\begin{aligned} &L_0 + Z(L_{int}) = L_0 + Z_1(L_{int}) + \frac{1}{2}Z_2(L_{int}, L_{int}) \\ =&L_0 + L_{int} + \frac{1}{2}Z_2(L_{int}, L_{int}) \overset{!}{=} \frac{1}{2}(\partial_\mu\phi_\rho\partial^\mu\phi_\rho - m_\rho^2\phi_\rho^2) - \frac{\lambda_\rho}{4!}\phi_\rho^4. \end{aligned} \tag{10.56}$$

Using:

$$\begin{aligned} \tilde{A} &= -\frac{\lambda^2\hbar^2}{96(2\pi)^4}\ln(\rho) \\ \tilde{B} &= \frac{\lambda^2\hbar^2 m^2\ln(\rho)}{8(2\pi)^4}\left(\gamma - \ln(2) - \frac{\ln(\rho)}{2} + \ln\left(\frac{m}{M}\right)\right) \\ \tilde{C} &= \frac{3\hbar\lambda^2}{8\pi^2}\ln(\rho). \end{aligned} \tag{10.57}$$

Thus:

- Wave function renormalization

$$\frac{1}{2}(1-2\tilde{A})\partial^\mu\phi\partial_\mu\phi \overset{!}{=} \frac{1}{2}\partial^\mu\phi_\rho\partial_\mu\phi_\rho = f^2(\rho)\frac{1}{2}\partial^\mu\phi\partial_\mu\phi$$
$$\Rightarrow f(\rho) = \sqrt{1-2\tilde{A}} = \sqrt{1+\frac{\lambda^2\hbar^2\ln\rho}{48(2\pi)^4}}. \tag{10.58}$$

- Mass renormalization:

$$\frac{1}{2}(m^2+\tilde{B})\phi^2 \overset{!}{=} \frac{1}{2}m_\rho^2\phi_\rho^2 = \frac{1}{2}m_\rho^2 f^2(\rho)\phi^2$$
$$\Rightarrow m_\rho = \frac{\sqrt{m^2+\tilde{B}}}{f(\rho)} = m\sqrt{\frac{1+\frac{\lambda^2\hbar^2\ln(\rho)}{8(2\pi)^4}\left(\gamma-\ln(2)-\frac{\ln(\rho)}{2}+\ln\left(\frac{m}{M}\right)\right)}{1+\frac{\lambda^2\hbar^2\ln\rho}{48(2\pi)^4}}}. \tag{10.59}$$

- Coupling constant renormalization

$$-\frac{1}{4!}(\lambda+\frac{\tilde{C}}{2})\phi^4 \overset{!}{=} -\frac{\lambda_\rho}{4!}\phi_\rho^4 = -\frac{\lambda_\rho}{4!}f^4(\rho)\phi^4$$
$$\Rightarrow \lambda_\rho = \frac{\lambda+\frac{\tilde{C}}{2}}{f^4(\rho)} = \lambda\frac{1+\frac{3\hbar\lambda\ln(\rho)}{16\pi^2}}{\left(1+\frac{\lambda^2\hbar^2\ln\rho}{48(2\pi)^4}\right)^2}. \tag{10.60}$$

Chapter 11

Miscellaneous of theories of physical interest

11.1 Introduction

This chapter is devoted to introduce a series of fields one usually works in physics. We introduce complex scalar fields, (abelian) gauge fields and fermion fields. We will discuss briefly the star product for these fields and an example of interaction to deduce the Feynman rules (in momentum space) for the theory. For now, we do not mix fields with different spins. We will do it in the next chapter, which discusses scalar and "usual" quantum electrodynamics (QED)

11.2 Complex scalar field

The complex scalar field $\phi^*(x)$ is very similar to the real scalar field field $\phi(x)$. It is also defined as a functional over $\mathbb{C}$ given by:

$$\begin{aligned} \phi^*(x) :& C^\infty(\mathbb{M}) \to \mathbb{C} \\ & h(x) \mapsto \overline{h}(x). \end{aligned} \tag{11.1}$$

Where $\overline{h}(x)$ indicates the complex conjugate. We can define the configuration space for a theory containing $\phi(x), \phi^*(x)$ as $\mathcal{C} := C^\infty(\mathbb{M})$. If we want to emphasize that the fields are independent, we can also define $\mathcal{C} := C^\infty(\mathbb{M}) \oplus C^\infty(\mathbb{M})$.

The space of fields is the set:

$$\mathcal{F} := \left\{ \sum_{n=0}^{N} \sum_{m=0}^{M} \int dX_n dY_m \, f_{n,m}(x_1, ...x_n, y_1, ..., y_m)\phi(x_1)...\phi(x_n)\phi^*(y_1)...\phi^*(y_m) \right\}. \tag{11.2}$$

The term corresponding to $n = m = 0$ is denoted by $f_0 \in \mathbb{C}$.

The functional derivative is also the same as for real scalar fields:

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \frac{\delta\phi^*(x)}{\delta\phi^*(y)} := \delta(x-y)$$
$$\frac{\delta\phi(x)}{\delta\phi^*(y)} = \frac{\delta\phi^*(x)}{\delta\phi(y)} := 0. \tag{11.3}$$

The free action for the complex scalar field is given by:

$$S_0 := \int dx\, \partial_\mu\phi(x)\partial^\mu\phi^*(x) - m^2\phi(x)\phi^*(x) \equiv \int dx\, \partial_\mu\phi\partial^\mu\phi^* - m^2|\phi|^2. \tag{11.4}$$

The equations of motion follows immediately:

$$\frac{\delta S_0}{\delta\phi(x)} = -\partial_\mu\partial^\mu\phi^*(x) - m^2\phi^*(x) = -(\partial^2+m^2)\phi^*(x)$$
$$\frac{\delta S_0}{\delta\phi^*(x)} = -\partial_\mu\partial^\mu\phi(x) - m^2\phi(x) = -(\partial^2+m^2)\phi(x). \tag{11.5}$$

The Poisson structure is "twice" (one for the scalar field, one for it's conjugate) the one of the real scalar field:

$$\{F,G\} = \frac{\delta F}{\delta\phi(x)}\Delta(x-y)\frac{\delta G}{\delta\phi^*(y)} + \frac{\delta F}{\delta\phi^*(x)}\Delta(x-y)\frac{\delta G}{\delta\phi(y)}. \tag{11.6}$$

We can check that we recover the usual commutation relation of classical fields computing the Poisson bracket of:

$$F = \phi(x), \quad G = \pi(x) := \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = \partial_{x^0}\phi^*(x)$$
$$F = \phi^*(x), \quad G = \pi^*(x) := \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*(x)} = \partial_{x^0}\phi(x). \tag{11.7}$$

The calculation is the same as for the real field (5.34).

The quantization procedure is also "twice" the one for a real scalar field:

$$
\begin{aligned}
F \star G :&= FG + \hbar \left(\int dx dy \, \frac{\delta F}{\delta \phi(x)} \Delta^+(x-y) \frac{\delta G}{\delta \phi^*(y)} + \frac{\delta F}{\delta \phi^*(x)} \Delta^+(x-y) \frac{\delta G}{\delta \phi(y)} \right) \\
&+ \frac{\hbar^2}{2} \left(\int dX_2 dY_2 \, \frac{\delta F}{\delta \phi(x_1) \delta \phi(x_2)} \Delta^+(x_1-y_1) \Delta^+(x_2-y_2) \frac{\delta G}{\delta \phi^*(y_1) \delta \phi^*(y_2)} \right. \\
&+ 2 \frac{\delta F}{\delta \phi(x_1) \delta \phi^*(x_2)} \Delta^+(x_1-y_1) \Delta^+(x_2-y_2) \frac{\delta G}{\delta \phi^*(y_1) \delta \phi(y_2)} \\
&\left. + \frac{\delta F}{\delta \phi^*(x_1) \delta \phi^*(x_2)} \Delta^+(x_1-y_1) \Delta^+(x_2-y_2) \frac{\delta G}{\delta \phi(y_1) \delta \phi(y_2)} \right) + ... \qquad (11.8)
\end{aligned}
$$

We can write the above formula in compact notation[59]:

$$
\begin{aligned}
F \star G := \sum_{n,m=0}^{\infty} \frac{\hbar^{n+m}}{n!m!} \int & dX_n dY_m d\tilde{X}_n d\tilde{Y}_m \\
& \frac{\delta^{n+m} F}{\delta \phi(x_1) ... \delta \phi(x_n) \delta \phi^*(y_1) ... \delta \phi^*(y_m)} \\
& \prod_{i=1}^{n} \prod_{k=1}^{m} \Delta^+(x_i - \tilde{x}_i) \Delta^+(y_i - \tilde{y}_i) \\
& \frac{\delta^{n+m} G}{\delta \phi^*(\tilde{x}_1) ... \delta \phi^*(\tilde{x}_n) \delta \phi(\tilde{y}_1) ... \delta \phi(\tilde{y}_m)}. \qquad (11.9)
\end{aligned}
$$

The construction of the T product is basically the same.

11.2.1 Example: Scattering of 2 particles $\to$ 2 particles consider $L_{int} = -\frac{\lambda}{4!} \int dx \, |\phi(x)|^4$

We compute the amplitude of two different processes $\phi\phi \to \phi^*\phi^*$ and $\phi\phi \to \phi\phi$ in the first order of perturbation theory in the momentum space. Since our theory is massive, we already took the adiabatic limit. The proof that the limit is well defined can be found in [23].

$\phi\phi \to \phi^*\phi^*$

The $T-$ product in the first order for this process is given by:

$$
T_1 = \frac{i}{\hbar} L_{int} = -\frac{i\lambda}{4!\hbar} \int dx \, (\phi(x)\phi^*(x))^2. \qquad (11.10)
$$

The amplitude is given by:

$$
\begin{aligned}
\mathcal{T}_1 &= \omega_0(\overline{\phi}(p_1)\overline{\phi}(p_2) \star T_1 \star \phi(p_3)\phi(p_4)) \\
&= -\frac{i\lambda}{4!\hbar} \int dx \, \omega_0(\overline{\phi}(p_1)\overline{\phi}(p_2) \star \phi^2(x)(\phi^*)^2(x) \star \phi^*(p_3)\phi^*(p_4)). \qquad (11.11)
\end{aligned}
$$

Where we have used the short notation:

$$\overline{\phi}(p_{1,2}) := \frac{2\omega_{p_{1,2}}}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{x}_{1,2} e^{ip_{1,2}x_{1,2}} \phi^*(x_{1,2})$$
$$\phi^*(p_{3,4}) := \frac{2\omega_{p_{3,4}}}{(2\pi)^{\frac{d-1}{2}}} \int d\vec{x}_{3,4} e^{-ip_{3,4}x_{3,4}} \phi^*(x_{3,4}). \tag{11.12}$$

We want to emphasize that the fact that the field is now complex does not change the sign on the exponential. The sign is a consequence of the interpretation of the creator operator acting on the vacuum, not on the field itself!

Essentially, we need to compute

$$\omega_0(\phi^*(x_1)\phi^*(x_2) \star \phi^2(x)(\phi^*(x))^2 \star \phi^*(x_3)\phi^*(x_4)). \tag{11.13}$$

However, we do not need to go through all the calculations to obtain the result. Since the star product only acts on a pair $\phi\phi^*$ and we have 6 ϕ^*'s and 2ϕ's, the final result is necessarily proportional to $(\phi^*)^4$. Hence,

$$\mathcal{T}_1 = 0. \tag{11.14}$$

The above result is one facet of charge conservation. Roughly speaking, we can attribute the field ϕ to a charge $+1$ and ϕ^* to a charge -1. Every process preserves the charge, that is, we have the same number of ϕ^*'s and ϕ's before and after scattering. A better explanation and further discussion can be found in [59].

$\phi\phi \to \phi\phi$

The tree-level amplitude is given by:

$$\mathcal{T}_1 = -i\frac{\lambda}{4!\hbar} \int dx\, \omega_0(\overline{\phi}(p_1)\overline{\phi}(p_2) \star \phi^2(x)(\phi^*)^2(x) \star \phi(p_3)\phi(p_4)). \tag{11.15}$$

We can write the star product as:

$$(\phi^*(x_1)\phi^*(x_2) \star \phi^2(x)((\phi^*(x))^2 \star \phi(x_3)\phi(x_4))$$
$$\overset{|\phi|=0}{=} \hbar^4 \Delta^+(x_1 - x)\Delta^+(x_2 - x)\Delta^+(x - x_3)\Delta^+(x - x_4). \tag{11.16}$$

The notation $\overset{|\phi|=0}{=}$ indicates we are only considering the terms that does not contain any powers of ϕ. Hence, the amplitude is:

$$\begin{aligned}\mathcal{T}_1 &= \frac{-i\lambda}{4!\hbar}\hbar^4 \left(\prod_{i=1}^{4} \frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right) \int d\vec{X}_4 dx\, e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4} \\ &\times \Delta^+(x_1-x)\Delta^+(x_2-x)\Delta^+(x-x_3)\Delta^+(x-x_4) \\ &= -\frac{i\lambda\hbar^3}{4!(2\pi)^{d-2}}\delta(p_1+p_2-p_3-p_4). \end{aligned} \tag{11.17}$$

11.3 (Abelian) Gauge fields

The next field we have to worry about is the spin-1 field called the "gauge field" or (specifically in our context) photon field $A^\mu(x)$:

$$A(x) : \begin{cases} \mathcal{C}_{\text{photon}} := C^\infty(\mathbb{M}, \mathbb{R}^d) \to \mathbb{R}^d \\ h \equiv (h^\mu)_{\mu=0,1\ldots,d-1} \mapsto A(x)(h) = h(x) \end{cases}. \tag{11.18}$$

The set of fields $\mathcal{F}_{\text{photon}}$ is defined as the set of functionals $F : \mathcal{C}_{\text{photon}} \to \mathbb{C}$ of the form:

$$F = \sum_{r=0}^{R} \int dY_r f_r^{\mu_1\ldots\mu_r}(y_1,\ldots,y_r)A_{\mu_1}(y_1)\ldots A_{\mu_r}(y_r) \equiv \langle f_r^{\mu_1\ldots\mu_r}, A_{\mu_1}\ldots A_{\mu_r}\rangle. \tag{11.19}$$

Where $R<\infty$, $f_0 \in \mathbb{C}$ and $f_r^{\mu_1\ldots\mu_r} \in \mathcal{F}'(\mathbb{M})$ is defined analogously to the scalar field (5.2.1). The $*$ operation is introduced as

$$F^* := \left\langle \overline{f_r^{\mu_1\ldots\mu_r}}, A_{\mu_1}\ldots A_{\mu_r}\right\rangle, \quad A^*(x) = A(x). \tag{11.20}$$

It transforms under action of $(\Lambda, a) \in \mathcal{P}_+^\uparrow$ as:

$$\beta_{\Lambda,a}A^\mu(x) := (\Lambda^{-1})^\mu_\nu A^\nu(\Lambda x + a). \tag{11.21}$$

We define the field strength tensor $F_{\mu\nu}$ as usual:

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{11.22}$$

And the free action as:

$$S_0 := -\frac{1}{4}\int dx\, F^{\mu\nu}(x)F_{\mu\nu}(x) = \frac{1}{2}\int dx\, A_\mu(x)D^{\mu\nu}A_\nu(x). \tag{11.23}$$

Where $D^{\mu\nu} := \eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu$. The functional derivative is done component by component:

$$\frac{\delta A_\nu(x)}{\delta A_\mu(y)} = \delta_{\mu\nu}\delta(x-y). \tag{11.24}$$

Hence, the free field equation is simply:

$$\frac{\delta S_0}{\delta A_\mu} = D^{\mu\nu}A_\nu(x) \stackrel{!}{=} 0. \tag{11.25}$$

As usual, the retarded field equation reads:

$$\begin{aligned}&\left(\frac{\delta S_0 + S}{\delta A_\mu}\right) \circ r_{S_0+S,S_0} \stackrel{!}{=} \frac{\delta S_0}{\delta A_\mu}\\ &\Rightarrow D^{\mu\nu}A_\nu^{\text{ret}}(x) = D^{\mu\nu}A_\nu(x) - \left(\frac{\delta S}{\delta A_\mu}(x)\right)^{\text{ret}}.\end{aligned} \tag{11.26}$$

The next step would be to invert $D^{\mu\nu}$. The problem is that $D^{\mu\nu}$ is not invertible. That is a direct consequence of the gauge invariance of the action. To avoid these problem, we introduce a gauge-fixing term:

$$S_0^{\text{gf}} := \frac{-\lambda}{2}\int dx(\partial_\mu A^\mu(x))^2. \tag{11.27}$$

Leading to a different differential operator $D_\lambda^{\mu\nu}$:

$$D_\lambda^{\mu\nu} = \eta^{\mu\nu}\partial^2 - (1-\lambda)\partial^\mu\partial^\nu. \tag{11.28}$$

Next we chose $\lambda = 1$, known as the "Feynman Gauge" leading to the massless Klein-Gordon field equation:

$$D_{\lambda=1}^{\mu\nu}A_\nu = \partial^2 A^\mu(x) \stackrel{!}{=} 0. \tag{11.29}$$

Hence, we have really 4 copies of scalar fields.

Last but not least, the star product is defined through:

$$F \star G = \sum_{n=0}^{\infty} \frac{(-\hbar)^n}{n!} dX_n dY_n$$
$$\frac{\delta^n F}{\delta A_{\mu_1}(x_1)...\delta A_{\mu_n}(x_n)} \prod_{l=1}^{n} D^+(x_l - y_l) \frac{\delta^n G}{\delta A^{\mu_1}(y_1)...\delta A^{\mu_n}(y_n)}. \tag{11.30}$$

As an example, let us compute $A^\mu(x) \star A^\nu(y)$:

$$A^\mu(x) \star A^\nu(y) = A^\mu(x)A^\nu(y) - \hbar \int d\tilde{x} d\tilde{y} \frac{\delta A^\mu(x)}{\delta A_\alpha(\tilde{x})} D^+(\tilde{x} - \tilde{y}) \frac{\delta A^\nu(y)}{\delta A^\alpha(\tilde{y})}. \tag{11.31}$$

To compute the derivative in the last line, we have to match the indices. This is done using the metric:

$$\frac{\delta A^\mu(x)}{\delta A_\alpha(\tilde{x})} = \eta^{\mu\rho} \frac{\delta A_\rho(x)}{A_\alpha(\tilde{x})} = \eta^{\mu\rho} \delta_{\alpha\rho} \delta(x - \tilde{x}) = \eta^{\mu\alpha} \delta(x - \tilde{x}). \tag{11.32}$$

Hence, the star product is simply:

$$A^\mu(x) \star A^\nu(y) = A^\mu(x)A^\nu(y) - \hbar \eta^{\mu\nu} D^+(x - y). \tag{11.33}$$

Remark: to do the quantization properly we also need ghost fields. Since they decouple in the examples we work in this dissertation, we will skip this discussion.

11.3.1 Free photons

If we naively quantize the photon field as

$$A^\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d\vec{p}}{2\omega_p} a^\mu(\vec{p}) e^{-ikx} + (a^\mu(\vec{p}))^* e^{ikx}. \tag{11.34}$$

We are, basically, quantizing 4 different scalar fields, one for each component of the photon field. However, the photon has only two degrees of freedom, corresponding to two different polarizations. To solve the problem, we have to separate the Fock space of particles in the physical Fock space (where only transverse polarized photons are allowed) and the nonphysical Fock space (containing longitudinal and scalar photons). The entire discussion is presented in [67] chapter 2.11. For our porpoises, it suffices to state that a physical photon with wave function $\chi(p)$ is created from the vacuum by:

$$a^*(\chi(\vec{p}, -t)) = \int \frac{d\vec{p}}{2\omega_p} \chi(\vec{p}) \epsilon_\mu(\vec{p}) a^*_\mu(\vec{p}) e^{ikx} \Big|_{p^0 = \omega_p}. \tag{11.35}$$

where $\epsilon_\mu(\vec{k})$ is called the polarization vector:

$$\epsilon_\mu(\vec{p}) := (0, \vec{\epsilon}) \quad \vec{p} \cdot \vec{\epsilon}(\vec{p}) \quad (\vec{\epsilon})^2 = 1. \tag{11.36}$$

With this choice of polarization vector, we guarantee that only physical photons are allowed. Note that the indices of the photon and the creation operator are "down", i.e,they do not represent a Lorenz contraction. Hence, external photons needed a different Feynman rule in our formalism:

- Incoming photons with momentum $\vec{p}$ and polarization $\epsilon_\mu(\vec{p})$ contributes with

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d\vec{p}}{2\omega_p} \epsilon^*_\mu(\vec{p}) A_\mu(\vec{p}) e^{ipx}. \tag{11.37}$$

- Outgoing photons with momentum $\vec{p}$ and polarization $\epsilon_\mu(\vec{p})$ contributes with

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d\vec{p}}{2\omega_p} \epsilon_\mu(\vec{p}) A_\mu(\vec{p}) e^{-ipx}. \tag{11.38}$$

11.4 Fermion Fields

This section is a simplified exposition of [24] Chapter 5.Just as in the reference, we will do the construction for fermion fields in only 3+1 dimensions.

In this chapter, we are going to mimic the construction of the scalar field. As introduced in the text of the scalar field, a single spinor field is characterized as a function $C^\infty(\mathbb{M}, \mathbb{C}^4) \to \mathbb{C}^4$:

$$\psi(x) : \begin{cases} C^\infty(\mathbb{M}, \mathbb{C}^4) \to \mathbb{C}^4 \\ h \equiv (h_k)_{k=1,\ldots,4} \mapsto \psi(x)(h) = h \end{cases} . \tag{11.39}$$

The field ψ is restricted to the Fermi statistic and transforms according to $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ under the action of $\mathcal{L}^\uparrow_+$. Associated with the field ψ we define the "complex ψ field" denoted by $\overline{\psi} := \psi^\dagger \gamma^0$:

$$\overline{\psi}(x) : \begin{cases} C^\infty(\mathbb{M}, \mathbb{C}^4) \to \mathbb{C}^{1\times 4} \\ h \equiv (h_k)_{k=1,\ldots,4} \mapsto \overline{\psi}(x)(h) = h^\dagger(x)\gamma^0 = \sum_{k=1}^4 h_k^c(x)\gamma^0_{kj} \end{cases} . \tag{11.40}$$

where γ^0 is the usual gamma matrix of the Dirac theory and the superscript c denotes the complex conjugation.

The product of fields **IS NOT** defined as the "usual QFT of textbooks". In these references, the product of the fields is defined as the product of two vectors in $\mathbb{C}^4 \to \mathbb{C}$:

$$(\overline{\psi}(y)\psi(x))(h) := \sum_{k,l=1}^{4} h_k^c(y)\gamma_{kl}^0 h_l(x) \in \mathbb{C}. \tag{11.41}$$

In our formalism, we define the product of two fields using the "wedge" product:

$$\overline{\psi}(y) \wedge \psi(x) \equiv 2!\overline{\psi}(y) \otimes_{as} \psi(x) : C(\mathbb{M}, \mathbb{C}^2)^{\times 2} \to \mathbb{C}^{16} \tag{11.42}$$

$$(h_1, h_2) \mapsto (h_1^\dagger(y)\gamma^0) \otimes h_2(x) - (h_2^\dagger(y)\gamma^0) \otimes h_1(x). \tag{11.43}$$

where $\otimes_{as}$ stands for antisymmetric product tensor. Note that by using this definition, we also have products of the form:

$$\psi(x) \wedge \psi(y), \overline{\psi}(x) \wedge \overline{\psi}(y) : C(\mathbb{M}, \mathbb{C}^2)^{\times 2} \to \mathbb{C}^{16}. \tag{11.44}$$

The usual objects of the theory involving products of $\overline{\psi}\psi$, for example, the current $j^\mu := \overline{\psi}(x) \wedge \gamma^\mu \psi(x) := \sum_{k,l=1}^{4} \gamma_{kl}^\mu \overline{\psi}_l(x) \wedge \psi_k(x)$, are no longer numbers but vectors in $\mathbb{C}^{4^n}$, $n \equiv$ number of fields. With this definition, we guarantee the anti-symmetry of fermion fields. For now on, we adopt the Einstein summation convention, that is, repeated indices must be summed. For example, the current $j^\mu = \gamma_{kl}^\mu \overline{\psi}_l(x) \wedge \psi_k(x)$.

The next step of our construction is the definition of the (functional) derivative. We define it as usual:

$$\begin{aligned}
&\frac{\delta}{\delta\overline{\psi}_r(y)}\overline{\psi}_{k_1}(x_1) \wedge ... \wedge \overline{\psi}_{k_l}(x_l) \wedge \psi_{k_{l+1}}(x_{l+1}) \wedge ... \wedge \psi_{k_n}(x_n)\\
&:= \sum_{j=1}^{l}(-1)^{j-1}\delta_{kr}\delta(x_j - y)\\
&\times \overline{\psi}_{k_1}(x_1) \wedge ... \wedge \widehat{\overline{\psi}_{k_j}(x_j)} \wedge \overline{\psi}_{k_l}(x_l) \wedge \psi_{k_{l+1}}(x_{l+1}) \wedge ... \wedge \psi_{k_n}(x_n)\\
&\frac{\delta}{\delta\psi_r(y)}\overline{\psi}_{k_1}(x_1) \wedge ... \wedge \overline{\psi}_{k_l}(x_l) \wedge \psi_{k_{l+1}}(x_{l+1}) \wedge ... \wedge \psi_{k_n}(x_n)\\
&:= \sum_{j=l+1}^{l}(-1)^{j-1}\delta_{kr}\delta(x_j - y)\\
&\overline{\psi}_{k_1}(x_1) \wedge ... \wedge \overline{\psi}_{k_l}(x_l) \wedge \psi_{k_{l+1}}(x_{l+1}) \wedge ... \wedge \widehat{\psi_{k_j}(x_j)} \wedge \psi_{k_n}(x_n)\\
&.
\end{aligned} \tag{11.45}$$

We also define the derivative acting on the right:

$$\frac{\delta_r}{\delta\psi_k(y)}\overline{\psi}(x_1)\wedge ... \wedge\psi(x_n) = (-1)^{n-1}\frac{\delta}{\delta\psi_k(y)}\overline{\psi}(x_1)\wedge ... \wedge\psi(x_n). \tag{11.46}$$

In practice, what we do when we need to compute a derivative is to "commute" the derivative with the fields. For example:

$$\begin{aligned}
&\frac{\delta}{\delta\psi_k(x)}(\overline{\psi}_{l_1}(x_1)\wedge\overline{\psi}_{l_2}(x_2)\wedge\psi_{l_3}(x)) = \left(\frac{\delta\overline{\psi}_{l_1}(x_1)}{\delta\psi_k(x)}\right)\wedge\overline{\psi}_{l_2}(x_2)\wedge\psi(x_3)\\
&-\overline{\psi}_{l_1}(x_1)\wedge\left(\frac{\delta\overline{\psi}_{l_2}(x_2)}{\delta\psi_k(x)}\right)\wedge\psi_{l_3}(x_3)+\overline{\psi}_{l_1}(x_1)\wedge\overline{\psi}_{l_2}(x_2)\left(\frac{\delta\psi_{l_3}(x_3)}{\delta\psi_k(x)}\right)\\
&= 0-0+\overline{\psi}_{l_1}(x_1)\wedge\overline{\psi}_{l_2}(x_2)\delta_{kl_3}\delta(x-x_3).
\end{aligned} \tag{11.47}$$

The right derivative is calculate "beginning with the derivative on the right":

$$\begin{aligned}
&\frac{\delta_r}{\delta\psi_k(x)}(\overline{\psi}_{l_1}(x_1)\wedge\overline{\psi}_{l_2}(x_2)\wedge\psi_{l_3}(x)) = \overline{\psi}_{l_1}(x_1)\wedge\overline{\psi}_{l_2}(x_2)\left(\frac{\delta\psi_{l_3}(x_3)}{\delta\psi_k(x)}\right)\\
&-\overline{\psi}_{l_1}(x_1)\wedge\left(\frac{\delta\overline{\psi}_{l_2}(x_2)}{\delta\psi_k(x)}\right)\wedge\psi_{l_3}(x_3)+\left(\frac{\delta\overline{\psi}_{l_1}(x_1)}{\delta\psi_k(x)}\right)\wedge\overline{\psi}_{l_2}(x_2)\wedge\psi(x_3)\\
&= \overline{\psi}_{l_1}(x_1)\wedge\overline{\psi}_{l_2}(x_2)\delta_{kl_3}\delta(x-x_3)-0+0.
\end{aligned} \tag{11.48}$$

We also define the "Spinor field space". We start with the classical configuration space:

$$\mathcal{C}_{\text{spinor}} := \bigoplus_{n=0}^{\infty} C^\infty(\mathcal{M},\mathcal{C}^4)^{\times n}. \tag{11.49}$$

The spinor field space $\mathcal{F}_{\text{spinor}}$ is defined as the set of functionals $F:\mathcal{C}_{\text{spinor}}\to\mathcal{C}$ of the form

$$\begin{aligned}
\mathcal{F}^{\text{spinor}} := \Bigg\{F = f_0 \oplus \bigoplus_{m=0}^{M}\bigoplus_{n=0}^{N}\int dX_n dY_m f^{k_1...k_n,l_1...l_m}(x_1,...,x_n,y_1,...,y_m)&\\
\overline{\psi}_{k_1}(x_1)\wedge ... \wedge\overline{\psi}_{k_n}(x_n)\wedge\psi_{l_1}(y_1)\wedge ... \wedge\psi_{l_m}(y_m)\Bigg\}&.
\end{aligned} \tag{11.50}$$

In the above definition, we need to exclude the term $n=m=0$ (it was already accounted for in f_0).

$$=: \bigoplus_{n=0}^{N}\sum_{l=0}^{n}\left\langle f_{n,l}^{k_1,...,k_n},\, \overline{\psi}_{k_1}\wedge\cdots\wedge\overline{\psi}_{k_l}\wedge\psi_{k_{l+1}}\wedge\cdots\wedge\psi_{k_n}\right\rangle,. \tag{11.51}$$

For $n \geq 1$, each $f_{n,l}^{k_1,\ldots,k_n}$ is a $\mathbb{C}$-valued distribution with compact support, $f_{n,l}^{k_1,\ldots,k_n}(x_1,\ldots,x_n)$ is totally antisymmetric under permutations of $(k_1,x_1),\ldots,(k_l,x_l)$ and permutations of $(k_{l+1},x_{l+1}),\ldots,(k_n,x$ and it fulfills the wave front set condition (5.2.1). We will write $\mathcal{F}'_{spinor}(\mathbb{M}^n)$ for the space of distributions of this kind. [24] page 354.

11.4.1 even-odd grading and the η trick:

Now we introduce some subsets of $\mathcal{F}_{\text{spinor}}$ with the objective of facilitating the construction of the retarded product and the matrix T. Essentially, we divide $\mathcal{F}_{\text{spinor}}$ into two subsets, one respecting the Bose-statistics and the other respecting the Fermi-statistics. This division is called "even-odd grading":

$$\begin{aligned}&\mathcal{F}_{\text{spinor}} = \mathcal{F}^{+}_{\text{spinor}} \oplus \mathcal{F}^{-}_{\text{spinor}} \text{ with}\\ &\mathcal{F}^{+}_{\text{spinor}} := \mathcal{C} \oplus [\{F_n | n \in \mathbb{N}^* \text{is even}\}] \quad \mathcal{F}^{+}_{\text{spinor}} := [\{F_n | n \in \mathbb{N}^* \text{is odd}\}] .\end{aligned} \tag{11.52}$$

where $F_n \in \mathcal{F}_{\text{spinor}}$ is defined above and $[-]$ denotes the linear span. Note that for $F, F_1, F_2 \in \mathcal{F}^{-}_{\text{spinor}}$ and $G, G_1, G_2 \in \mathcal{F}^{+}_{\text{spinor}}$:

$$\begin{cases} F_1 \wedge F_2, G_1 \wedge G_2 \in \mathcal{F}^{+}_{\text{spinor}} \\ F \wedge G, G \wedge F \in \mathcal{F}^{-}_{\text{spinor}} \end{cases} . \tag{11.53}$$

The η trick consists of transforming every field into a Bose field. To do it, we introduce some Grassmann variables $\eta_{i,j}$ respecting:

$$\eta_j \eta_k = -\eta_k \eta_j \quad \forall j, k \in \mathbb{N}. \tag{11.54}$$

Defining the fields

$$\tilde{F}_j := \begin{cases} 1 \otimes F_j, \quad F_j \in \mathcal{F}^{+}_{\text{spinor}} \\ \eta_j \otimes F_j, \quad F_j \in \mathcal{F}^{-}_{\text{spinor}} \end{cases} . \tag{11.55}$$

$\tilde{F}_j$ obeys the Fermi statistics. Defining:

$$(a_1 \otimes F_1)(a_1 \otimes F_1) := (a_1 a_2) \otimes (F_1 F_2) \text{ where } a_j \in \{1, \eta_j\}. \tag{11.56}$$

we can derive all the properties of the R product described for the scalar field for the fermion fields and recover the sign at the end. The idea will become transparent further in the text (see

11.4.4).

11.4.2 *-operation and Poincaré group

The last feature we have to impose on the Dirac spinors is how it transforms under the orthochronus Poincaré group $\mathcal{P}^+$ and the $*-$ operation.

The (linear) action of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ on $\overline{\psi}(x_1) \wedge ... \wedge \overline{\psi}(x_n) \wedge \psi(y_1) \wedge ...\psi(y_n)$ is defined by [67] Chapter 1.2, notation page 356 [24]:

$$\begin{aligned}&\beta_{\Lambda,a}\overline{\psi}(x_1) \wedge ... \wedge \overline{\psi}(x_n) \wedge \psi(y_1) \wedge ...\psi(y_n)\\&:= (\overline{\psi}(\Lambda(A)x_1 + a)S(A)) \wedge ... \wedge (\overline{\psi}(\Lambda(A)x_n + a)S(A))\\&\wedge (S(A^{-1})\psi(\Lambda(A)y_1 + a)) \wedge ... \wedge (S(A^{-1})\psi(\Lambda(A)y_n + a)).\end{aligned} \tag{11.57}$$

We use the following representation:

$$D\left(\frac{1}{2}, 0\right) \oplus D\left(0, \frac{1}{2}\right) : SL(2, \mathbb{C}) \to \mathbb{C}^{4\times4} \quad A \mapsto S(A) := \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^\dagger \end{pmatrix}, . \tag{11.58}$$

and $A \mapsto \Lambda(A)$ is the usual group homomorphism from $SL(2, \mathbb{C})$ onto $\mathcal{L}_\uparrow^+$, see, e.g., [72] [67]. The $*$-operation can also be defined

$$\psi(x)^* := \psi^\dagger(x) = \overline{\psi}(x)\gamma^0 \quad \text{and} \quad \overline{\psi}(x)^* := \left(\psi^\dagger(x)\gamma^0\right)^\dagger = \gamma^0\psi(x), . \tag{11.59}$$

The definition is extended using:

$$(F \wedge G)^* := G^* \wedge F^*.. \tag{11.60}$$

11.4.3 Field equation

The free Lagrangian for spinor fields is

$$S_0^{\text{spinor}} := \int dx\, \overline{\psi}_k(x) \wedge (i\partial_\mu\gamma^\mu_{kj} - m1_{kj})\psi_j(x) \equiv \int dx\, \overline{\psi}(x) \wedge (i\partial\!\!\!/ - m)\psi(x). \tag{11.61}$$

We omit the identity matrix multiplying m and $a\!\!\!/ := a^\mu\gamma_\mu = \gamma^\mu a_\mu$. From the free Lagrangian we read the field equation for both spinors:

$$\frac{\delta S_0}{\delta\psi} = i\partial_\mu\overline{\psi}_k(x)\gamma^\mu_{kj} + m\overline{\psi}(x) \quad \frac{\delta S_0}{\delta\overline{\psi}} = (i\partial\!\!\!/ - m)\psi(x). \tag{11.62}$$

11.4.4 Retarded propagator

Now we can construct the retarded product of the spinor field using the same machinery developed to the scalar field. The retarded field equation reads:

$$\frac{\delta(S+S_0)}{\delta\psi}\circ r_{S_0+S,S_0}=\frac{\delta S_0}{\delta\psi}\Rightarrow\overline{\psi}^{\mathrm{ret}}(x)(i\overleftarrow{\not\partial}+m)=\overline{\psi}(x)(i\overleftarrow{\not\partial}+m)-\left(\frac{\delta S}{\delta\psi}(x)\right)^{\mathrm{ret}} \tag{11.63}$$

$$\frac{\delta(S+S_0)}{\delta\overline{\psi}}\circ r_{S_0+S,S_0}=\frac{\delta S_0}{\delta\overline{\psi}}\Rightarrow(i\not\partial-m)\psi^{\mathrm{ret}}(x)=(i\not\partial-m)\psi(x)-\left(\frac{\delta S}{\delta\overline{\psi}}(x)\right)^{\mathrm{ret}}. \tag{11.64}$$

where $\overleftarrow{\not\partial}$ means the derivative is acting on the function on the left (in that case $\overline{\psi}$).

As mentioned in the case of the scalar field, to solve the retarded field equation, we have to calculate the "inverse" of $(i\not\partial-m)$ and $(i\overleftarrow{\not\partial}+m)$. Since these operators are matrices in spinor space, the inverse of them will also be matrices. They are denoted by $S^{\mathrm{ret}}(x)$ and $S^{\mathrm{adv}}(x)$. $S^{\mathrm{ret}}(x)$ is the "inverse" of $(i\not\partial-m)$ and it is given by:

$$S^{\mathrm{ret}}(z):=(i\not\partial_z+m)\Delta^{\mathrm{ret}}(z)\overset{d=4}{=}\frac{1}{(2\pi)^4}\int d^4p\,\frac{\not p+m}{p^2-m^2+ip^00}e^{-ipx}. \tag{11.65}$$

Using the definitions above and

$$(i\not\partial+m)(i\not\partial-m)=-(\partial^2+m^2). \tag{11.66}$$

It is easy to show that:

$$(i\not\partial-m)S^{\mathrm{ret}}(x)=\delta(x),\quad \operatorname{supp}S^{\mathrm{ret}}\subseteq\overline{V}_+. \tag{11.67}$$

$S^{\mathrm{adv}}(x)$ is the "inverse" of $(i\overleftarrow{\not\partial}+m)$ and is given by:

$$\begin{aligned}S^{\mathrm{adv}}(z)&:=\Delta^{\mathrm{ret}}(z)(i\overleftarrow{\not\partial}-m)\overset{z\to-z}{=}\Delta^{\mathrm{ret}}(-z)(i\overleftarrow{\not\partial}+m)=(i\not\partial+m)\Delta^{\mathrm{ret}}(-z)\\&\overset{d=4}{=}\frac{1}{(2\pi)^4}\int d^4p\,\frac{\not p+m}{p^2-m^2-ip^00}e^{ipx}.\end{aligned} \tag{11.68}$$

Again, we can use

$$(i\overleftarrow{\not\partial}+m)(i\overleftarrow{\not\partial}-m)=-(\overleftarrow{\partial}^2+m^2). \tag{11.69}$$

To show that

$$S^{\mathrm{adv}}(x)(i\overleftarrow{\not\partial}+m)=\delta(x),\quad \operatorname{supp}S^{\mathrm{adv}}\subseteq\overline{V}_{-}. \tag{11.70}$$

Note that this propagator has the support in the past causal cone. That justifies the "slang" used by the physicist that an antiparticle is the particle traveling backwards in time.

The propagator of the classical Poisson bracket is defined as:

$$S(z):=S^{\mathrm{ret}}(z)-S^{\mathrm{adv}}(z). \tag{11.71}$$

Since $\psi,\overline{\psi}$ have different propagators, we have two different Wightman two-point functions page 358 [24]:

$$\begin{aligned} S^{+}(x)&:=(i\not\partial+m)\Delta^{+}(x)\overset{d=4}{=}\frac{1}{(2\pi)^3}\int d^4p(\not p+m)\theta(p^0)\delta(p^2-m^2)e^{-ipx}\\ S^{-}(x)&:=-(i\not\partial+m)\Delta^{+}(-x)\overset{d=4}{=}\frac{1}{(2\pi)^3}\int d^4p(\not p-m)\theta(p^0)\delta(p^2-m^2)e^{ipx}. \end{aligned} \tag{11.72}$$

The Feynman propagator is defined by page 369 [24]:

$$\begin{aligned} S^{F}_{jk}(x)&:=S^{+}_{jk}(x)\theta(x^0)-S^{-}_{jk}(x)\theta(-x^0)=(i\not\partial+m)\Delta^{F}(x)\\ &=\frac{i}{(2\pi)^4}\int d^4p\,\frac{\not p+m}{p^2-m^2+i0}e^{-ipx}. \end{aligned} \tag{11.73}$$

Just as in the case of complex scalar field, the star product is defined through:

$$\begin{aligned} &F\star G=F\wedge G\\ &+\hbar\int dxdy\left(\frac{\delta_r F}{\delta\psi_t(x)}S^{+}_{tk}(x-y)\wedge\frac{\delta G}{\overline{\psi}_k(x)}+\frac{\delta_r F}{\delta\overline{\psi}_u(x)}S^{-}_{vu}(y-x)\wedge\frac{\delta G}{\delta\psi_v(y)}\right)\\ &+\frac{\hbar^2}{2!}\int dX_2dY_2\Bigg(\frac{\delta_r^2F}{\delta\psi_{t_1}(x_1)\delta\psi_{t_2}(x_2)}S^{+}_{t_1k_1}(x_1-y_1)S^{+}_{t_2k_2}(x_2-y_2)\wedge\frac{\delta^2G}{\delta\overline{\psi}_{k_1}(y_1)\delta\overline{\psi}_{k_2}(y_2)}\\ &+\frac{\delta_r^2F}{\delta\psi_{t_1}(x_1)\delta\overline{\psi}_{u_2}(x_2)}S^{+}_{t_1k_1}(x_1-y_1)S^{-}_{v_2u_2}(y_2-x_2)\wedge\frac{\delta^2G}{\delta\overline{\psi}_{k_1}(y_1)\delta\psi_{v_2}(y_2)}\\ &+\frac{\delta_r^2F}{\delta\overline{\psi}_{u_1}(x_1)\delta\overline{\psi}_{u_2}(x_2)}S^{-}_{v_1u_1}(y_1-x_1)S^{-}_{v_2vu2}(y_2-x_2)\wedge\frac{\delta^2G}{\delta\psi_{v_1}(y_1)\delta\psi_{v_2}(y_2)}\Bigg)+.... \end{aligned} \tag{11.74}$$

In compact notation page 359 [24]:

$$
\begin{aligned}
F \star G := \sum_{n,m=0}^{\infty} \int dX_n dY_m \frac{\hbar^{n+m}}{n!m!} dX_n dY_n d\tilde{X}_n d\tilde{Y}_m \\
\frac{\delta_r^{n+m} F}{\delta\Psi_t(x_{1,n})\delta\overline{\Psi}_u(y_{1,m})} \prod_{j=1}^{n} S^{+}_{t_j s_j}(x_j - \tilde{x}_j) \\
\wedge \prod_{l=1}^{m} S^{-}_{v_l u_l}(\tilde{y}_l - y_l) \frac{\delta^{n+m} G}{\delta\overline{\Psi}_s(\tilde{x}_{1,n})\delta\Psi_v(\tilde{y}_{1,m})}.
\end{aligned}
\tag{11.75}
$$

Where:

$$
\begin{aligned}
\delta\Psi_t(x_{1,n}) &:= \delta\psi_{t_1}(x_1)....\delta\psi_{t_n}(x_n) \\
\delta\overline{\Psi}_u(y_{1,m}) &:= \delta\overline{\psi}_{u_1}(y_1)...\delta\overline{\psi}_{u_m}(y_m).
\end{aligned}
\tag{11.76}
$$

Once the retarded/advanced propagators and the star product are defined, we can define the Poisson bracket and the commutators. Unlike the Bose particles, the sign of the commutators is not always the same. To avoid the problem, we use the *eta trick* and transforms all the fields in Bosonic fields and compute the Poisson bracket/commutator as usual.

$$
\begin{aligned}
[F, G]_\star &:= \tilde{F} \star \tilde{G} - \tilde{G} \star \tilde{F} \\
\{F, G\} &:= \lim_{\hbar \to 0} \frac{[F, G]}{i\hbar} = \lim_{\hbar \to 0} \frac{\tilde{F} \star \tilde{G} - \tilde{G} \star \tilde{F}}{i\hbar}.
\end{aligned}
\tag{11.77}
$$

where $\tilde{F}, \tilde{G}$ means the field considering the eta trick (11.55).

Let us try some examples:

Example 1: $F = \overline{\psi}_j(x), G = \psi_k(y)$

Since F and G have odd powers of fields, the modified fields read:

$$
\tilde{F} = \eta_j \overline{\psi}_j(x) \quad \tilde{G} = \eta_k \psi_k(x).
\tag{11.78}
$$

We omit the symbol of the tensor product.

Hence:

$$
\begin{aligned}
F \star G &= (\eta_j \overline{\psi}_j(x)) \star (\eta_k \psi_k(y)) = (\eta_j \eta_k)(\overline{\psi}_j(x) \wedge \psi_k(y) + \hbar S^-_{kj}(y-x)) \\
G \star F &= (\eta_k \eta_j)(\psi_k(y) \wedge \overline{\psi}_j(x) + \hbar S^+_{kj}(y-x)) \\
[F, G] &= \eta_j \eta_k \overline{\psi}_j(x) \wedge \psi_k(y) - \eta_k \eta_j \psi_k(y) \wedge \overline{\psi}_j(x) \\
&\quad + \hbar((\eta_j \eta_k) S^-_{kj}(y-x) - (\eta_k \eta_j) S^+_{kj}(y-x)) \\
&\quad = (\eta_j \eta_k)(\overline{\psi}_j(x) \wedge \psi_k(y) + \psi_k(y) \wedge \overline{\psi}_j(x) + \hbar(S^-_{kj}(y-x) + S^+_{kj}(y-x)) \\
&\quad = \hbar(S^-_{kj}(y-x) + S^+_{kj}(y-x)).
\end{aligned} \tag{11.79}
$$

The Poisson bracket is simply:

$$
\{F, G\} = -i(S^-_{kj}(y-x) + S^+_{kj}(y-x)) \equiv S_{kj}(y-x) = (i\partial\!\!\!/_y + m)_{kj} \Delta(y-x). \tag{11.80}
$$

Note that in this case, if we were to define the commutator using only the F, G fields, we would have defined it as:

$$
[F, G] = F \star G + G \star F. \tag{11.81}
$$

Due to the eta trick, that confusion with signs is not a big deal.

Example 2: $F = \overline{\psi}_j(x)$, $G = \psi_{k_1}(y_1) \wedge \psi_{k_2}(y_2)$

The new fields are simply:

$$
\tilde{F} = \eta_j \overline{\psi}_j(x) \quad , \quad \tilde{G} = G = \psi_{k_1}(y_1) \wedge \psi_{k_2}(y_2). \tag{11.82}
$$

Hence:

$$
\begin{aligned}
F \star G &= \eta_j \bigg(\overline{\psi}_j(x) \wedge \psi_{k_1}(y_1) \wedge \psi_{k_2}(y_2) \\
&\qquad + \hbar(S^-_{k_1 j}(y_1 - x)\psi_{k_2}(y_2) + S^-_{k_2 j}(y_2 - x)\psi_{k_1}(y_1) \bigg) \\
G \star F &= \eta_j \bigg(\psi_{k_1}(y_1) \wedge \psi_{k_2}(y_2) \wedge \overline{\psi}_j(x) \\
&\qquad + \hbar(S^+_{k_1 j}(y_1 - x)\psi_{k_2}(y_2) + S^+_{k_2 j}(y_2 - x)\psi_{k_1}(y_1) \bigg) \\
[F, G] &= F \star G - G \star F = \hbar S_{k_1 k}(y_1 - x)\psi_{k_2}(y_2) + \hbar S_{k_2 j}(y_2 - x)\psi_{k_1}(y_1).
\end{aligned} \tag{11.83}
$$

As one can see, the eta trick was not necessary in the above expression. One can summarize the change of the sign using:

$$[F,G] = \begin{cases} F \star G + G \star F & F, G \in \mathcal{F}^- \\ F \star G - G \star F & \text{otherwise} \end{cases}. \tag{11.84}$$

Using the eta-trick, one can construct the $T-$ and $R-$ product exactly as done for the scalar field.

11.4.5 Free-Fermion states

Just as in the case of the scalar field, to compute scattering states, we need the Fock space representation. The construction of Fock space is basically the same as done for scalar fields, with the difference we need to consider anti-symmetric products:

$$\mathfrak{F}_{\text{fermion}} = \bigoplus_{n=0}^{\infty} \mathfrak{A}\mathcal{H}^{\otimes n}. \tag{11.85}$$

Where $\mathfrak{A}$ stands for anti-symmetrization. We assume the reader is familiar with the concept of Fermionic Fock space. If it is not the case, we refer to [35] chapter 3 and/or [67] chapter 2.2.

The Dirac spinor (operator) is defined through:

$$\begin{aligned}
\psi^{\text{op}}(x) &:= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\vec{p}}{2\omega_p} \sum_s a_s(\vec{p}) u_s(\vec{p}) e^{-ipx} + b_s^*(\vec{p}) v_s(\vec{p}) e^{ipx} \Big|_{p_0=\omega_p} \\
(\psi^{\text{op}})^\dagger &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\vec{p}}{2\omega_p} \sum_s a_s^*(\vec{p}) u_s^\dagger(\vec{p}) e^{ipx} + b_s(\vec{p}) v_s^\dagger(\vec{p}) e^{-ipx} \Big|_{p_0=\omega_p} \\
\overline{\psi}^{\text{op}} &:= (\psi^{\text{op}})^\dagger \gamma^0.
\end{aligned} \tag{11.86}$$

Where $s = \frac{1}{2}, -\frac{1}{2}$ represents the spin, $b_s^*(\vec{p})$ is related to the **creation operator of an anti-particle with wave function $\zeta(\vec{p})$ and spin s** by:

$$b_s^*(\psi) = \int \frac{d\vec{p}}{2\omega_p} \zeta(\vec{p}) b_s^*(\vec{p}). \tag{11.87}$$

$a_s^*(\vec{p})$ on the other hand,is related to the **creation operator of a particle with wave function $\zeta(\vec{p})$ and spin s** by:

$$a_s^*(\psi) = \int \frac{d\vec{p}}{2\omega_p} \zeta(\vec{p}) a_s^*(\vec{p}). \tag{11.88}$$

The operators $a_s(\vec{p}), b_s(\vec{p})$ are related to the annihilation operators of particle/anti-particle just as in the case of bosonic field. The only difference is that they obey anti-commutation relation, instead of commutation relations:

$$
\begin{aligned}
\{a_s^*(\vec{p}), a_l(\vec{q})\} &:= a_s^*(\vec{p})a_l(\vec{q}) + a_l(\vec{q})a_s^*(\vec{p}) = 2\hbar\omega_p\delta(\vec{p}-\vec{q})\delta_{sl} \\
\{b_s^*(\vec{p}), b_l(\vec{q})\} &:= b_s^*(\vec{p})b_l(\vec{q}) + b_l(\vec{q})b_s^*(\vec{p}) = 2\hbar\omega_p\delta(\vec{p}-\vec{q})\delta_{sl}.
\end{aligned}
\tag{11.89}
$$

u, v are eingenvectors of the free Hamiltonian. To properly introduce then, we need an auxiliary notation:

$$
\chi_s := \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & s = +\frac{1}{2} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & s = -\frac{1}{2} \end{cases}.
\tag{11.90}
$$

Then:

$$
\begin{aligned}
u_s(\vec{p}) &:= \sqrt{\omega_p + m}\begin{pmatrix} \chi_S \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_S \end{pmatrix} \\
v_s(\vec{p}) &:= \sqrt{\omega_p + m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_S \\ \chi_S \end{pmatrix}.
\end{aligned}
\tag{11.91}
$$

where $\vec{\sigma}\cdot\vec{p}$ is given by:

$$
\vec{\sigma}\cdot\vec{p} = p_x\sigma_x + p_y\sigma_y + p_z\sigma_z, \quad \sigma_i \text{ Pauli matrix } i.
\tag{11.92}
$$

From the above definitions, it can easily proven that:

$$
\begin{aligned}
\mathbf{u}_s^\dagger(\vec{p})\mathbf{u}_l(\vec{p}) &= \mathbf{v}_s^\dagger(\vec{p})\mathbf{v}_l(\vec{p}) = 2\omega_p\delta_{sl} \\
\mathbf{v}_s^\dagger(\vec{p})\mathbf{u}_l(\vec{q}) &= \mathbf{u}_s^\dagger(\vec{p})\mathbf{v}_l(\vec{q}) = 0.
\end{aligned}
\tag{11.93}
$$

Where the bold letters represent the matrix product in 4 dimensions. For $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$

$$\mathbf{a}^\dagger \mathbf{b} = \sum_{i=1}^{4} \overline{a}_i b_i. \tag{11.94}$$

Last but not least, using $\gamma^0 := \begin{pmatrix} \mathbf{1}_{2\times 2} & 0 \\ 0 & -\mathbf{1}_{2\times 2} \end{pmatrix}$ is easy to show that:

$$\sum_s \mathbf{u}_s(\vec{p})\overline{\mathbf{u}}_s(\vec{p}) = \not{p} + m \tag{11.95}$$

$$\sum_s \mathbf{v}_s(\vec{p})\overline{\mathbf{v}}_s(\vec{p}) = \not{p} - m. \tag{11.96}$$

As usual, the goal of when computing scattering amplitudes is to compute

$$\langle \Omega (b_s^*)^n (a_l^*)^m | S (b_t^*)^t (a_r^*)^r \Omega \rangle. \tag{11.97}$$

Just as we have done in the case of bosonic field, let us compute some simple examples in the free theory to obtain the correct Feynman rules.

$e^+ \to e^+$

The corresponding expression in the Fock space is given by:

$$\begin{aligned} \mathcal{T}_1 &= \langle \Omega b_s^*(\vec{p}) | b_l^*(\vec{q})\Omega \rangle = \langle \Omega | b_s(\vec{p}) b_l^*(\vec{q})\Omega \rangle \\ &= \langle \Omega | (2\omega_p \delta(\vec{p}-\vec{q}) - b_l^*(\vec{q}) b_s(\vec{p}))\Omega \rangle = 2\omega_p \delta_{sl} \delta(\vec{p}-\vec{q}). \end{aligned} \tag{11.98}$$

We compute the action of the field ψ acting on the vacuum:

$$\begin{aligned} \psi(x)\Omega &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d\vec{p}}{2\omega_p} \sum_s (a_s(\vec{p}) u_s(\vec{p}) e^{-ipx} + b_s^*(\vec{p}) v_s(\vec{p}) e^{ipx})\Omega \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d\vec{p}}{2\omega_p} \sum_s v_s(\vec{p}) e^{ipx} (b_s^*(\vec{p})\Omega). \end{aligned} \tag{11.99}$$

We can perform the inverse Fourier transformation:

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int dx\, e^{i\vec{p}\vec{x}} \psi(x) = \frac{e^{i\omega_p x^0}}{2\omega_p} \sum_s v_s(\vec{p}) (b_s^*(\vec{p})\Omega). \tag{11.100}$$

To completely isolate $b_s^*(\vec{p})\Omega$ in the equation above, we use (11.93) to write:

$$b_s^*(\vec{p})\Omega = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}\, e^{-ipx} \mathbf{v}_s^\dagger(\vec{p}) \boldsymbol{\psi}(x) \Big|_{p^0=\omega_p} . \tag{11.101}$$

Hence, the expression in our formalism reads:

$$\begin{aligned} \mathcal{T}_1 &= \omega_0 \left(\overline{\frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}\, e^{-ipx} \mathbf{v}_s^\dagger(\vec{p}) \boldsymbol{\psi}(x) \Big|_{p^0=\omega_p}} \star \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{y}\, e^{-iqy} \mathbf{v}_l^\dagger(\vec{q}) \boldsymbol{\psi}(y) \Big|_{q^0=\omega_q} \right) \\ &= \frac{1}{(2\pi)^3} \int d\vec{x} d\vec{y}\, e^{ipx-iqy} \omega_0 \left(\boldsymbol{\psi}^\dagger(x) \mathbf{v}_s(\vec{p}) \star \mathbf{v}_l^\dagger(\vec{q}) \boldsymbol{\psi}(y) \right) . \end{aligned} \tag{11.102}$$

The problem now is that the star product is only defined to $\overline{\psi} \star \psi$, not $\psi^\dagger \star \psi$. To solve the problem we introduce:

$$\psi^\dagger = \boldsymbol{\psi}^\dagger \mathbf{1}_{4\times 4} = \boldsymbol{\psi}^\dagger \boldsymbol{\gamma}^0 \boldsymbol{\gamma}^0 = \overline{\boldsymbol{\psi}} \boldsymbol{\gamma}^0 . \tag{11.103}$$

We also introduce the vector indices to make the calculations clearer, i.e.:

$$\begin{aligned} \boldsymbol{\psi}^\dagger(x) \mathbf{v}_s(\vec{p}) &\equiv \sum_{t,q} \overline{\psi}_t(x) \gamma^0_{tq} (v_s(\vec{p}))_q \\ \mathbf{v}_l^\dagger(\vec{q}) \boldsymbol{\psi}(y) &\equiv \sum_r (v_l^\dagger(\vec{q}))_r \psi_r(y). \end{aligned} \tag{11.104}$$

We adopt the Einstein summation convention: repeated indices must be summed. Thus:

$$\begin{aligned} \mathcal{T}_1 &= \frac{1}{(2\pi)^3} \int d\vec{x} d\vec{y}\, e^{ipx-iqy} \gamma^0_{tq} (v_s(\vec{p}))_q (v_l^\dagger(\vec{q}))_r \omega_0 \left(\overline{\psi}_t(x) \star \psi_r(y) \right) \\ &= \frac{1}{(2\pi)^3} \int d\vec{x} d\vec{y}\, e^{ipx-iqy} \gamma^0_{tq} (v_s(\vec{p}))_q (v_l^\dagger(\vec{q}))_r \left(\hbar S^-_{rt}(y-x) \right) . \end{aligned} \tag{11.105}$$

We can write the expression above as a product of matrices:

$$\begin{aligned} \mathcal{T}_1 &= \frac{\hbar}{(2\pi)^3} \int d\vec{x} d\vec{y}\, e^{ipx-iqy} \mathbf{v}_s^\dagger(\vec{q}) \mathbf{S}^-(y-x) \boldsymbol{\gamma}^0 \mathbf{v}_l(\vec{p}) \\ &= \frac{\hbar}{(2\pi)^6} \int \frac{d\vec{k}}{2\omega_k} d\vec{x} d\vec{y} e^{ipx-iqy} \mathbf{v}_s^\dagger(\vec{q}) (\mathbf{k} - m) e^{ik(y-x)} \gamma^0 \mathbf{v}_l(\vec{p}). \end{aligned} \tag{11.106}$$

Note that we can perform the integral over $d\vec{x} d\vec{y}$ just by integrating the exponential function:

$$
\begin{aligned}
\mathcal{T}_1 &= \frac{\hbar}{(2\pi)^6}\int \frac{d\vec{k}}{2\omega_k}(2\pi)^6\delta(\vec{p}-\vec{k})\delta(\vec{k}-\vec{q})\mathbf{v}_s^\dagger(\vec{q})(\mathbf{k}-m)\gamma^0\mathbf{v}_l(\vec{p}) \\
&= \frac{\hbar}{2\omega_p}\delta(\vec{p}-\vec{q})\mathbf{v}_s^\dagger(\vec{p})(\mathbf{p}-m)\gamma^0\mathbf{v}_l(\vec{p}).
\end{aligned}
\tag{11.107}
$$

The final trick we have to use to compute the product of matrices is to write:

$$
\mathbf{p}-m = \sum_k \mathbf{v}_k(\vec{p})\overline{\mathbf{v}}_k(\vec{p}).
\tag{11.108}
$$

From which we obtain:

$$
\begin{aligned}
\mathcal{T}_1 &= \frac{\hbar}{2\omega_p}\delta(\vec{p}-\vec{q})\sum_k \mathbf{v}_s^\dagger(\vec{p})\mathbf{v}_k(\vec{p})\overline{\mathbf{v}}_k(\vec{p})\gamma^0\mathbf{v}_l(\vec{p}) \\
&= \frac{\hbar}{2\omega_p}\delta(\vec{p}-\vec{q})\sum_k \mathbf{v}_s^\dagger(\vec{p})\mathbf{v}_k(\vec{p})\mathbf{v}_k^\dagger(\vec{p})\mathbf{v}_l(\vec{p}).
\end{aligned}
\tag{11.109}
$$

From (11.93):

$$
\mathbf{v}_s^\dagger(\vec{p})\mathbf{v}_k(\vec{p}) = \delta_{sk}2\omega_p \quad \mathbf{v}_k^\dagger(\vec{p})\mathbf{v}_l(\vec{p}) = \delta_{kl}2\omega_p.
\tag{11.110}
$$

We conclude:

$$
\mathcal{T}_1 = \frac{\hbar}{2\omega_p}\delta(\vec{p}-\vec{q})\sum_k \delta_{lk}\delta_{ks}(2\omega_p)^2 = 2\hbar\omega_p\delta(\vec{p}-\vec{q})\delta_{ls}.
\tag{11.111}
$$

$e^- \to e^-$

Now we repeat the calculation with aa particle going to a particle in the free theory. In the formalism of Fock space:

$$
\begin{aligned}
\mathcal{T}_1 &= \langle \Omega a_s^*(\vec{p}) | a_l^*(\vec{q})\Omega \rangle \\
&= \langle \Omega | a_s(\vec{p}) a_l^*(\vec{q})\Omega \rangle = 2\hbar\omega_p\delta(\vec{p}-\vec{q})\delta_{sl}.
\end{aligned}
\tag{11.112}
$$

The corresponding expression can be found if we apply $\psi^\dagger\Omega$:

$$
\begin{aligned}
(\psi^{\mathrm{op}})^\dagger\Omega &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d\vec{p}}{2\omega_p}\sum_s(a_s^*(\vec{p})u_s^\dagger(\vec{p})e^{ipx}+b_s(\vec{p})v_s^\dagger(\vec{p})e^{-ipx})\Omega \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d\vec{p}}{2\omega_p}\sum_s u_s^\dagger(\vec{p})e^{ipx}(a_s^*(\vec{p})\Omega). \qquad (11.113)
\end{aligned}
$$

We can compute the inverse Fourier transformation:

$$
\begin{aligned}
\sum_s u_s^\dagger(\vec{p})(a_s^*(\vec{p})\Omega) &= \frac{2\omega_p}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}\, e^{-ipx}(\psi^{\mathrm{op}})^\dagger(x) \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}e^{-ipx}(\boldsymbol{\psi}^{\mathrm{op}})^\dagger(x)\mathbf{u}_s(\vec{p}). \qquad (11.114)
\end{aligned}
$$

In the above equation we have used (11.93). Hence:

$$
\begin{aligned}
\mathcal{T}_1 &= \omega_0\left(\overline{\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}e^{-ipx}\boldsymbol{\psi}^\dagger(x)\mathbf{u}_s(\vec{p})}\star\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{y}e^{-iqy}\boldsymbol{\psi}^\dagger(x)\mathbf{u}_l(\vec{q})\right) \\
&= \frac{1}{(2\pi)^3}\int d\vec{x}d\vec{y}\, e^{ipx-iqy}\omega_0\left(\mathbf{u}_s^\dagger(\vec{p})\boldsymbol{\psi}(x)\star\overline{\boldsymbol{\psi}}(y)\boldsymbol{\gamma}^0\mathbf{u}_l(\vec{q})\right). \qquad (11.115)
\end{aligned}
$$

Now we repeat the same kind of calculation done in the case of the free e^- field.

$$
\begin{aligned}
\mathcal{T}_1 &= \frac{1}{(2\pi)^3}\int d\vec{x}d\vec{y}\, e^{ipx-iqy}u_{s,a}^\dagger(\vec{p})\gamma_{bc}^0 u_{l,c}\omega_0(\psi_a(x)\star\overline{\psi}_b(y)) \\
&= \frac{\hbar}{(2\pi)^3}\int d\vec{x}d\vec{y}\, e^{ipx-iqy}u_{s,a}^\dagger(\vec{p})\gamma_{bc}^0 u_{l,c}(\vec{q})S_{ab}^+(x-y) \\
&= \frac{\hbar}{(2\pi)^6}\int\frac{d\vec{x}d\vec{y}d\vec{k}}{2\omega_k}\, e^{ipx-iqy}u_{s,a}^\dagger(\vec{p})\gamma_{bc}^0 u_{l,c}(\vec{q})(\not{k}+m)_{ab}e^{-ik(x-y)} \\
&= \frac{\hbar}{2\omega_k}\delta(\vec{p}-\vec{q})\mathbf{u}_s^\dagger(\vec{p})(\not{\mathbf{p}}+\mathbf{m})\boldsymbol{\gamma}^0\mathbf{u}_l(\vec{p}). \qquad (11.116)
\end{aligned}
$$

Using (11.96) and (11.93) we can write:

$$
\begin{aligned}
\mathbf{u}_s^\dagger(\vec{p})(\not{\mathbf{p}}+\mathbf{m})\boldsymbol{\gamma}^0\mathbf{u}_l(\vec{p}) &= \sum_k \mathbf{u}_s^\dagger(\vec{p})\mathbf{u}_k(\vec{p})\overline{\mathbf{u}}_k(\vec{p})\boldsymbol{\gamma}^0\mathbf{u}_l(\vec{p}) \\
&= \sum_k(2\omega_p)^2\delta_{sk}\delta_{kl} = 4\omega_p^2\delta_{sl}. \qquad (11.117)
\end{aligned}
$$

Thus:

$$
\mathcal{T}_1 = \frac{\hbar}{2\omega_p}\delta(\vec{p}-\vec{q})4\omega_p^2\delta_{sl} = 2\hbar\omega_p\delta(\vec{p}-\vec{q})\delta_{sl}. \qquad (11.118)
$$

As expected.

We can summarize the Feynman rule for incoming/outgoing Fermions/ anti-fermions as follows:

- Incoming anti-fermions with momentum $\vec{p}$ and spin s are described by a term

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}\, e^{ipx}\overline{\boldsymbol{\psi}}(x)\boldsymbol{\gamma}^0\mathbf{v}_s(\vec{p}). \tag{11.119}$$

- Outgoing anti-fermions with momentum $\vec{p}$ and spin s are described by a term

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}\, e^{-ipx}\mathbf{v}_s^{\dagger}(\vec{p})\boldsymbol{\psi}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}\, e^{-ipx}\overline{\mathbf{v}}_s(\vec{p})\boldsymbol{\gamma}^0\boldsymbol{\psi}(x). \tag{11.120}$$

- Incoming fermions with momentum $\vec{p}$ and spin s are described by a term

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}\, e^{ipx}\mathbf{u}_s^{\dagger}(\vec{p})\boldsymbol{\psi}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}\, e^{ipx}\overline{\mathbf{u}}_s(\vec{p})\boldsymbol{\gamma}^0\boldsymbol{\psi}(x). \tag{11.121}$$

- Outgoing fermions with momentum $\vec{p}$ and spin s are described by a term

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}\, e^{-ipx}\overline{\boldsymbol{\psi}}(x)\boldsymbol{\gamma}^0\mathbf{u}_s(\vec{p}). \tag{11.122}$$

11.4.6 Fermi four fermion interaction

As an example of a model involving only fermionic fields, we consider an adaptation of the Fermi four fermion interaction [30]

$$S = \int dx\, \overline{\psi}_j(x) \wedge (i\partial_\mu\gamma^\mu_{jk} - m)\psi_k(x) - \lambda(\overline{\psi}_k(x) \wedge \psi_k(x))^2. \tag{11.123}$$

The original theory is composed with more than a type of fermion, and here we consider only one type to make it as simple as possible. Although the above theory is non-renormalizable, it was an important model for the β decay [62]. To avoid terms of the form $\overline{\overline{\psi}}$ in the formulas, we denote $\overline{\psi} \equiv \psi^+$ and $\psi \equiv \psi^-$.

Let us compute the vertex of the theory given by the amplitude of $\psi^+\psi^- \to \psi^+\psi^-$:

$$\mathcal{T}_1 = \frac{i}{\hbar}\omega_0(\overline{\psi^+_{s_1}}(p_1) \wedge \overline{\psi^-_{s_2}}(p_2) \star L_{int} \star \psi^+_{s_3}(p_3) \wedge \psi^-_{s_4}(p_4)). \tag{11.124}$$

Where:

$$
\begin{aligned}
\overline{\psi_{s_1}^{-}}(p_1) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}_1\, e^{ip_1x_1} \mathbf{u}^{\dagger}_{s_1}(\vec{p_1}) \boldsymbol{\psi}(x_1) \\
&\equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}_1\, e^{ip_1x_1} u^{\dagger}_{s_1,k_1}(\vec{p_1}) \psi_{k_1}(x_1) \\
\overline{\psi_{s_2}^{+}}(x_2) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}_2\, e^{ip_2x_2} \overline{\boldsymbol{\psi}}(x_2) \boldsymbol{\gamma}^0 \mathbf{v}_{s_2}(\vec{p}_2) \\
&\equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}_2\, e^{ip_2x_2} \overline{\psi}_{k_2}(x_2) \gamma^0_{k_2k_2'} v_{s_2,k_2'}(\vec{p}_2) \\
\psi_{s_3}^{-}(p_3) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}_3\, e^{-ip_3x_3} \overline{\boldsymbol{\psi}}(x_3) \boldsymbol{\gamma}^0 \mathbf{u}_{s_3}(\vec{p}_3) \\
&\equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}_3\, e^{-ip_3x_3} \overline{\psi}_{k_3}(x_3) \gamma^0_{k_3k_3'} u_{s_3,k_3'}(\vec{p}_3) \\
\psi_{s_4}^{+}(p_4) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}_4\, e^{-ip_4x_4} \mathbf{v}^{\dagger}_{s_4}(\vec{p}_4) \boldsymbol{\psi}(x_4) \\
&\equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}_4\, e^{-ip_4x_4}\, v^{\dagger}_{s_4,k_4}(\vec{p}_4) \psi_{k_4}(x_4).
\end{aligned}
\tag{11.125}
$$

Now we compute $\overline{\psi}_k(x) \wedge \psi_k(x) \wedge \overline{\psi}_j(x) \wedge \psi_j(x) \star \overline{\psi}_{k_3}(x_3) \wedge \psi_{k_4}(x_4)$. The final expression is long; therefore, we change the notation in the hope that we manage to increase clarity. We omit the wedge symbol, that is, $\overline{\psi} \wedge \psi \equiv \overline{\psi}\psi$ and the arguments of $\overline{\psi}_k(x), \psi_k(x), \overline{\psi}_j(x), \psi_j(x), \overline{\psi}_{k_3}(x_3), \psi_{k_4}(x_4)$. Since the argument of the function is related to the spinorial index, we should not have a problem reading the final expression. Remark: By omitting $\wedge$ we do not mean that we are computing the inner product in $\mathbb{C}^4$!!

$$
\begin{aligned}
&\overline{\psi}_k(x) \wedge \psi_k(x) \wedge \overline{\psi}_j(x) \wedge \psi_j(x) \star \overline{\psi}_{k_3}(x_3) \wedge \psi_{k_4}(x_4) \\
&\equiv \overline{\psi}_k \psi_k \overline{\psi}_j \psi_j \star \overline{\psi}_{k_3} \psi_{k_4} = \overline{\psi}_k \psi_k \overline{\psi}_j \psi_j \overline{\psi}_{k_3} \psi_{k_4} \\
&+ \hbar \left(\frac{\delta_r \overline{\psi}_k \psi_k \overline{\psi}_j \psi_j}{\delta \psi_t(x)} S^{+}_{ts}(x - \tilde{x}) \frac{\delta \overline{\psi}_{k_3} \psi_{k_4}}{\delta \overline{\psi}_s(\tilde{x})} + \frac{\delta_r \overline{\psi}_k \psi_k \overline{\psi}_j \psi_j}{\delta \overline{\psi}_u(y)} S^{-}_{vu}(\tilde{y} - y) \frac{\delta \overline{\psi}_{k_3} \psi_{k_4}}{\psi_v(\tilde{y})} \right) \\
&+ \hbar^2 \left(\frac{\delta^2_r \overline{\psi}_k \psi_k \overline{\psi}_j \psi_j}{\delta \psi_t(x) \delta \overline{\psi}_u(y)} S^{+}_{ts}(x - \tilde{x}) S^{-}_{vu}(\tilde{y} - y) \frac{\delta^2 \overline{\psi}_{k_3} \psi_{k_4}}{\delta \overline{\psi}_s(\tilde{x}) \delta \psi_v(\tilde{y})} \right).
\end{aligned}
\tag{11.126}
$$

The term proportional to $\hbar$ is:

$$
\begin{aligned}
&\overline{\psi}_k \psi_k \overline{\psi}_j S^{+}_{jk_3}(x - x_3) \psi_{k_4} + \overline{\psi}_k \overline{\psi}_j \psi_j S^{+}_{jk_3}(x - x_3) \psi_{k_4} \\
&+ \overline{\psi}_k \psi_k \psi_j S^{-}_{k_4 j}(x_4 - x) \overline{\psi}_{k_3} + \psi_k \overline{\psi}_j \psi_j S^{-}_{k_4 k}(x_4 - x) \overline{\psi}_{k_3}.
\end{aligned}
\tag{11.127}
$$

The term proportional to $\hbar^2$ is:

$$S^+_{jk_3}(x-x_3)S^-_{k_4j}(x_4-x)\overline{\psi}_k\psi_k - S^+_{kk_3}(x-x_3)S^-_{k_4j}(x_4-x)\overline{\psi}_k\psi_j$$
$$+S^+_{jk_3}(x-x_3)S^-_{k_4k}(x_4-x)\psi_k\overline{\psi}_j + S^+_{kk_3}(x-x_3)S^-_{k_4k}(x_4-x)\overline{\psi}_j\psi_j. \qquad (11.128)$$

We can simplify the result by changing $j \leftrightarrow k$ in the second order of (11.128). Then:

$$2\hbar^2(S^+_{jk_3}(x-x_3)S^-_{k_4j}(x_4-x)\overline{\psi}_k\psi_k - S^+_{kk_3}(x-x_3)S^-_{k_4j}(x_4-x)\overline{\psi}_k\psi_j). \qquad (11.129)$$

Computing $\psi_{k_1}(x_1) \wedge \overline{\psi}_{k_2}(x_2)\star$equation above, we obtain:

$$\overset{|\psi|=0}{=} -2\hbar^4 S^+_{jk_3}(x-x_3)S^-_{k_4j}(x_4-x)S^-_{kk_2}(x-x_2)S^+_{k_1k}(x_1-x)$$
$$+2\hbar^4 S^+_{kk_3}(x-x_3)S^-_{k_4j}(x_4-x)S^-_{jk_2}(x-x_2)S^+_{k_1k}(x_1-x). \qquad (11.130)$$

To complete the calculation, we need to use the same kind of trick we use to compute the amplitude of free-fermions. The first line reads:

$$\begin{aligned}\mathcal{T}_1 = -2i\lambda\hbar^3 \int \frac{dx\, d\vec{X}_4}{(2\pi)^6}\, e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4} \\ \times \left(\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\mathbf{S}^+(x_1-x)\mathbf{S}^-(x-x_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2)\right) \\ \times \left(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\mathbf{S}^-(x_4-x)\mathbf{S}^+(x-x_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\right).\end{aligned} \qquad (11.131)$$

Using

$$S^+(z) = \frac{1}{(2\pi)^3}\int \frac{d\vec{q}}{2\omega_q}(\not{q}+m)e^{-iqz} = \frac{1}{(2\pi)^3}\int \frac{d\vec{q}}{2\omega_q}\sum_s \mathbf{u}_s(\vec{q})\overline{\mathbf{u}}_s(\vec{q})e^{-iqz}$$
$$S^-(z) = \frac{1}{(2\pi)^3}\int \frac{d\vec{q}}{2\omega_q}(\not{q}-m)e^{iqz} = \frac{1}{(2\pi)^3}\int \frac{d\vec{q}}{2\omega_q}\sum_s \mathbf{v}_s(\vec{q})\overline{\mathbf{v}}_s(\vec{q})e^{iqz} \qquad (11.132)$$

We can write the contribution to the amplitude as:

$$\begin{aligned}\mathcal{T}_1 = i\lambda\hbar^3 \int \frac{dx\, d\vec{X}_4 d\vec{Q}_4}{(2\pi)^{18}\left(\prod_{i=1}^4 2\omega_{q_i}\right)} \\ e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4-iq_1(x_1-x)+iq_2(x-x_2)-iq_3(x-x_3)+iq_4(x_4-x)} \\ \sum_{k_1,k_2,k_3,k_4}\left(\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\mathbf{u}_{k_1}(\vec{q}_1)\overline{\mathbf{u}}_{k_1}(\vec{q}_1)\mathbf{v}_{k_2}(\vec{q}_2)\overline{\mathbf{v}}_{k_2}(\vec{q}_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2)\right) \\ \times\left(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\mathbf{v}_{k_4}(\vec{q}_4)\overline{\mathbf{v}}_{k_4}(\vec{q}_4)\mathbf{u}_{k_3}(\vec{q}_3)\overline{\mathbf{u}}_{k_3}(\vec{q}_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\right)\end{aligned} \qquad (11.133)$$

The integrals can easily be done:

$$
\begin{aligned}
& i\lambda\hbar^3 \int \frac{dxdy\, d\vec{X}_4 d\vec{Q}_4}{(2\pi)^{18}\left(\prod_{i=1}^4 2\omega_{q_i}\right)} \\
& \qquad e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4-iq_1(x_1-x)+iq_2(x-x_2)-iq_3(x-x_3)+iq_4(x_4-x)} \\
&= \frac{i\lambda\hbar^3}{(2\pi)^6} \int \frac{dx\, d\vec{Q}_4}{\left(\prod_{i=1}^4 2\omega_{q_i}\right)} \left(\prod_{i=1}^4 \delta(\vec{p}_i - \vec{q}_i)\right) e^{i(q_1+q_2-q_3-q_4)x} = \frac{i\lambda\hbar^3}{(2\pi)^3} \frac{\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)}{\left(\prod_{i=1}^4 2\omega_{q_i}\right)}
\end{aligned}
\tag{11.134}
$$

Using $\vec{p}_i = \vec{q}_i$, the matrix product can be computed using (11.93):

$$
\begin{aligned}
&\sum_{k_1,k_2} \mathbf{u}^\dagger_{s_1}(\vec{p}_1)\mathbf{u}_{k_1}(\vec{p}_1)\overline{\mathbf{u}}_{k_1}(\vec{p}_1)\mathbf{v}_{k_2}(\vec{p}_2)\overline{\mathbf{v}}_{k_2}(\vec{p}_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2) \\
&= \sum_{k_1,k_2} 2\omega_{p_1} 2\omega_{p_2} \delta_{s_1k_1}\delta_{s_2k_2}\overline{\mathbf{u}}_{k_1}(\vec{p}_1)\mathbf{v}_{k_2}(\vec{p}_2) = 4\omega_{p_1}\omega_{p_2}\overline{\mathbf{u}}_{s_1}(\vec{p}_1)\mathbf{v}_{s_2}(\vec{p}_2)
\end{aligned}
\tag{11.135}
$$

$$
\begin{aligned}
&\sum_{k_3,k_4} \mathbf{v}^\dagger_{s_4}(\vec{p}_4)\mathbf{v}_{k_4}(\vec{p}_4)\overline{\mathbf{v}}_{k_4}(\vec{p}_4)\mathbf{u}_{k_3}(\vec{p}_3)\overline{\mathbf{u}}_{k_3}(\vec{p}_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3) \\
&4\omega_{p_3}\omega_{p_4} \sum_{k_3,k_4} \delta_{k_3s_3}\delta_{k_4s_4}\overline{\mathbf{v}}_{k_4}(\vec{p}_4)\mathbf{u}_{k_3}(\vec{p}_3) = 4\omega_{p_3}\omega_{p_4}\overline{\mathbf{v}}_{s_4}(\vec{p}_4)\mathbf{u}_{s_3}(\vec{p}_3)
\end{aligned}
\tag{11.136}
$$

Thus, the first term correspond to:

$$
-2\frac{i\lambda\hbar^3}{(2\pi)^3}\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)(\overline{\mathbf{u}}_{s_1}(\vec{p}_1)\mathbf{v}_{s_2}(\vec{p}_2))(\overline{\mathbf{v}}_{s_4}(\vec{p}_4)\mathbf{u}_{s_3}(\vec{p}_3)).
\tag{11.137}
$$

The second term is analogous and contributes with:

$$
2\frac{i\lambda\hbar^3}{(2\pi)^3}(\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\mathbf{u}_{s_3}(\vec{p}_3))(\mathbf{v}^\dagger_{s_2}(\vec{p}_2)\mathbf{v}_{s_4}(\vec{p}_4))\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4).
\tag{11.138}
$$

Thus, the total amplitude is simply:

$$
2\frac{i\lambda\hbar^3}{(2\pi)^3}((\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\mathbf{u}_{s_3}(\vec{p}_3))(\mathbf{v}^\dagger_{s_2}(\vec{p}_2)\mathbf{v}_{s_4}(\vec{p}_4)) - (\overline{\mathbf{u}}_{s_1}(\vec{p}_1)\mathbf{v}_{s_2}(\vec{p}_2))(\overline{\mathbf{v}}_{s_4}(\vec{p}_4)\mathbf{u}_{s_3}(\vec{p}_3)))\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4).
\tag{11.139}
$$

The corresponding Feynman diagrams

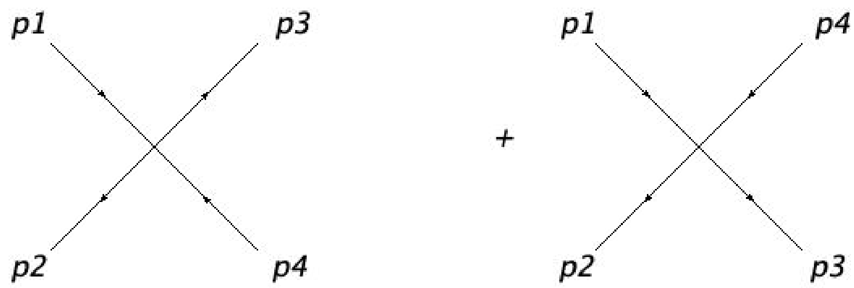


Figure 11.1: We follow the usual convention: Antiparticles are represented going in the opposite direction they are actually traveling. Note that since are working with Fermions, changing $p_3 \leftrightarrow p_4$ implies a change in the sign.

Chapter 12

Examples of more sophisticated perturbative theories

12.1 Warm-up: effective theory for 2 scalar fields

The first theory we want to consider that contains more than one field is the simplest possible: a theory with two different types of scalar fields ϕ_1, ϕ_2. The space of fields is defined as

$$\mathcal{F} := \mathcal{F}_1 \otimes \mathcal{F}_2. \tag{12.1}$$

The multiplication and star product are simply defined by:

$$(F_1 \otimes F_2) \cdot (G_1 \otimes G_2) := (F_1 \cdot G_1) \otimes (F_2 \cdot G_2) \tag{12.2}$$
$$(F_1 \otimes F_2) \star (G_1 \otimes G_2) := (F_1 \star G_1) \otimes (F_2 \star G_2). \tag{12.3}$$

All the constructions done for a single real scalar field also work for the product of scalar fields. We are interested in studying the action:

$$\begin{aligned} S = \int dx \, \{ & \frac{1}{2} \partial_\mu \phi_1(x) \partial^\mu \phi_1(x) - m^2 \phi_1^2(x) \\ & + \partial_\mu \phi_2(x) \partial^\mu \phi_2(x) - M^2 \phi_2^2(x) - \mu \phi_2(x) \phi_1^2(x) \}. \end{aligned} \tag{12.4}$$

Once again, we work in the adiabatic limit since for massive theories it is well defined. In the limit $M^2 \gg$ any energy scale involved in the problem. The problem is known in the literature as effective field theory (Chapter 33 [68]). The idea is to change the full theory that contains two types of field to a simpler one that contains only one type of field. Typically, this is done by matching the amplitudes of the original theory to those of the new theory. There are more sophisticated methods to perform the calculation [52], but since the goal here is to introduce how to work with two different types of fields, the tree level matching is already good enough.

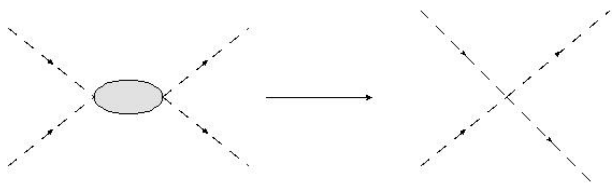


Figure 12.1: We want to exchange diagram for two types of fields to a tree-level diagram of ϕ_1^4 theory. The dashed line indicates ϕ_1 and the bubble interactions involving ϕ_1 and ϕ_2. We compute only the first order.

Let us get down to business. We want to compute $\phi_1\phi_1 \to \phi_1\phi_1$ scattering. Considering the form of our interaction, this process corresponds to $T_2(L_{int}, L_{int})$. The construction of $T-$product respects the same rules, hence:

$$\begin{aligned} T_2(L_{int}, L_{int}) &= -\frac{\mu^2}{\hbar^2}\int dxdy\, \phi_1^2(x_1)\phi_2(x) \star_F \phi_1^2(y)\phi_2(y) \\ &= -\frac{\mu^2}{\hbar^2}\int dxdy\, (\phi_1^2(x_1) \star_F \phi_1^2(y))(\phi_2(x) \star_F \phi_2(y)). \end{aligned} \tag{12.5}$$

The corresponding amplitude is given by:

$$\mathcal{T}_2 = \omega_0(\overline{\phi}_1(p_1)\overline{\phi}_1(p_2) \star T_2(L_{int}, L_{int}) \star \phi_1(p_3)\phi_1(p_4)). \tag{12.6}$$

Again:

$$\begin{aligned} \overline{\phi(p_1)\phi(p_2)} &\equiv \frac{4\omega_{p_1}\omega_{p_2}}{(2\pi)^{d-1}}\int d\vec{x}_1 d\vec{x}_2\, e^{ip_1x_1+ip_2x_2}\phi(x_1)\phi(x_2) \\ \phi(p_3)\phi(p_4) &\equiv \frac{4\omega_{p_3}\omega_{p_4}}{(2\pi)^{d-1}}\int d\vec{x}_3 d\vec{x}_4\, e^{-ip_3x_3-ip_4x_4}\phi(x_3)\phi(x_4). \end{aligned} \tag{12.7}$$

Since we have the contraction of 4 ϕ_1's and none ϕ_2 in the amplitude, the term that has a non-zero contribution for the amplitude is:

$$-\frac{\mu^2}{\hbar^2}\phi_1^2(x)\phi_1^2(y)\hbar\Delta_2^F(x-y) = -\frac{\mu^2}{\hbar}\phi_1^2(x)\phi_1^2(y)\left(i\int\frac{dp}{(2\pi)^d}\frac{e^{-ip(x-y)}}{p^2-M^2+i0}\right). \tag{12.8}$$

Plugging the expression above into the amplitude we obtain:

$$\begin{aligned} \mathcal{T}_2 = -\frac{\mu^2}{\hbar}\left(\prod_{i=1}^{4}\frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\int d\vec{X}_4 e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4} \\ \times\int dxdy\,\omega_0(\phi_1(x_1)\phi_1(x_2) \star \phi_1^2(x)\phi_1^2(y) \star \phi_1(x_3)\phi_1(x_4))\Delta_2^F(x-y). \end{aligned} \tag{12.9}$$

The star product can be easily computed:

$$
\begin{aligned}
&\omega_0(\phi_1(x_1)\phi_1(x_2)\star\phi_1^2(x)\phi_1^2(y)\star\phi_1(x_3)\phi_1(x_4)) = \\
&2\hbar^4\Delta^+(x_1-x)\Delta^+(x_2-x)\Delta^+(y-x_3)\Delta^+(y-x_4) \quad (12.10)\\
&2\hbar^4\Delta^+(x_1-x)\Delta^+(x_2-y)\Delta^+(x-x_3)\Delta^+(y-x_4) \quad (12.11)\\
&2\hbar^4\Delta^+(x_1-x)\Delta^+(x_2-y)\Delta^+(y-x_3)\Delta^+(x-x_4)\\
&. \quad (12.12)
\end{aligned}
$$

And they correspond precisely to the diagrams:

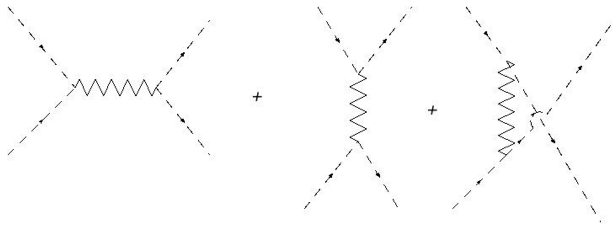


Figure 12.2: There are 3 ways of connecting 4 scalar fields with a propagator in between.

Consider the calculation of $2\hbar^4\Delta^+(x_1-x)\Delta^+(x_2-x)\Delta^+(y-x_3)\Delta^+(y-x_4)$. The remaining terms represent a permutation of the variables and need not be done in full.

$$
\begin{aligned}
\mathcal{T}_2 = -2i\mu^2\hbar^3 &\left(\prod_{i=1}^{4}\frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\frac{1}{(2\pi)^{4(d-1)+d}}\int d\vec{X}_4 e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4}\\
&\int dxdydp\left(\prod_{i=1}^{4}\frac{d\vec{k}_i}{2\omega_{k_i}}\right)\frac{e^{-ik_1(x_1-x)-ik_2(x_2-x)+ik_3(y-x_3)+ik_3(y-x_4)-ip(x-y)}}{p^2-M^2+i0}. \quad (12.13)
\end{aligned}
$$

The calculation above is very simple if we perform the integrals in the right order. We star with the integrals on the x and y variables:

$$
\begin{aligned}
\mathcal{T}_2 = -2i\mu^2\hbar^3 &\left(\prod_{i=1}^{4}\frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)\frac{1}{(2\pi)^{4(d-1)+d}}\int d\vec{X}_4 e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4}\\
&\int (2\pi)^{2d}dp\left(\prod_{i=1}^{4}\frac{d\vec{k}_i}{2\omega_{k_i}}\right)\frac{e^{-ik_1x_1-ik_2x_2-k_3x_3-ik_4x_4}\delta(p-k_1-k_2)\delta(p+k_3+k_4)}{p^2-M^2+i0}. \quad (12.14)
\end{aligned}
$$

Now we perform the integral in the p variable:

$$\mathcal{T}_2 = -2i\mu^2\hbar^3 \left(\prod_{i=1}^{4} \frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right) \frac{1}{(2\pi)^{4(d-1)-d}} \int d\vec{X}_4 e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4}$$
$$\int \left(\prod_{i=1}^{4} \frac{d\vec{k}_i}{2\omega_{k_i}}\right) \frac{\delta(-k_1-k_2-k_3-k_4)}{(k_1+k_2)^2-M^2+i0} e^{-ik_1x_1-ik_2x_2-ik_3x_3-ik_4x_4}. \tag{12.15}$$

The integral over $d\vec{X}_4$ will contribute to changing the expontentials by:

$$(2\pi)^{4(d-1)}\delta(k_1-p_1)\delta(k_2-p_2)\delta(k_3+p_3)\delta(k_4+p_4). \tag{12.16}$$

hence, we can perform the remaining integrals very easily:

$$\mathcal{T}_2 = -\frac{2i\mu^2\hbar^3}{(2\pi)^{d-2}} \frac{\delta(p_1+p_2-p_3-p_4)}{(p_1+p_2)^2-M^2+i0} \xrightarrow{M\gg(p_1+p_2)^2} \frac{2i\mu^2\hbar^3}{(2\pi)^{d-2}M^2}\delta(p_1+p_2-p_3-p_4). \tag{12.17}$$

The contributions of the other terms are the same. Hence, the final form of the amplitude is:

$$\mathcal{T}_2 = i\frac{6\mu^2\hbar^3}{(2\pi)^{d-2}M^2}\delta(p_1+p_2-p_3-p_4). \tag{12.18}$$

Hence, comparing the result with (9.83) we can change the interaction by an effective interaction of the form:

$$L_{int} = -\frac{\lambda}{4!}\int dx\, \phi^4(x), \quad \lambda = -\frac{6\mu^2}{M^2}. \tag{12.19}$$

12.2 Scalar QED

We will follow [68] Chapter 9.

The scalar QED also involves two types of fields: A complex scalar field and a gauge photon. A nice and intuitive deduction of it's Lagrangian consists in starting with the free theory and imposing invariance under local $U(1)$ transformations $\phi \to e^{i\alpha(x)}\phi, \alpha : \mathbb{M} \to \mathbb{R}$ [17]. The procedure described leads to:

$$S = \int dx\, \partial_\mu\phi(x)\partial^\mu\phi^*(x) - m^2|\phi(x)|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}(x)$$
$$- ieA^\mu(\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x)) + e^2A_\mu(x)A^\mu(x)|\phi(x)|^2. \tag{12.20}$$

Here e is the coupling constant. We denote the gauge field by the Greek letter γ. As an example, we compute the Feynman rules for some vertices present in the theory:

I) Particle scattering: $\phi \to \phi\gamma$.

II) Antiparticle scattering: $\phi^* \to \phi^*\gamma$

III) Pair creation: $\gamma \to \phi\phi^*$

IV) Pair annihilate and create one photon $\phi\phi^* \to \gamma$

V) Pair annihilate and create two photons $\phi\phi^* \to \gamma\gamma$

VI) Particle/antiparticle light scattering $\phi\gamma \to \phi\gamma$

We can compute the Feynman rules (in momentum space) for all those processes. Note that in the first four the only relevant term of the interaction Lagrangian is $-ieA^\mu(\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x))$. To simplify the notation, we define $d\vec{\sigma}_n := d\vec{X}_n \left(\prod_{i=1}^n \frac{2\omega_{p_i}}{(2\pi)^{\frac{d-1}{2}}}\right)$.

Remark: Compared to our reference [68], the sign of the first four vertices is changed. That happened because the book chose to create a particle/ annihilate anti-particle with ϕ^* and create an anti-particle/ annihilate a particle with ϕ. That is the opposite convention we have followed. The results are recovered if we simply change $\phi \to \phi^*$. Then the interaction term $-ieA^\mu(\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x)) \to ieA^\mu(\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x))$ and the sign are recovered.

12.2.1 I) Particle scattering: $\phi \to \phi\gamma$

$$\begin{aligned}\mathcal{T}_1 = \frac{i}{\hbar}\omega_0(\overline{\phi}(p_1) \star L_{int} \star A^\nu(p_2)\phi(p_3)) = \frac{e}{\hbar}\int dx d\vec{\sigma}_3\, e^{ip_1x_1 - ip_2x_2 + ip_3x_3} \\ \omega_0(\phi^*(x_1) \star (A^\mu(\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x))) \star A^\nu(x_2)\phi(x_3)).\end{aligned} \tag{12.21}$$

Let us compute the star product separately:

$$\begin{aligned}&\phi^*(x_1) \star A^\mu(x)\phi^*(x)\partial_\mu\phi(x) \star A^\nu(x_2)\phi(x_3) \\ =&(\phi^*(x_1) \star \phi^*(x)\partial_\mu\phi(x) \star \phi(x_3))(A^\mu(x) \star A^\nu(x_2)) \\ \overset{|\text{fields}|=0}{=}&(-\hbar^2\partial_{x,\mu}\Delta_\phi^+(x_1 - x)\Delta_\phi^+(x - x_3))(-\hbar\eta^{\mu\nu}\Delta_A^+(x - x_2)) \\ =&\hbar^3\Delta_A^+(x - x_2)\Delta_\phi^+(x - x_3)\partial_x^\nu\Delta_\phi^+(x_1 - x).\end{aligned} \tag{12.22}$$

In the above equation, we have used the result of star products with derivative fields discussed in Section (9.7). The notation $|\text{field}| = 0$ means we do not have any powers of fields.

The second term is easily computed:

$$
\begin{aligned}
&\phi^*(x_1) \star A^\mu(x)\phi(x)\partial_\mu\phi^*(x) \star A^\nu(x_2)\phi(x_3) \\
&= (\phi^*(x_1) \star \phi(x)\partial_\mu\phi^*(x) \star \phi(x_3))(A^\mu(x) \star A^\nu(x_2)) \\
&\overset{|\text{fields}|=0}{=} (-\hbar^2\Delta^+_\phi(x_1 - x)\partial_{x,\mu}\Delta^+_\phi(x - x_3))(-\hbar\eta^{\mu\nu}\Delta^+_A(x - x_2)) \\
&= \hbar^3\Delta^+_A(x - x_2)\Delta^+_\phi(x_1 - x)\partial^\nu_x\Delta^+_\phi(x - x_3).
\end{aligned} \tag{12.23}
$$

Now we just have to perform the integrals:

$$
\begin{aligned}
&e\hbar^2 \int dxd\vec{\sigma}_3\, e^{ip_1x_1 - ip_2x_2 + ip_3x_3}\Delta^+_A(x - x_2)\Delta^+_\phi(x - x_3)\partial^\nu_x\Delta^+_\phi(x_1 - x) \\
=&e\hbar^2 \int dxd\vec{\sigma}_3 \frac{d\vec{k}_3}{\prod_{i=1}^3 2\omega_{p_i}} \frac{1}{(2\pi)^{3(d-1)}} \\
&\times e^{ip_1x_1 - ip_2x_2 + ip_3x_3}(\partial^\nu_x e^{-ik_1(x_1 - x)})e^{-ik_2(x - x_2) - ik_3(x - x_3)} \\
=&\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}(ip^\nu_1\delta(p_1 - p_2 - p_3)) = i\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}p^\nu_1\delta(p_1 - p_2 - p_3).
\end{aligned} \tag{12.24}
$$

$$
\begin{aligned}
&-e\hbar^2 \int dxd\vec{\sigma}_3\, e^{ip_1x_1 - ip_2x_2 + ip_3x_3}\Delta^+_A(x - x_2)\Delta^+_\phi(x_1 - x)\partial^\nu_x\Delta^+_\phi(x - x_3) \\
= &-e\hbar^2 \int dxd\vec{\sigma}_3 \frac{d\vec{k}_3}{\prod_{i=1}^3 2\omega_{p_i}} \frac{1}{(2\pi)^{3(d-1)}} \\
&\times e^{ip_1x_1 - ip_2x_2 + ip_3x_3}e^{-ik_1(x_1 - x) - ik_2(x - x_2)}(\partial^\nu_x e^{-ik_3(x - x_3)}) \\
=&\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}(-ip^\nu_3\delta(p_1 - p_2 - p_3)) = i\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}p^\nu_1\delta(p_1 - p_2 - p_3).
\end{aligned} \tag{12.25}
$$

Hence:

$$
\mathcal{T}_1 = i\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}(p^\nu_1 + p^\nu_2)\delta(p_1 - p_2 - p_3). \tag{12.26}
$$

The corresponding diagram reads:

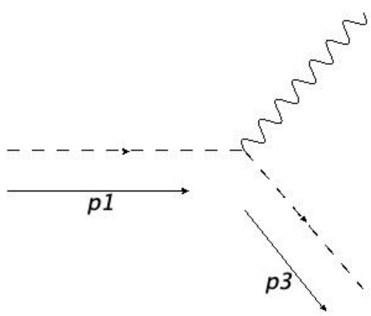


Figure 12.3: The Feynman rule is $i\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}(p^\nu_1 + p^\nu_2)$

The process $II-IV$ have a similar Feynman rule. For that reason, we go a bit faster through the calculations:

12.2.2 II) Antiparticle scattering: $\phi^* \to \phi^*\gamma$

The amplitude:

$$\mathcal{T}_1 = \frac{i}{\hbar}\omega_0(\overline{\phi}^*(p_1) \star L_{int} \star A^\nu(p_2)\phi^*(p_3)) = \frac{e}{\hbar}\int dx d\vec{\sigma}_3\, e^{ip_1x_1 - ip_2x_2 + ip_3x_3}$$
$$\omega_0(\phi(x_1) \star (A^\mu(\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x))) \star A^\nu(x_2)\phi^*(x_3)). \tag{12.27}$$

Star product of the first term:

$$\begin{aligned}
&\phi(x_1) \star A^\mu(x)\phi^*(x)\partial_\mu\phi(x) \star A^\nu(x_2)\phi^*(x_3)\\
&= (\phi(x_1) \star \phi^*(x)\partial_\mu\phi(x) \star \phi^*(x_3))(A^\mu(x) \star A^\nu(x_2))\\
&\overset{|\text{fields}|=0}{=} (-\hbar^2\partial_{x,\mu}\Delta^+_\phi(x-x_3)\Delta^+_\phi(x_1-x))(-\hbar\eta^{\mu\nu}\Delta^+_A(x-x_2))\\
&= \hbar^3\Delta^+_A(x-x_2)\Delta^+_\phi(x_1-x)\partial^\nu_x\Delta^+_\phi(x-x_3).
\end{aligned} \tag{12.28}$$

Star product of the second term:

$$\begin{aligned}
\phi(x_1) \star A^\mu(x)\phi(x)\partial_\mu\phi^*(x) \star A^\nu(x_2)\phi^*(x_3) &= (\phi(x_1) \star \phi(x)\partial_\mu\phi^*(x) \star \phi^*(x_3))(A^\mu(x) \star A^\nu(x_2))\\
&\overset{|\text{fields}|=0}{=} (-\hbar^2\Delta^+_\phi(x_3-x)\partial_{x,\mu}\Delta^+_\phi(x_1-x))(-\hbar\eta^{\mu\nu}\Delta^+_A(x-x_2))\\
&= \hbar^3\Delta^+_A(x-x_2)\Delta^+_\phi(x-x_3)\partial^\nu_x\Delta^+_\phi(x_1-x).
\end{aligned} \tag{12.29}$$

Note the expressions above are the same as in the first case, but exchanged first $\leftrightarrow$ second. Hence:

$$\mathcal{T}_1 = -i\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}(p_1^\nu + p_2^\nu)\delta(p_1 - p_2 - p_3). \tag{12.30}$$

The corresponding diagram reads:

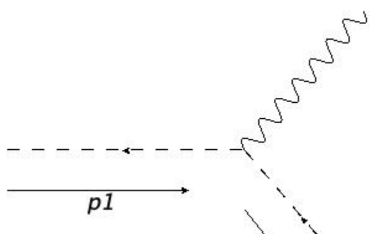


Figure 12.4: The Feynman rule is $-i\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}(p_1^\nu + p_3^\nu)$

12.2.3 III) Pair creation: $\gamma \to \phi\phi^*$

Amplitude:

$$\mathcal{T}_1 = \frac{i}{\hbar}\omega_0(\overline{A}^\nu(p_1) \star L_{int} \star \phi^*(p_2)\phi(p_3)) = \frac{e}{\hbar}\int dx d\vec{\sigma}_3\, e^{ip_1x_1 - ip_2x_2 + ip_3x_3}$$
$$\omega_0((A^\nu(x_1) \star A^\mu(x))((\phi^*(x)\partial_\mu\phi(x) - \phi\partial_\mu\phi^*(x)) \star \phi^*(x_2)\phi(x_3)). \tag{12.31}$$

Star product:

$$\begin{aligned}&(A^\nu(x_1) \star A^\mu(x))((\phi^*(x)\partial_\mu\phi(x) - \phi\partial_\mu\phi^*(x)) \star \phi^*(x_2)\phi(x_3))\\ \overset{|\text{fields}|=0}{=}&\hbar^3\eta^{\mu\nu}\Delta_A^+(x_1 - x)(\partial_{x,\mu}\Delta_\phi^+(x - x_2)\Delta_\phi^+(x - x_3)\\ &- \Delta_\phi^+(x - x_2)\partial_{x,\mu}\Delta_\phi^+(x - x_3)).\end{aligned} \tag{12.32}$$

The integrals

$$\begin{aligned}\mathcal{T}_1 = &\frac{\hbar^2 e}{(2\pi)^{3(d-1)}}\int dx d\vec{\sigma}_3 \frac{d\vec{K}_3}{\prod_{i=1}^3 2\omega_{p_i}}\, e^{ip_1x_1 - ip_2x_2 - ip_3x_3}\\ &\times e^{-ik_1(x_1-x) - ik_2(x-x_2) - ik_3(x-x_3)}(-ik_2^\nu + ik_3^\nu)\\ = &\, i\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}(p_3^\nu - p_2^\nu)\delta(p_1 - p_2 - p_3).\end{aligned} \tag{12.33}$$

The corresponding diagram reads:

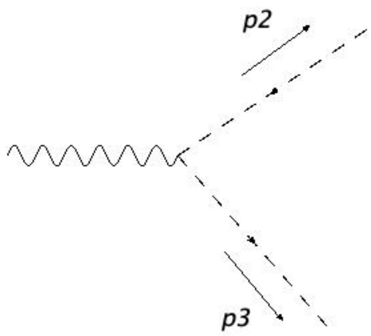


Figure 12.5: The Feynman rule is $i\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}(p_3^\nu - p_2^\nu)(p_2^\nu - p_3^\nu)$

12.2.4 IV) Pair annihilate and create one photon $\phi\phi^* \to \gamma$

Amplitude:

$$\mathcal{T}_1 = \frac{i}{\hbar}\omega_0(\overline{\phi}^*(p_1)\overline{\phi}(p_2) \star L_{int} \star A^\nu(p_3)) = \frac{e}{\hbar}\int dx d\vec{\sigma}_3\, e^{ip_1x_1+ip_2x_2-ip_3x_3}$$
$$\omega_0(\phi(x_1)\phi^*(x_2) \star (\phi^*(x)\partial_\mu\phi(x) - \phi\partial_\mu\phi^*(x)))(A^\mu(x) \star A^\nu(x_3)). \tag{12.34}$$

Star product:

$$\begin{aligned}&(\phi(x_1)\phi^*(x_2) \star (\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x)))(A^\mu(x) \star A^\nu(x_3))\\ =&\hbar^3\eta^{\mu\nu}(\Delta^+_\phi(x_1 - x)\partial_{x,\mu}\Delta^+_\phi(x_2 - x) - \partial_{x,\mu}\Delta^+_\phi(x_1 - x)\Delta^+(x_2 - x))\Delta^+_A(x - x_3).\end{aligned} \tag{12.35}$$

The integrals

$$\begin{aligned}\mathcal{T}_1 &= \frac{\hbar^2 e}{(2\pi)^{3(d-1)}}\int dx d\vec{\sigma}_3 \frac{d\vec{K}_3}{\prod_{i=1}^3 2\omega_{p_i}}, e^{ip_1x_1+ip_2x_2-ip_3x_3}\\ &\qquad \times e^{-ik_1(x_1-x)-ik_2(x_2-x)-ik_3(x-x_3)}(ik_2^\nu - ik_1^\nu)\\ &= i\frac{e\hbar^2}{(2\pi)^{\frac{d-3}{2}}}(p_2^\nu - p_1^\nu)\delta(p_1 + p_2 - p_3).\end{aligned} \tag{12.36}$$

The corresponding diagram reads:

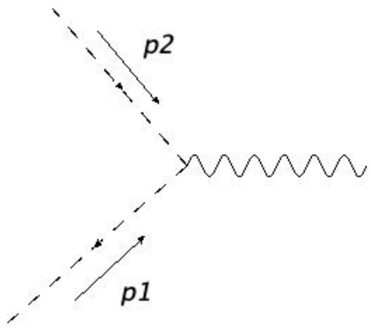


Figure 12.6: The Feynman rule is $ie(2\pi)^{\frac{d-1}{2}}\hbar^2(p_1^\nu - p_2^\nu)$

12.2.5 V) Pair annihilate and create two photons $\phi\phi^* \to \gamma\gamma$

This process involves the other part of the interaction Lagrangian $L_{int} = e^2\int dx A_\mu(x)A^\mu(x)|\phi(x)|^2$. Nevertheless the calculations are not much different from the other processes.

Amplitude:

$$\begin{aligned}\mathcal{T}_1 &= \frac{i}{\hbar}\omega_0(\overline{\phi}^*(p_1)\overline{\phi}(p_2) \star L_{int} \star A^\mu(p_3)A^\nu(p_4))\\ &= \frac{ie^2}{\hbar}\int dx d\vec{\sigma}_4\, e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4}\\ &\quad \omega_0(\phi(x_1)\phi^*(x_2) \star \phi^*(x)\phi(x))(A^\alpha(x)A_\alpha(x) \star A^\mu(x_3)A^\nu(x_4)).\end{aligned} \tag{12.37}$$

Star product:

$$
\begin{aligned}
&\phi(x_1)\phi^*(x_2) \star \phi^*(x)\phi(x))(A^\alpha(x)A_\alpha(x) \star A^\mu(x_3)A^\nu(x_4)) \\
&\overset{|\text{fields}|=0}{=} \hbar^2 \Delta^+_\phi(x_1 - x)\Delta^+_\phi(x_2 - x)(2\hbar^2 \eta^{\nu\mu}\Delta^+_A(x - x_3)\Delta^+_A(x - x_4)) \\
&= 2\hbar^4 \delta^{\mu\nu}\Delta^+_\phi(x_1 - x)\Delta^+_\phi(x_2 - x)\Delta^+_A(x - x_3)\Delta^+_A(x - x_4).
\end{aligned} \tag{12.38}
$$

The star product of the photon fields deserves more attention:

$$
\begin{aligned}
A^\alpha(x)A_\alpha(x) \star A^\mu(x_3)A^\nu(x_4) &= \eta^{\alpha\beta}A_\alpha(x)A_\beta(x) \star A^\mu(x_3)A^\nu(x_4) \\
&= \eta^{\alpha\beta}A_\alpha(x)A_\beta(x)A^\mu(x_3)A^\nu(x_4) \\
&- \eta^{\alpha\beta}\hbar\left(\int dwdz \frac{\delta A_\alpha(x)A_\beta(x)}{\delta A_\rho(w)}\Delta^+_A(w - z)\frac{\delta A^\mu(x_3)A^\nu(x_4)}{\delta A^\rho(z)}\right) \\
&+ \frac{\hbar^2}{2}\eta_{\alpha\beta}\left(\int dW_2 dZ_2 \frac{\delta^2 A_\alpha(x)A_\beta(x)}{\delta A_\sigma(z_1)\delta A_\rho(z_2)}\Delta^+_A(z_1 - w_1)\Delta^+_A(z_2 - w_2)\right. \\
&\times \left.\frac{\delta A^\mu(x_3)A^\nu(x_4)}{\delta A^\sigma(w_1)\delta A^\rho(w_2)}\right).
\end{aligned} \tag{12.39}
$$

The important term for us is the second one. Now we can use our functional derivative rule to compute:

$$
\begin{aligned}
\frac{\delta^2 A_\alpha(x)A_\beta(x)}{\delta A_\sigma(z_1)\delta A_\rho(z_2)} &= \frac{\delta}{\delta A_\sigma(z_1)}\left(\frac{\delta A_\alpha(x)A_\beta(x)}{\delta A_\rho(z_2)}\right) \\
&= \frac{\delta}{\delta A_\sigma(z_1)}\delta_{\alpha\rho}(\delta(x - z_2)A_\beta(x) + \delta_{\beta\rho}\delta(x - z_2)A^\alpha(x)) \\
&= (\delta_{\alpha\rho}\delta_{\beta\sigma} + \delta_{\alpha\sigma}\delta_{\beta\rho})\delta(x - z_2)\delta(x - z_1).
\end{aligned} \tag{12.40}
$$

Similarly for the other derivative:

$$
\frac{\delta A^\mu(x_3)A^\nu(x_4)}{\delta A^\sigma(w_1)\delta A^\rho(w_2)} = \delta^{\mu\sigma}\delta^{\rho\nu}\delta(x_3 - w_1)\delta(x_4 - w_2) + \delta^{\mu\rho}\delta^{\nu\sigma}\delta(x_3 - w_2)\delta(x_4 - w_1). \tag{12.41}
$$

Now we plug the derivatives in the last line of (12.39)

$$
\begin{aligned}
&\frac{\hbar^2}{2}\eta^{\alpha\beta}\int dW_2 dZ_2 (\delta_{\alpha\rho}\delta_{\beta\sigma} + \delta_{\alpha\sigma}\delta_{\beta\rho})\delta(x - z_2)\delta(x - z_1)\Delta^+_A(z_1 - w_1)\Delta^+_A(z_2 - w_2) \\
&\quad\times (\delta^{\mu\sigma}\delta^{\rho\nu}\delta(x_3 - w_1)\delta(x_4 - w_2) + \delta^{\mu\rho}\delta^{\nu\sigma}\delta(x_3 - w_2)\delta(x_4 - w_1)) \\
&\quad= \frac{\hbar^2}{2}\eta^{\alpha\beta}\Delta^+_A(x - x_3)\Delta^+_A(x - x_4)(\delta^{\alpha\nu}\delta^{\beta\mu} + \delta^{\alpha\mu}\delta^{\beta\nu} + \delta^{\alpha\mu}\delta^{\beta\nu} + \delta^{\alpha\nu}\delta^{\beta\mu}) \\
&\quad= \hbar^2\Delta^+_A(x - x_3)\Delta^+_A(x - x_4)\eta^{\alpha\beta}(\delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\mu\beta}\delta^{\nu\alpha}) \\
&\quad= \hbar^2\Delta^+_A(x - x_3)\Delta^+_A(x - x_4)(\eta^{\mu\nu} + \eta^{\nu\mu}) = 2\hbar^2\eta^{\mu\nu}\Delta^+_A(x - x_3)\Delta^+_A(x - x_4).
\end{aligned} \tag{12.42}
$$

In the above equation we have used $\eta^{\mu\nu} = \eta^{\nu\mu}$.
Hence, the amplitude is:

$$
\begin{aligned}
\mathcal{T}_1 &= 2ie^2\hbar^3\eta^{\mu\nu}\frac{1}{(2\pi)^{4(d-1)}}\int dx d\vec{\sigma}_4 \frac{d\vec{K}_4}{\prod_{i=1}^4 2\omega_{p_i}} \\
&\quad e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4} e^{-ik_1(x_1-x)-ik_2(x_2-x)-ik_3(x-x_3)-ik_4(x-x_4)} \\
&= 2i\frac{e^2\hbar^3}{(2\pi)^{d-2}}\eta^{\mu\nu}\delta(p_1+p_2-p_3-p_4). \qquad (12.43)
\end{aligned}
$$

The corresponding diagram reads:

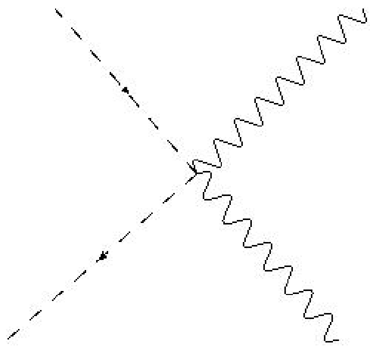


Figure 12.7: The Feynman rule is $2i\frac{e^2\hbar^3}{(2\pi)^{d-2}}\eta^{\mu\nu}$

12.2.6 VI) Particle/antiparticle light scattering $\phi\gamma \to \phi\gamma$

The amplitude:

$$
\begin{aligned}
\mathcal{T}_1 &= \omega_0(\overline{\phi}(p_1)\overline{A}^{\mu}(p_2) \star L_{int} \star \phi(p_3)\phi A^{\nu}(p_4)) \\
&= \frac{ie^2}{\hbar}\int dx d\vec{\sigma}_4\, e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4} \\
&\quad \omega_0((\phi^*(x_1) \star \phi^*(x)\phi(x) \star \phi(x_3))(A^{\mu}(x_2) \star A^{\alpha}(x)A_{\alpha}(x) \star A^{\nu}(x_4)). \qquad (12.44)
\end{aligned}
$$

Star product:

$$
\begin{aligned}
&(\phi^*(x_1) \star \phi^*(x)\phi(x) \star \phi(x_3))(A^{\mu}(x_2) \star A^{\alpha}(x)A_{\alpha}(x) \star A^{\nu}(x_4)) \\
&\overset{|\text{fields}|=0}{=} \hbar^2\Delta^+_{\phi}(x_1-x)\Delta^+_{\phi}(x-x_3)(2\hbar^2\eta^{\mu\nu}\Delta^+_A(x_2-x)\Delta^+_A(x-x_4)) \\
&= 2\hbar^4\eta^{\mu\nu}\Delta^+_{\phi}(x_1-x)\Delta^+_{\phi}(x-x_3)\Delta^+_A(x_2-x)\Delta^+_A(x-x_4). \qquad (12.45)
\end{aligned}
$$

The amplitude:

$$\mathcal{T}_1 = 2i\frac{e^2\hbar^3}{(2\pi)^{d-2}}\eta^{\mu\nu}\delta(p_1 + p_2 - p_3 - p_4). \tag{12.46}$$

The corresponding diagram reads:

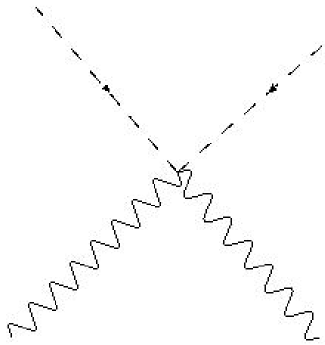


Figure 12.8: The Feynman rule is $= 2i\frac{e^2\hbar^3}{(2\pi)^{d-2}}\eta^{\mu\nu}$

12.3 QED

As discussed in the scalar QED chapter, one can obtain the action of QED starting with the free theory of fermions and imposing invariance under local $U(1)$ charge [17]. Long story short, the action is:

$$S = S_0^{\text{fermion}} + S^{\text{gauge}} + e\int dx\,\overline{\psi}_j(x) \wedge A_\mu(x)\gamma^\mu_{jk}\psi_k(x) \tag{12.47}$$

where

$$\begin{aligned}
&S_0^{\text{fermion}} := \int dx\,\overline{\psi}_j(x) \wedge (i\partial_\mu\gamma^\mu_{jk} + m)\psi_k(x)\\
&S^{\text{gauge}} := \int dx - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu(x))^2\\
&\overline{\psi}_j(x) \wedge A_\mu(x)\gamma^\mu_{jk}\psi_k(x) \equiv A_\mu(x)j^\mu(x), \quad j^\mu(x) := \overline{\psi}_j(x) \wedge \gamma^\mu_{jk}\psi_k(x).
\end{aligned} \tag{12.48}$$

and e is the coupling constant. As usual when working with QED, we denote the particle as e^- (in analogy to the electron), the antiparticle as e^+ (in analogy to the positron), and the gauge field as γ (in analogy to the photon). As an example of application, we compute some amplitudes of QED:

I) Moeller Scattering with anti-particles $e^+e^+ \to e^+e^+$

II) Compton Scattering $e^-\gamma \to e^-\gamma$ (page 195 [67])

III) Bhabha scattering $e^-e^+ \to e^-e^+$

Remark: Due to the fixing of the gauge, we also have the presence of ghosts in full Lagrangian. Nevertheless, since they decouple, we ignore them here.

12.3.1 I)

Since we have two particles in the initial state and two in the final state, we need the second order of the $T-$ matrix to compute the tree-level diagram. It is given by:

$$\begin{aligned}&T_2(L_{int}, L_{int}) = \frac{i^2}{2!\hbar^2} L_{int} \star_F L_{int} = -\frac{e^2}{2\hbar^2} \int dxdy \\ &(\overline{\psi}_j(x) \wedge \gamma^\mu_{jk}\psi_k(x) \star_F \overline{\psi}_l(y) \wedge \gamma^\nu_{lm}\psi_m(y))(A_\mu(x) \star_F A_\nu(y)).\end{aligned} \tag{12.49}$$

Note that in the scattering process, we need two powers of $\overline{\psi}$ and two powers of ψ, hence, we can consider only the first term of the star product:

$$\overline{\psi}_j(x) \wedge \gamma^\mu_{jk}\psi_k(x) \star_F \overline{\psi}_l(y) \wedge \gamma^\nu_{lm}\psi_m(y) = \overline{\psi}_j(x) \wedge \gamma^\mu_{jk}\psi_k(x)\overline{\psi}_l(y) \wedge \gamma^\nu_{lm}\psi_m(y). \tag{12.50}$$

The star product of the photon field can be easily calculated:

$$A_\mu(x) \star_F A_\nu(y) = A_\mu(x)A_\nu(y) - \hbar\eta_{\mu\nu}\Delta^F_A(x-y). \tag{12.51}$$

Since in the scattering process we do not have any photons, the relevant term is $\hbar\eta_{\mu\nu}D^+(x-y)$.

Hence, the second-order $T-$ product can be written as:

$$\overline{\psi}_j(x) \wedge \gamma^\mu_{jk}\psi_k(x)\overline{\psi}_l(y) \wedge \gamma^\nu_{lm}\psi_m(y)\eta_{\mu\nu}D^F(x-y). \tag{12.52}$$

The scattering amplitude is given by:

$$\begin{aligned}\mathcal{T}_2 &= \omega_0(\overline{\psi^-}_{s_1}(p_1)\overline{\psi^-}_{s_2}(p_2) \star T_2 \star \psi^-_{s_3}(p_3)\psi^-_{s_4}(p_4)) \\ &= \frac{1}{(2\pi)^6}\int d\vec{X}_4 dxdy e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4}(-\frac{e^2}{2\hbar^2})(-\hbar D^F(x-y)\eta_{\mu\nu}\gamma^\mu_{jk}\gamma^\nu_{lm}) \\ &\times \gamma^0_{k_1,k_1'}v_{s_1,k_1'}(\vec{p}_1)\gamma^0_{k_2,k_2'}v_{s_2k_2'}(\vec{p}_2)v^\dagger_{s_3,k_3}(\vec{p}_3)v^\dagger_{s_4,k_4}((\vec{p}_4) \\ &\times \omega_0(\overline{\psi}_{k_1}(x_1)\overline{\psi}_{k_2}(x_2) \star \overline{\psi}_j(x)\psi_k(x)\overline{\psi}_l(y)\psi_m(y) \star \psi(x_3)\psi(x_4)).\end{aligned} \tag{12.53}$$

Where we have used the following notation:

$$
\begin{aligned}
\overline{\psi^+}_{s_1}(p_1) &= \int \frac{d\vec{x}_1}{(2\pi)^{\frac{3}{2}}}\, e^{ip_1x_1}\overline{\boldsymbol{\psi}}(x_1)\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1) \equiv \int \frac{d\vec{x}_1}{(2\pi)^{\frac{3}{2}}}\, e^{ip_1x_1}\gamma^0_{k_1,k_1'}v_{s_1,k_1'}(\vec{p}_1)\overline{\psi}_{k_1}(x_1)\\
\overline{\psi^+}_{s_2}(p_2) &= \int \frac{d\vec{x}_2}{(2\pi)^{\frac{3}{2}}}\, e^{ip_2x_2}\overline{\boldsymbol{\psi}}(x_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2) \equiv \int \frac{d\vec{x}_2}{(2\pi)^{\frac{3}{2}}}\, e^{ip_2x_2}\gamma^0_{k_2,k_2'}v_{s_2,k_2'}(\vec{p}_2)\overline{\psi}_{k_2}(x_2)\\
\psi^+_{s_3}(p_3) &= \int \frac{d\vec{x}_3}{(2\pi)^{\frac{3}{2}}} e^{-ip_3x_3}\mathbf{v}^\dagger_{s_3}(\vec{p}_3)\boldsymbol{\psi}(x_3) \equiv \int \frac{d\vec{x}_3}{(2\pi)^{\frac{3}{2}}}\, e^{-ip_3x_3}v^\dagger_{s_3,k_3}(\vec{p}_3)\psi_{k_3}(x_3)\\
\psi^+_{s_4}(p_4) &= \int \frac{d\vec{x}_4}{(2\pi)^{\frac{3}{2}}} e^{-ip_4x_4}\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\boldsymbol{\psi}(x_4) \equiv \int \frac{d\vec{x}_4}{(2\pi)^{\frac{3}{2}}}\, e^{-ip_4x_4}v^\dagger_{s_4,k_4}(\vec{p}_4)\psi_{k_4}(x_4).
\end{aligned}
\tag{12.54}
$$

The relevant star product can be computed in two steps:

$$
\begin{aligned}
&\overline{\psi}_{k_1}(x_1)\overline{\psi}_{k_2}(x_2) \star \overline{\psi}_j(x)\psi_k(x)\overline{\psi}_l(y)\psi_m(y) \overset{|\psi|=2}{=} \frac{\hbar^2}{2}\int dW_2 dZ_2\\
&\left(\frac{\delta_r^2\overline{\psi}_{k_1}(x_1)\overline{\psi}_{k_2}(x_2)}{\delta\overline{\psi}_{u_1}(w_1)\delta\overline{\psi}_{u_2}(w_2)}S^-_{v_1u_1}(z_1-w_1)S^-_{v_2u_2}(z_2-w_2)\frac{\delta^2\overline{\psi}_j(x)\psi_k(x)\overline{\psi}_l(y)\psi_m(y)}{\delta\psi_{v_1}(z_1)\delta\psi_{v_2}(z_2)}\right)\\
&= \frac{\hbar^2}{2}(S^-_{kk_1}S^-_{mk_2} - S^-_{mk_1}S^-_{kk_2} - S^-_{kk_2}S^-_{mk_1} + S^-_{mk_1}S^-_{kk_2})\overline{\psi}_j(x)\overline{\psi}_l(y)\\
&= \hbar^2(S^-_{kk_1}S^-_{mk_2} - S^-_{mk_1}S^-_{kk_2})\overline{\psi}_j(x)\overline{\psi}_l(y).
\end{aligned}
\tag{12.55}
$$

On the other hand:

$$
\begin{aligned}
&\overline{\psi}_j(x)\overline{\psi}_l(y) \star \psi_{k_3}(x_3)\psi_{k_4}(x_4) \overset{|\psi|=0}{=} \frac{\hbar^2}{2}\int dW_2 dZ_2\\
&\times \frac{\delta_r^2\overline{\psi}_j(x)\overline{\psi}_l(y)}{\delta\overline{\psi}_{u_1}(w_1)\delta\overline{\psi}_{u_2}(w_2)}S^-_{v_1u_1}(z_1-w_1)S^+_{v_2u_2}(z_2-w_2)\frac{\delta\psi_{k_3}(x_3)\delta\psi_{k_4}(x_4)}{\delta\psi_{v_1}(z_1)\delta\psi_{v_2}(z_2)}\\
&= \frac{\hbar^2}{2}(S^-_{k_4j}S^-_{k_3l} - S^-_{k_3j}S^-_{k_4l} - S^-_{k_4l}S^-_{k_3j} + S^-_{k_3l}S^-_{k_4j})\\
&= \hbar^2(S^-_{k_4j}S^-_{k_3l} - S^-_{k_3j}S^-_{k_4l}).
\end{aligned}
\tag{12.56}
$$

Hence:

$$
\begin{aligned}
&\omega_0(\overline{\psi}_{k_1}(x_1)\overline{\psi}_{k_2}(x_2) \star \overline{\psi}_j(x)\psi_k(x)\overline{\psi}_l(y)\psi_m(y) \star \psi(x_3)\psi(x_4))\\
&= \hbar^4(S^-_{kk_1}S^-_{mk_2} - S^-_{mk_1}S^-_{kk_2})(S^-_{k_4j}S^-_{k_3l} - S^-_{k_3j}S^-_{k_4l}).
\end{aligned}
\tag{12.57}
$$

Next we insert the gamma matrices to write the amplitude as a product of matrices:

$$
\begin{aligned}
&\gamma^\mu_{jk}\gamma^\nu_{lm}(S^-_{kk_1}S^-_{mk_2} - S^-_{mk_1}S^-_{kk_2})(S^-_{k_4j}S^-_{k_3l} - S^-_{k_3j}S^-_{k_4l}) \\
&\quad = (\mathbf{S}^-(x_4 - x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(x - x_1))_{k_4k_1}(\mathbf{S}^-(x_3 - y)\boldsymbol{\gamma}^\nu\mathbf{S}^-(y - x_2))_{k_3k_2} \\
&\quad - (\mathbf{S}^-(x_3 - x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(x - x_1))_{k_3k_1}(\mathbf{S}^-(x_4 - y)\boldsymbol{\gamma}^\nu\mathbf{S}^-(y - x_2))_{k_4k_2} \\
&\quad - (\mathbf{S}^-(x_4 - x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(y - x_2))_{k_4k_2}(\mathbf{S}^-(x_3 - y)\boldsymbol{\gamma}^\nu\mathbf{S}^-(x - x_1))_{k_3k_1} \\
&\quad + (\mathbf{S}^-(x_3 - x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(x - x_2))_{k_3k_2}(\mathbf{S}^-(x_4 - y)\boldsymbol{\gamma}^\nu\mathbf{S}^-(y - x_1))_{k_4k_1}.
\end{aligned}
\tag{12.58}
$$

Using the above result, we conclude the amplitude is:

$$
\begin{aligned}
\mathcal{T}_2 = \frac{e^2\hbar^3}{2(2\pi)^6} &\int d\vec{X}_4 dx dy \, e^{ip_1x_1 + ip_2x_2 - ip_3x_3 - ip_4x_4} D^+(x - y) \\
&\times \Big((\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\mathbf{S}^-(x_4 - x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(x - x_1)\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1)) \\
&\times (\mathbf{v}^\dagger_{s_3}(\vec{p}_3)\mathbf{S}^-(x_3 - y)\boldsymbol{\gamma}_\mu\mathbf{S}^-(y - x_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2)) \\
&- (\mathbf{v}^\dagger_{s_3}(\vec{p}_3)\mathbf{S}^-(x_3 - x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(x - x_1)\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1)) \\
&\times (\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\mathbf{S}^-(x_4 - y)\boldsymbol{\gamma}_\mu\mathbf{S}^-(y - x_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2)) \\
&- (\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\mathbf{S}^-(x_4 - x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(y - x_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2)) \\
&\times (\mathbf{v}^\dagger_{s_3}(\vec{p}_3)\mathbf{S}^-(x_3 - y)\boldsymbol{\gamma}_\mu\mathbf{S}^-(x - x_1)\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1)) \\
&+ (\mathbf{v}^\dagger_{s_3}(\vec{p}_3)\mathbf{S}^-(x_3 - x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(x - x_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2)) \\
&\times (\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\mathbf{S}^-(x_4 - y)\boldsymbol{\gamma}_\mu\mathbf{S}^-(y - x_1)\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1)) \Big).
\end{aligned}
\tag{12.59}
$$

Before solving the matrix multiplication, we perform the integrals. We do compute explicitly the first term, the others are essentially the same computation. Since $v, v^\dagger$ do not depend on x, x_i we omit then in the following.

$$
\begin{aligned}
&\frac{e^2\hbar^3}{2(2\pi)^6}\int d\vec{X}_4 dx dy \, e^{ip_1x_1 + ip_2x_2 - ip_3x_3 - ip_4x_4} D^F(x - y) \\
&\quad \times (\mathbf{S}^-(x_4 - x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(x - x_1))_{k_4k_1}(\mathbf{S}^-(x_3 - y)\boldsymbol{\gamma}_\mu\mathbf{S}^-(y - x_2))_{k_3k_2} \\
&\quad = \frac{e^2\hbar^3}{2(2\pi)^6}\int d\vec{X}_4 dx dy \, e^{ip_1x_1 + ip_2x_2 - ip_3x_3 - ip_4x_4}\frac{1}{(2\pi)^4}\int dq\,\frac{e^{-iq(x-y)}}{q^2 - i0} \\
&\quad = \frac{e^2\hbar^3}{2(2\pi)^{12}}\int \frac{d\vec{Q}_4}{\prod_{i=1}^4 2\omega_{q_i}}((\not{q}_4 - \mathbf{m})e^{iq_4(x_4 - x)}\boldsymbol{\gamma}^\mu(\not{q}_1 - \mathbf{m})e^{iq_1(x - x_1)})_{k_4k_1} \\
&\quad \times (\not{q}_3 - \mathbf{m})e^{iq_3(x_3 - y)}\boldsymbol{\gamma}_\mu(\not{q}_2 - \mathbf{m})e^{iq_2(y - x_2)})_{k_3k_2}.
\end{aligned}
\tag{12.60}
$$

Note that we can "ignore" the matrices and perform the integral over the exponential, as long as it is done in the right order. The integrals over $dqd\vec{Q}_4$ must be the last to be done, since their result is not simply a delta function. The integrals over the exponential reads:

$$
\begin{aligned}
&\int dxdydqd\vec{X}_4 d\vec{Q}_4 \, e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4-iq(x-y)} \\
&\times e^{iq_1(x-x_1)+iq_2(y-x_2)+iq_3(x_3-y)+iq_4(x_4-x)} \\
=&\int dxdydqd\vec{X}_4 d\vec{Q}_4 \, \times e^{i(q_1-q_4-q)x+i(q_2-q_3+q)y} \\
&\times e^{i(p_1-q_1)x_1+i(p_2-q_2)x_2+i(q_3-p_3)x_3+i(q_4-p_4)x_4}
\end{aligned}
\tag{12.61}
$$

Now we perform the integrals over $dxdyd\vec{X}$ leading to:

$$
\int dqd\vec{Q}_4 (2\pi)^8 (2\pi)^{3\cdot 4} \left(\prod_{i=1}^{4} \delta(\vec{p}_i - \vec{q}_i) \right) \delta(q_1 - q_4 - q)\delta(q + q_2 - q_3). \tag{12.62}
$$

In the expression above, we already cancel the exponential of the form $e^{i(\omega_p - \omega_q)x^0}$. Inserting the result in the amplitude we obtain:

$$
\begin{aligned}
&\frac{\delta(p_1 + p_2 - p_3 - p_4)}{\left(\prod_{i=1}^4 2\omega_{p_i}\right)(2\pi)^2 (p_1 - p_4)^2 - i0} \\
&\times ((\not{\mathbf{p}}_4 - \mathbf{m})\boldsymbol{\gamma}^\mu(\not{\mathbf{p}}_1 - \mathbf{m}))_{k_4k_1}((\not{\mathbf{p}}_3 - \mathbf{m})\boldsymbol{\gamma}_\mu(\not{\mathbf{p}}_2 - \mathbf{m}))_{k_3k_2}.
\end{aligned}
\tag{12.63}
$$

The integral over the other terms are essentially the same; one just has to be careful with the fraction. Thus, the amplitude reads:

$$
\begin{aligned}
\mathcal{T}_2 =& \frac{e^2\hbar^3}{2\left(\prod_{i=1}^4 2\omega_{p_i}\right)(2\pi)^2}\delta(p_1 + p_2 - p_3 - p_4)\Bigg(\\
&\frac{(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)(\not{\mathbf{p}}_4 - \mathbf{m})\boldsymbol{\gamma}^\mu(\not{\mathbf{p}}_1 - \mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1))(\mathbf{v}^\dagger_{s_3}(\vec{p}_3)(\not{\mathbf{p}}_3 - \mathbf{m})\boldsymbol{\gamma}_\mu(\not{\mathbf{p}}_2 - \mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2))}{(p_1 - p_4)^2 - i0} \\
-&\frac{(\mathbf{v}^\dagger_{s_3}(\vec{p}_3)(\not{\mathbf{p}}_3 - \mathbf{m})\boldsymbol{\gamma}^\mu(\not{\mathbf{p}}_1 - \mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1))(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)(\not{\mathbf{p}}_4 - \mathbf{m})\boldsymbol{\gamma}_\mu(\not{\mathbf{p}}_2 - \mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2))}{(p_3 - p_1)^2 - i0} \\
-&\frac{(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)(\not{\mathbf{p}}_4 - \mathbf{m})\boldsymbol{\gamma}^\mu(\not{\mathbf{p}}_2 - \mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2))(\mathbf{v}^\dagger_{s_3}(\vec{p}_3)(\not{\mathbf{p}}_3 - \mathbf{m})\boldsymbol{\gamma}_\mu(\not{\mathbf{p}}_1 - \mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1))}{(p_2 - p_4)^2 - i0} \\
&\frac{(\mathbf{v}^\dagger_{s_3}(\vec{p}_3)(\not{\mathbf{p}}_3 - \mathbf{m})\boldsymbol{\gamma}^\mu(\not{\mathbf{p}}_2 - \mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2))(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)(\not{\mathbf{p}}_4 - \mathbf{m})\boldsymbol{\gamma}_\mu(\not{\mathbf{p}}_1 - \mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1))}{(p_3 - p_2)^2 - i0}\Bigg).
\end{aligned}
\tag{12.64}
$$

Using conservation of momentum $p_1 - p_4 = p_3 - p_2$ and $p_3 - p_1 = p_2 - p_4$ we can simplify the expression to:

$$\mathcal{T}_2 = 2\frac{e^2\hbar^3}{2\left(\prod_{i=1}^4 2\omega_{p_i}\right)(2\pi)^2}\delta(p_1+p_2-p_3-p_4)\Bigg(\frac{(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)(\not{\mathbf{p}}_4-\mathbf{m})\boldsymbol{\gamma}^\mu(\not{\mathbf{p}}_1-\mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1))(\mathbf{v}^\dagger_{s_3}(\vec{p}_3)(\not{\mathbf{p}}_3-\mathbf{m})\boldsymbol{\gamma}_\mu(\not{\mathbf{p}}_2-\mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2))}{(p_1-p_4)^2-i0}$$
$$-\frac{(\mathbf{v}^\dagger_{s_3}(\vec{p}_3)(\not{\mathbf{p}}_3-\mathbf{m})\boldsymbol{\gamma}^\mu(\not{\mathbf{p}}_1-\mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1))(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)(\not{\mathbf{p}}_4-\mathbf{m})\boldsymbol{\gamma}_\mu(\not{\mathbf{p}}_2-\mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2))}{(p_3-p_1)^2-i0}\Bigg). \quad (12.65)$$

Now we use (11.96) and (11.93) to write:

$$\begin{aligned}
\mathbf{v}^\dagger_{s_4}(\vec{p}_4)(\not{\mathbf{p}}_4-\mathbf{m})\boldsymbol{\gamma}^\mu(\not{\mathbf{p}}_1-\mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1) &= 4\omega_{p_1}\omega_{p_4}\overline{\mathbf{v}}_{s_4}(\vec{p}_4)\boldsymbol{\gamma}^\mu\mathbf{v}_{s_1}(\vec{p}_1)\\
\mathbf{v}^\dagger_{s_3}(\vec{p}_3)(\not{\mathbf{p}}_3-\mathbf{m})\boldsymbol{\gamma}_\mu(\not{\mathbf{p}}_2-\mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2) &= 4\omega_{p_2}\omega_{p_3}\overline{\mathbf{v}}_{s_3}(\vec{p}_3)\boldsymbol{\gamma}_\mu\mathbf{v}_{s_2}(\vec{p}_2)\\
\mathbf{v}^\dagger_{s_3}(\vec{p}_3)(\not{\mathbf{p}}_3-\mathbf{m})\boldsymbol{\gamma}^\mu(\not{\mathbf{p}}_1-\mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_1}(\vec{p}_1) &= 4\omega_{p_1}\omega_{p_3}\overline{\mathbf{v}}_{s_3}(\vec{p}_3)\boldsymbol{\gamma}_\mu\mathbf{v}_{s_1}(\vec{p}_1)\\
\mathbf{v}^\dagger_{s_4}(\vec{p}_4)(\not{\mathbf{p}}_4-\mathbf{m})\boldsymbol{\gamma}_\mu(\not{\mathbf{p}}_2-\mathbf{m})\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2) &= 4\omega_{p_2}\omega_{p_4}\overline{\mathbf{v}}_{s_4}(\vec{p}_4)\boldsymbol{\gamma}^\mu\mathbf{v}_{s_2}(\vec{p}_2).
\end{aligned} \quad (12.66)$$

And simplify the expression tremendously:

$$\mathcal{T}_2 = \frac{e^2\hbar^3}{(2\pi)^2}\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$
$$\left((\overline{\mathbf{v}}_{s_4}(\vec{p}_4)\boldsymbol{\gamma}^\mu\mathbf{v}_{s_1}(\vec{p}_1))(\overline{\mathbf{v}}_{s_3}(\vec{p}_3)\boldsymbol{\gamma}_\mu\mathbf{v}_{s_2}(\vec{p}_2))-(\overline{\mathbf{v}}_{s_3}(\vec{p}_3)\boldsymbol{\gamma}_\mu\mathbf{v}_{s_1}(\vec{p}_1))(\overline{\mathbf{v}}_{s_4}(\vec{p}_4)\boldsymbol{\gamma}^\mu\mathbf{v}_{s_2}(\vec{p}_2))\right). \quad (12.67)$$

One can find a totally independent expression from the indices considering the square of the expression above (disconsider the delta function) and averaging over the spins just as done in [67] page 193. The corresponding diagrams are:

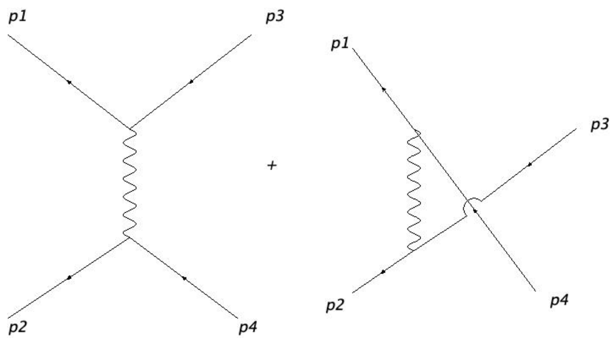


Figure 12.9: The t and u channel

12.3.2 II)

Before computing the amplitude, let us explicitly compute $T_2(j^\mu(x), j^\nu(y))$ using the causal factorization (9.1):

$$
\begin{aligned}
&T_2(j^\mu(x), j^\nu(y)) = \\
&\frac{i^2}{2\hbar^2}\left(T_1(j^\mu(x)) \star T_1(j^\nu(y))\theta(x^0-y^0) + T_1(j^\nu(y)) \star T_1(j^\mu(x))\theta(y^0-x^0)\right).
\end{aligned} \tag{12.68}
$$

Remark: in the formula above we implicitly use the eta trick to get the right sign in the second term. In the process we are studying, we are interested in a term that has both $\overline{\psi}$ and ψ:

$$
\begin{aligned}
&j^\mu(x) \star j^\nu(y) = \gamma^\mu_{k_1k_2}\gamma^\nu_{k_3k_4}(\overline{\psi}_{k_1}(x) \wedge \psi_{k_2}(x) \star \overline{\psi}_{k_3}(y) \wedge \psi_{k_4}(y)) \\
&\overset{|\overline{\psi}|=|\psi|=1}{=} \hbar\gamma^\mu_{k_1k_2}\gamma^\nu_{k_3k_4}\left(S^+_{k_2k_3}(x-y)\overline{\psi}_{k_1}(x) \wedge \psi_{k_4}(y) + S^-_{k_4k_1}(y-x)\psi_{k_2}(x) \wedge \overline{\psi}_{k_3}(y)\right).
\end{aligned} \tag{12.69}
$$

Hence:

$$
\begin{aligned}
&T_2(j^\mu(x), j^\nu(y)) \overset{|\overline{\psi}|=|\psi|=1}{=} \frac{-1}{2\hbar}\hbar\gamma^\mu_{k_1k_2}\gamma^\nu_{k_3k_4} \\
&\Big(S^+_{k_2k_3}(x-y)\overline{\psi}_{k_1}(x)\psi_{k_4}(y)\theta(x^0-y^0) + S^-_{k_2k_3}(x-y)\psi_{k_4}(y)\overline{\psi}_{k_1}(x)\theta(y^0-x^0) \\
&+ S^-_{k_4k_1}(y-x)\psi_{k_2}(x)\overline{\psi}_{k_3}(x)\theta(x^0-y^0) + S^+_{k_4k_1}(y-x)\overline{\psi}_{k_3}(y)\psi_{k_2}(x)\theta(y^0-x^0)\Big) \\
&= \frac{-1}{2\hbar}\hbar\gamma^\mu_{k_1k_2}\gamma^\nu_{k_3k_4}\left(S^F_{k_2k_3}(x-y)\overline{\psi}_{k_1}(x)\psi_{k_4}(y) + S^F_{k_4k_1}(y-x)\overline{\psi}_{k_3}(y)\psi_{k_1}(x)\right) \\
&= \frac{-1}{2\hbar}j^\mu(x) \star_F j^\nu(y) \\
&\equiv -\frac{1}{2\hbar}\left(\overline{\boldsymbol{\psi}}(x)\boldsymbol{\gamma}^\mu\mathbf{S}^F(x-y)\boldsymbol{\gamma}^\nu\boldsymbol{\psi}(y) + \overline{\boldsymbol{\psi}}(y)\boldsymbol{\gamma}^\nu\mathbf{S}^F(y-x)\boldsymbol{\gamma}^\mu\boldsymbol{\psi}(x)\right).
\end{aligned} \tag{12.70}
$$

Now we can compute the amplitude. The incoming electron is represented by:

$$
\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}_1\, e^{ip_1x_1}\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\boldsymbol{\psi}(x) \equiv \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}_1\, e^{ip_1x_1}u^\dagger_{s_1,k_1}(\vec{p}_1)\psi_{k_1}(x). \tag{12.71}
$$

The outgoing electron is represented by

$$
\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}_3\, e^{-ip_3x_3}\overline{\boldsymbol{\psi}}(x_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3) \equiv \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}_3\, e^{-ip_3x_3}\gamma^0_{k_3k_3'}\overline{\psi}_{k_3}(x_3)u_{k_3'}(\vec{p}_3). \tag{12.72}
$$

The incoming photon is represented by:

$$
\frac{2\omega_{p_2}}{(2\pi)^{\frac{3}{2}}}\int d\vec{x}_2\, e^{ip_2x_2}\epsilon_{\mu_2}(\vec{p}_2)A_{\mu_2}(x_2). \tag{12.73}
$$

The outgoing photon is represented by:

$$\frac{2\omega_{p_4}}{(2\pi)^{\frac{3}{2}}} \int d\vec{x}_4 \, e^{-ip_4x_4} \epsilon_{\mu_4}(\vec{p}_4) A_{\mu_4}(x_4). \tag{12.74}$$

Hence, the amplitude for the process $e^-\gamma \to e^-\gamma$ is given by:

$$\begin{aligned} \mathcal{T}_2 = \frac{4\omega_{p_2}\omega_{p_4}}{(2\pi)^6} \int d\vec{X}_4 \, e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4} \epsilon_{\mu_2}(\vec{p}_2)\epsilon_{\mu_4}(\vec{p}_4) \\ \times \left(u^\dagger_{s_1,k_1}(\vec{p}_1)\gamma^0_{k_3k_3'} u_{s_3,k_3'}(\vec{p}_3)\omega_0(\psi_{k_1}(x_1)A_{\mu_2}(x_2) \star T_2(L_{int}, L_{int}) \star \overline{\psi}_{k_3}(x_3)A_{\mu_4}(x_4)) \right) \end{aligned} \tag{12.75}$$

where

$$\begin{aligned} &T_2(L_{int}, L_{int}) \\ &= -\frac{e^2}{2\hbar} \left(\overline{\boldsymbol{\psi}}(x)\boldsymbol{\gamma}^\mu \mathbf{S}^F(x-y)\boldsymbol{\gamma}^\nu \boldsymbol{\psi}(y) + \overline{\boldsymbol{\psi}}(y)\boldsymbol{\gamma}^\nu \mathbf{S}^F(y-x)\boldsymbol{\gamma}^\mu \boldsymbol{\psi}(x) \right) A_\mu(x)A_\nu(y). \end{aligned} \tag{12.76}$$

Now we have to compute the star product. The star product for the photon field is given by:

$$\begin{aligned} &A_{\mu_2}(x_2) \star A_\mu(x)A_\nu(y) \star A_{\mu_4}(x_4) \\ &= -\hbar(\eta_{\mu_2\mu}D^+(x_2-x)A_\nu(y) + \eta_{\mu_2\nu}D^+(x_2-y)A_\mu(x)) \star A_{\mu_4}(x_4) \\ &\overset{|A|=0}{=} \hbar^2 \left(\eta_{\mu_2\mu}\eta_{\nu\mu_4}D^+(x_2-x)D^+(y-x_4) + \eta_{\mu_2\nu}\eta_{\mu\mu_4}D^+(x_2-y)D^+(x-x_4) \right). \end{aligned} \tag{12.77}$$

The star product of the fermion fields (already considering the contractions) leads to:

$$\begin{aligned} &\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\boldsymbol{\psi}(x_1) \star \overline{\boldsymbol{\psi}}(x)\boldsymbol{\gamma}^\mu \mathbf{S}^F(x-y)\boldsymbol{\gamma}^\nu \boldsymbol{\psi}(y) \star \boldsymbol{\gamma}^0 \overline{\boldsymbol{\psi}}(x_3)\mathbf{u}_{s_3}(\vec{p}_3) \\ &\overset{|\psi|=|\overline{\psi}|=0}{=} \hbar^2 \mathbf{u}^\dagger_{s_1}(\vec{p}_1)\mathbf{S}^+(x_1-x)\boldsymbol{\gamma}^\mu \mathbf{S}^F(x-y)\boldsymbol{\gamma}^\nu \mathbf{S}^+(y-x_3)\boldsymbol{\gamma}^0 \mathbf{u}_{s_3}(\vec{p}_3). \end{aligned} \tag{12.78}$$

And similarly for the second term. Hence, the amplitude can be written as:

$$
\begin{aligned}
\mathcal{T}_2 = & \frac{-e^2\hbar^3 4\omega_{p_2}\omega_{p_4}}{2(2\pi)^4}\int d\vec{X}_4 dxdy\, e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4} \\
& \mathbf{u}^{\dagger}_{s_1}(\vec{p}_1)\mathbf{S}^{+}(x_1-x)\epsilon_\mu(\vec{p}_2)\boldsymbol{\gamma}_\mu\mathbf{S}^F(x-y)\epsilon_\nu(\vec{p}_4)\boldsymbol{\gamma}_\nu \\
& \times\mathbf{S}^{+}(y-x_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p_3})D^{+}(x_2-x)D^{+}(y-x_4) \\
& +\mathbf{u}^{\dagger}_{s_1}(\vec{p}_1)\mathbf{S}^{+}(x_1-x)\epsilon_\mu(\vec{p}_4)\boldsymbol{\gamma}_\mu \\
& \times\mathbf{S}^F(x-y)\epsilon_\nu(\vec{p}_2)\boldsymbol{\gamma}_\nu\mathbf{S}^{+}(y-x_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p_3})D^{+}(x_2-y)D^{+}(x-x_4) \\
& +\mathbf{u}^{\dagger}_{s_1}(\vec{p}_1)\mathbf{S}^{+}(x_1-y)\epsilon_\mu(\vec{p}_4)\boldsymbol{\gamma}_\mu\mathbf{S}^F(y-x)\epsilon_\nu(\vec{p}_2)\boldsymbol{\gamma}_\nu \\
& \times\mathbf{S}^{+}(x-x_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p_3})D^{+}(x_2-x)D^{+}(y-x_4) \\
& +\mathbf{u}^{\dagger}_{s_1}(\vec{p}_1)\mathbf{S}^{+}(x_1-y)\epsilon_\mu(\vec{p}_2)\boldsymbol{\gamma}_\mu\mathbf{S}^F(y-x)\epsilon_\nu(\vec{p}_4)\boldsymbol{\gamma}_\nu \\
& \times\mathbf{S}^{+}(x-x_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p_3})D^{+}(x_2-y)D^{+}(x-x_4)
\end{aligned}
\tag{12.79}
$$

The expression above is big, but can be simplified. First of all, note that since $\epsilon_0 = 0$:

$$
\epsilon_\mu\gamma_\mu = -\epsilon_\mu\gamma^\mu = -\sum_{i=1}^{3}\epsilon_i\gamma^i = \not{\epsilon}. \tag{12.80}
$$

Since we have two contractions, we can change $\epsilon_\mu\gamma_\mu \to \not{\epsilon}$ and the overall sign is correct.

The integrals that have to be done are essentially integrals over exponential. For example, the integrals of the first line are:

$$
\begin{aligned}
& \frac{1}{(2\pi)^{12}(2\pi)^4}\int dxdyd\vec{X}_4\, e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4} \\
& \times\int\frac{d\vec{K}_4dk}{\left(\prod_{i=1}^4 2\omega_{p_i}\right)}e^{-ik_1(x_1-x)-ik(x-y)-ik_3(y-x_3)-ik_2(x_2-x)-ik_4(y-x_4)} \\
= & \frac{(2\pi)^4}{\left(\prod_{i=1}^4 2\omega_{p_i}\right)}\int d\vec{K}_4dk\left(\prod_{i=1}^4\delta(\vec{p}_i-\vec{k}_i)\right)\delta(\vec{k}_1+\vec{k}_2-\vec{k})\delta(\vec{k}-\vec{k}_3-\vec{k}_4)
\end{aligned}
\tag{12.81}
$$

The other integrals are essentially the same. Hence, the result is

$$
\begin{aligned}
\mathcal{T}_2 = &\frac{-e^2 i\hbar^3 4\omega_{p_2}\omega_{p_4}}{2(2\pi)^2\left(\prod_{i=1}^4 2\omega_{p_i}\right)}\\
&\mathbf{u}^\dagger_{s_1}(\vec{p}_1)(\not{\mathbf{p}}_1+\mathbf{m})\not{\epsilon}(\vec{p}_2)\frac{\not{\mathbf{p}}_1+\not{\mathbf{p}}_2+\mathbf{m}}{(p_1+p_2)^2-m^2}\not{\epsilon}(\vec{p}_4)(\not{\mathbf{p}}_3+\mathbf{m})\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\\
&+\mathbf{u}^\dagger_{s_1}(\vec{p}_1)(\not{\mathbf{p}}_1+\mathbf{m})\not{\epsilon}(\vec{p}_4)\frac{\not{\mathbf{p}}_1-\not{\mathbf{p}}_4+\mathbf{m}}{(p_1-p_4)^2-m^2}\not{\epsilon}(\vec{p}_2)(\not{\mathbf{p}}_3+\mathbf{m})\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\\
&+\mathbf{u}^\dagger_{s_1}(\vec{p}_1)(\not{\mathbf{p}}_1+\mathbf{m})\not{\epsilon}(\vec{p}_4)\frac{\not{\mathbf{p}}_1-\not{\mathbf{p}}_4+\mathbf{m}}{(p_1-p_4)^2-m^2}\not{\epsilon}(\vec{p}_2)(\not{\mathbf{p}}_3+\mathbf{m})\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\\
&+\mathbf{u}^\dagger_{s_1}(\vec{p}_1)(\not{\mathbf{p}}_1+\mathbf{m})\not{\epsilon}(\vec{p}_2)\frac{\not{\mathbf{p}}_1+\not{\mathbf{p}}_2+\mathbf{m}}{(p_1+p_2)^2-m^2}\not{\epsilon}(\vec{p}_4)(\not{\mathbf{p}}_3+\mathbf{m})\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\\
&= -\frac{e^2 i\hbar^3}{(2\pi)^2 4\omega_{p_1}\omega_{p_3}}\\
&\times\left(\mathbf{u}^\dagger_{s_1}(\vec{p}_1)(\not{\mathbf{p}}_1+\mathbf{m})\not{\epsilon}(\vec{p}_2)\frac{\not{\mathbf{p}}_1+\not{\mathbf{p}}_2+\mathbf{m}}{(p_1+p_2)^2-m^2}\not{\epsilon}(\vec{p}_4)(\not{\mathbf{p}}_3+\mathbf{m})\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\right.\\
&\left.+\mathbf{u}^\dagger_{s_1}(\vec{p}_1)(\not{\mathbf{p}}_1+\mathbf{m})\not{\epsilon}(\vec{p}_4)\frac{\not{\mathbf{p}}_1-\not{\mathbf{p}}_4+\mathbf{m}}{(p_1-p_4)^2-m^2}\not{\epsilon}(\vec{p}_2)(\not{\mathbf{p}}_3+\mathbf{m})\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\right).
\end{aligned} \tag{12.82}
$$

Finally, we use (11.93) and (11.96) to write:

$$
\begin{aligned}
&\mathbf{u}^\dagger_{s_1}(\vec{p}_1)(\not{\mathbf{p}}_1+\mathbf{m}) = 2\omega_{p_1}\overline{\mathbf{u}}_{s_1}(\vec{p}_1)\\
&(\not{\mathbf{p}}_3+\mathbf{m})\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3) = 2\omega_{p_3}\mathbf{u}_{s_3}(\vec{p}_3).
\end{aligned} \tag{12.83}
$$

and obtain:

$$
\begin{aligned}
\mathcal{T}_2 = -\frac{ie^2\hbar^3}{(2\pi)^2}&\left(\overline{\mathbf{u}}_{s_1}(\vec{p}_1)\not{\epsilon}(\vec{p}_2)\frac{\not{p}_1+\not{p}_2+\mathbf{m}}{(p_1+p_2)^2-m^2}\not{\epsilon}(\vec{p}_4)\mathbf{u}_{s_3}(\vec{p}_3)\right.\\
&\left.+\,\overline{\mathbf{u}}_{s_1}(\vec{p}_1)\not{\epsilon}(\vec{p}_4)\frac{\not{p}_1-\not{p}_4+\mathbf{m}}{(p_1-p_4)^2-m^2}\not{\epsilon}(\vec{p}_2)\mathbf{u}_{s_3}(\vec{p}_3)\right)\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4).
\end{aligned} \tag{12.84}
$$

The calculation of the cross-section from the amplitude above can be found at page 198 [67].

The diagrams are:

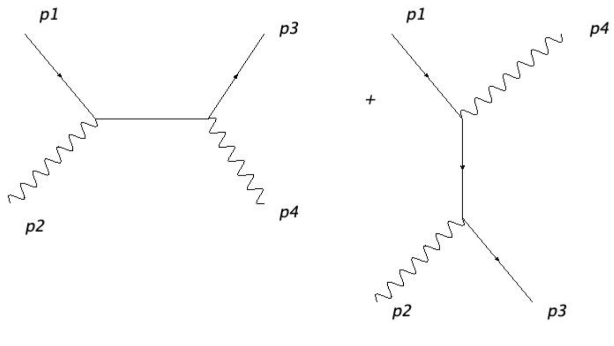


Figure 12.10: They are respectively the s and u channels

Remark: The above formula is slightly different from the ones in the usual QFT books because we compute S_{if} instead of S_{fi}. However, it is easy to recover the formula S_{fi} from S_{if}, just change the indices $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ in the formula above. Then we recover the usual result displayed, for example, in [18].

12.3.3 III)

The computation of Bhabha scattering is very similar to Möller scattering. We consider:

- The incoming electron is represented by

$$\int \frac{d\vec{x}_1}{(2\pi)^{\frac{3}{2}}}\, e^{ip_1x_1}\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\boldsymbol{\psi}(x_1) \equiv \int \frac{d\vec{x}_1}{(2\pi)^{\frac{3}{2}}}\, e^{ip_1x_1}u^\dagger_{s_1,k_1}(\vec{p}_1)\psi_{k_1}(x_1). \tag{12.85}$$

- The outgoing electron is represented by

$$\int \frac{d\vec{x}_3}{(2\pi)^{\frac{3}{2}}}\, e^{-ip_3x_3}\overline{\boldsymbol{\psi}}(x_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3) \equiv \int \frac{d\vec{x}_3}{(2\pi)^{\frac{3}{2}}}\, e^{-ip_3x_3}\gamma^0_{k_3k'_3}\overline{\psi}_{k_3}(x_3)u_{s_3,k'_3}(\vec{p}_3). \tag{12.86}$$

- The incoming positron is represented by

$$\int \frac{d\vec{x}_2}{(2\pi)^{\frac{3}{2}}}\, e^{ip_2x_2}\overline{\boldsymbol{\psi}}(x_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2) \equiv \int \frac{d\vec{x}_2}{(2\pi)^{\frac{3}{2}}}\, e^{ip_2x_2}\gamma^0_{k_2,k'_2}v_{s_2,k'_2}(\vec{p}_2)\overline{\psi}_{k_2}(x_2). \tag{12.87}$$

- The outgoing positron is represented by:

$$\int \frac{d\vec{x}_4}{(2\pi)^{\frac{3}{2}}} e^{-ip_4x_4}\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\boldsymbol{\psi}(x_4) \equiv \int \frac{d\vec{x}_4}{(2\pi)^{\frac{3}{2}}}\, e^{-ip_4x_4}v^\dagger_{s_4,k_4}(\vec{p}_4)\psi_{k_4}(x_4). \tag{12.88}$$

- The relevant term of the $T-$ product is

$$\frac{-e^2}{2\hbar}\int dxdy\, \left(\overline{\psi}_j(x)\wedge\gamma^\mu_{jk}\psi_k(x)\overline{\psi}_l(y)\wedge\gamma^\nu_{lm}\psi_m(y)\eta_{\mu\nu}D^F(x-y)\right). \tag{12.89}$$

The amplitude is given by:

$$\begin{aligned}
\mathcal{T}_2 = & \frac{-e^2}{2\hbar(2\pi)^4}\int d\vec{X}_4dxdy\, e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4} \\
& \times u^\dagger_{s_1,k_1}(\vec{p}_1)\gamma^0_{k_3k'_3}u_{s_3,k'_3}(\vec{p}_3)\gamma^\mu_{jk}D^F(x-y)\gamma_{\mu,lm}\gamma^0_{k_2,k'_2}v_{s_2,k'_2}(\vec{p}_2)v^\dagger_{s_4,k_4}(\vec{p}_4) \\
& \times \omega_0\left(\psi_{k_1}(x_1)\overline{\psi}_{k_2}(x_2)\star\overline{\psi}_j(x)\psi_k(x)\overline{\psi}_l(y)\psi_m(y)\star\overline{\psi}_{k_3}(x_3)\psi_{k_4}(x_4)\right).
\end{aligned} \tag{12.90}$$

The computation of the star product is immediate:

$$
\begin{aligned}
&\psi_{k_1}(x_1)\overline{\psi}_{k_2}(x_2)\star\overline{\psi}_j(x)\psi_k(x)\overline{\psi}_l(y)\psi_m(y)\\
\overset{|\psi|=|\overline{\psi}|=1}{=}\ & -\hbar^2\left(S^-_{kk_2}S^+_{k_1j}\overline{\psi}_l\psi_m - S^-_{kk_2}S^+_{k_1l}\overline{\psi}_j\psi_m + S^-_{mk_2}S^+_{k_1j}\psi_k\overline{\psi}_l + S^-_{mk_2}S^+_{k_1l}\overline{\psi}_j\psi_k\right).
\end{aligned} \tag{12.91}
$$

Once again, we use the indices to omit the argument of the fields and propagators. The expression above can be simplified if we consider the symmetry $x \leftrightarrow y$, leading to:

$$
-2\hbar^2(S^-_{kk_2}(x-x_2)S^+_{k_1j}(x_1-x)\overline{\psi}_l(y)\psi_m(y) - S^-_{mk_2}(y-x_2)S^+_{k_1j}(x_1-x)\overline{\psi}_l(y)\psi_k(x)). \tag{12.92}
$$

Now we compute the star product of the above expression with $\overline{\psi}_{k_3}(x_3)\psi_{k_4}(x_4)$. The result with $|\psi| = |\overline{\psi}|$ is:

$$
\begin{aligned}
-2\hbar^4\Bigg(& S^-_{kk_2}(x-x_2)S^+_{k_1j}(x_1-x)S^+_{mk_3}(y-x_3)S^-_{k_4l}(x_4-y)\\
& -S^-_{mk_2}(y-x_2)S^+_{k_1j}(x_1-x)S^+_{kk_3}(x-x_3)S^-_{k_4l}(x_4-y)\Bigg).
\end{aligned} \tag{12.93}
$$

Thus, the amplitude is:

$$
\begin{aligned}
\mathcal{T}_2 = \frac{e^2\hbar^3}{(2\pi)^4}\int d\vec{X}_4 dx dy\, & e^{ip_1x_1+ip_2x_2-ip_3x_3-ip_4x_4}D^F(x-y)\times\\
\Bigg(&\left(\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\mathbf{S}^+(x_1-x)\boldsymbol{\gamma}^\mu\mathbf{S}^-(x-x_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2)\right)\\
&\times\left(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\mathbf{S}^-(x_4-y)\boldsymbol{\gamma}_\mu\mathbf{S}^+(y-x_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\right)\\
&-\left(\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\mathbf{S}^+(x_1-x)\boldsymbol{\gamma}^\mu\mathbf{S}^+(x-x_3)\boldsymbol{\gamma}^0\mathbf{u}_{s_3}(\vec{p}_3)\right)\\
&\times\left(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\mathbf{S}^-(x_4-y)\boldsymbol{\gamma}_\mu\mathbf{S}^-(y-x_2)\boldsymbol{\gamma}^0\mathbf{v}_{s_2}(\vec{p}_2)\right)\Bigg).
\end{aligned} \tag{12.94}
$$

The next steps are to compute the integrals and use (11.93) and (11.96) to simplify the equation. Since it is basically the same computation as done for Möller scattering, we allow ourselves to skip the computation and present only the final result:

$$
\begin{aligned}
\mathcal{T}_2 =& \frac{-ie^2\hbar^3}{(2\pi)^2}\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\\
&\times\left(\frac{\left(\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\boldsymbol{\gamma}^\mu\mathbf{v}_{s_2}(\vec{p}_2)\right)\left(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\boldsymbol{\gamma}_\mu\mathbf{u}_{s_3}(\vec{p}_3)\right)}{(p_1+p_2)^2-i0}\right.\\
&\left.-\frac{\left(\mathbf{u}^\dagger_{s_1}(\vec{p}_1)\boldsymbol{\gamma}^\mu\mathbf{u}_{s_3}(\vec{p}_3)\right)\left(\mathbf{v}^\dagger_{s_4}(\vec{p}_4)\boldsymbol{\gamma}_\mu\mathbf{v}_{s_2}(\vec{p}_2)\right)}{(p_1-p_3)^2-i0}\right).
\end{aligned} \tag{12.95}
$$

The cross section can be obtained once the formula above is multiplied by its conjugate. This can be fount at [77].

The diagrams are:

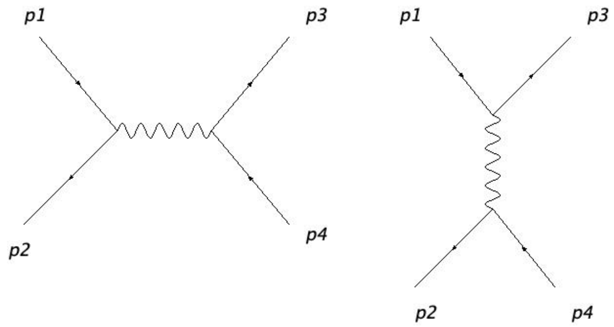


Figure 12.11: The s and t channel representing the annihilation and scattering process

Chapter 13

Conclusion

This dissertation discussed the basics of perturbative expansion of fields and the construction of the scattering matrix using the formalism first developed by Stückelberg, improved by Bogoliubov and finally brought to solid mathematical foundations by Epstein-Glaser. The passage from classical field theory to quantum field theory was done using deformation quantization, a method mostly unknown in the main-stream theoretical physics community. The perturbative expansion, both in the retarded product as well as in the S-matrix, was done inductively and fixed by clear axioms; nevertheless, the formal steps were omitted and the we focus in practical calculations. The same philosophy guides us in the renormalization and computation that involve more sophisticated fields.

13.1 What is new in the present work

This work contributed to the literature by explicit computing the scattering amplitudes to the most usual theories studied in a quantum field theory course. We also highlighted how one can start from the usual Fock space in quantum field theory and "translate" the formulas to the aforementioned formalism allowing one to expand the formalism to theories that are not present in the dissertation.

13.2 What is left to do

Now that we have computed a lot of scattering amplitudes, it is time to talk about important aspects of quantum field theory that we have not mentioned in this work. Firs (and possible mos important): we have not verify that the axioms of T-product are valid for theories that are not the scalar field (specially QED). That is done in the first part of chapter 5 of [24].

In this work, we also did not discuss symmetries or anomalies. As a reference, we suggest [24] Chapter 4 and references therein.

Another important topic taught in a QFT course that was not mentioned here is non-abelian fields and spontaneous symmetry breaking. Non-abelian theories can be studied adapting the notation introduced for fermionic fields. One reference to this subject is [48]. Regarding spontaneous symmetry breaking, we cite [27].

Last but not least, we want to emphasize the relation of the material presented here with algebraic quantum field theory [25] and the last chapter of [34].

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Appendix A

Proof of formula (A.1)

We want to prove that given $\mathcal{V}, \mathcal{W}$ vector spaces, $f(v_1 \otimes ... \otimes v_n) : \mathcal{V}^{\times n} \to \mathcal{W}$ symmetric and linear, than f can be written as

$$f(v_1 \otimes ... \otimes v_n) = \frac{1}{n!} \frac{\partial^n}{\partial\lambda_1 ... \partial\lambda_n}\bigg|_{\lambda_1 = ... = \lambda_n = 0} f((\sum_{k=1}^{n} \lambda_k v_k)^{\otimes n}). \tag{A.1}$$

Where $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$. These can be easily done using induction. For $n = 1$ the statement is trivial:

$$f(\lambda v) = \lambda f(v) \Rightarrow \frac{\partial}{\partial\lambda}\bigg|_{\lambda=0} f(\lambda v) = f(v). \tag{A.2}$$

For pedagogical reasons, we also write the second term explicitly. For simplicity, we use the notation $f(\lambda_1 v_1 + \lambda_2 v_2, \lambda_1 v_1 + \lambda_2 v_2) \equiv f((\lambda_1 v_1 + \lambda_2 v_2) \otimes (\lambda_1 v_1 + \lambda_2 v_2))$:

$$\begin{aligned} &\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}\bigg|_{\lambda_1=\lambda_2=0} f(\lambda_1 v_1 + \lambda_2 v_2, \lambda_1 v_1 + \lambda_2 v_2) \\ &= \frac{\partial}{\partial\lambda_2}\bigg|_{\lambda_2=0} \frac{\partial}{\partial\lambda_2}\bigg|_{\lambda_1=0} f(\lambda_1 v_1 + \lambda_2 v_2, \lambda_1 v_1 + \lambda_2 v_2) \\ &= \frac{\partial}{\partial\lambda_2}\bigg|_{\lambda_2=0} (f(v_1 + \lambda v_2, \lambda v_2) + f(\lambda v_2, v_1 + \lambda v_2)) \\ &= 2\frac{\partial}{\partial\lambda_2}\bigg|_{\lambda_2=0} f(v_1 + \lambda v_2, \lambda v_2) \\ &= 2\bigg(\underbrace{f(v_1 + v_2, 0)}_{=0} + f(v_1, v_2)\bigg) = 2f(v_1, v_2). \end{aligned} \tag{A.3}$$

In the above equation, we have used linearity $f(0, v) = 0 \forall v \in \mathcal{V}$.

Now we assume that for $n - 1$ the formula ((A.1) holds true. Then for n we can compute:

$$\frac{\partial^n}{\partial\lambda_1 ... \partial\lambda_n}\bigg|_{\lambda_1=...=\lambda_n=0} f((\sum_{k=1}^{n}\lambda_k v_k)^{\otimes n})$$
$$= \frac{\partial^{n-1}}{\partial\lambda_1 ... \partial\lambda_{n-1}}\bigg|_{\lambda_1=...=\lambda_{n-1}=0} \frac{\partial}{\partial\lambda_n}\bigg|_{\lambda_n=0} f((\sum_{k=1}^{n}\lambda_k v_k)^{\otimes n}). \tag{A.4}$$

We can "act" in $n-$ different entries with the derivative $\frac{\partial}{\partial\lambda_n}$. Since f is symmetric in the arguments, we can write:

$$\begin{aligned}
&\frac{\partial}{\partial\lambda_n}\bigg|_{\lambda_n=0} f\left[(\sum_{k=1}^{n}\lambda_k v_k)^{\otimes n}\right] \\
&= f\left[(\sum_{k=1}^{n-1}\lambda_k v_k + v_n)\otimes(\sum_{k=1}^{n}\lambda_k v_k)^{\otimes n-1}\right] \\
&+ f\left[(\sum_{k=1}^{n}\lambda_k v_k)\otimes(\sum_{k=1}^{n-1}\lambda_k v_k + v_n)\otimes(\sum_{k=1}^{n}\lambda_k v_k)^{\otimes n-2}\right] + ... \\
&= nf\left[(\sum_{k=1}^{n}\lambda_k v_k)^{\otimes n-1}\otimes(\sum_{k=1}^{n-1}\lambda_k v_k + v_n)\right] \\
&\overset{\lambda_n=0}{=} nf\left[(\sum_{k=1}^{n-1}\lambda_k v_k)^{\otimes n-1}\otimes(\sum_{k=1}^{n-1}\lambda_k v_k + v_n)\right].
\end{aligned} \tag{A.5}$$

The tricky part of the demonstration is the following argument: The terms that the derivatives "act" on the last entry do not contribute, just like the case $n = 2$. Writing the entire proof is a bit complicated and, since it is not the main point of the thesis, not strictly necessary. We will just present the argument: If a derivative "act" on the last entry, there will be $n-2$ derivatives left and $n-1$ λ' to eliminate. Since there are more λ's than derivatives, after applying all derivatives, using the linearity of f, the contribution will be proportional to $\lambda_j f(....) \to 0$. Therefore, to avoid these mismatches the derivatives have to "act" just in the firs $n-1$ terms. That is equivalent to setting:

$$nf\left[(\sum_{k=1}^{n-1}\lambda_k v_k)^{\otimes n-1}\otimes(\sum_{k=1}^{n-1}\lambda_k v_k + v_n)\right] \to nf\left[(\sum_{k=1}^{n-1}\lambda_k v_k)^{\otimes n-1}\otimes v_n\right]. \tag{A.6}$$

With that being said, we can use the induction hypotheses:

$$\frac{\partial^{n-1}}{\partial\lambda_1 ... \partial\lambda_{n-1}}\bigg|_{\lambda_1=...=\lambda_{n-1}=0} nf\left[(\sum_{k=1}^{n-1}\lambda_k v_k)^{\otimes n-1}\otimes v_n\right]$$
$$= n(n-1)! f(v_1, ..., v_n) = n! f(v_1, ..., v_n). \tag{A.7}$$

Just like we wanted to show. An alternative proof of the result can be found at [3] (portuguese only).

Appendix B

Some notions on distributions

B.1 Introduction

The goal of this appendix is to present some notion of the mathematical background underling the physics exposed in the dissertation. We will follow mostly [2].

B.2 Schwartz functions and test functions

We begin the discussion by presenting the definition of a **Schwarz function**. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called a Schwartz function if and only if it is smooth (i.e., infinitely differentiable $\equiv \in C^{\infty}(\mathbb{R}^n)$) and the functions as well as its derivatives decay faster than any polynomial at infinity. The aforementioned condition can be stated as

$$\forall m \in \mathbb{N}, \beta \text{ multi-index} : \quad \lim_{\|x\| \to \infty} (1 + \|x\|)^m |D^{\beta} f(x)| = 0. \tag{B.1}$$

Where D is a differential operator given by:

$$D^{\beta} := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} ... \partial_{x_n}^{\beta_n}}. \tag{B.2}$$

It is easy to see that the set of all Schwartz functions forms a vector space. This vector space is denoted by $\mathcal{J}(\mathbb{R}^n)$. The space is endowed with a norm:

$$\|f\|_{m,\beta} := \sup \left\{ (1 + \|x\|)^m |D^{\beta} f(x)|, \, x \in \mathbb{R}^n \right\}. \tag{B.3}$$

We say that a sequence converges f_n converges to f if and only if $\lim_{n \to \infty} \|f_n - f\|_{m,\beta} = 0$ for all $m \in \mathbb{N}$ and β multi-indexes.

An important subset of Schwartz functions is the set of **test functions**, already mentioned in the first section of the present work. The notion of convergence in the space of test functions is different from the convergence in the space of Schwartz functions. We say $g_n \to g$ if

1 there is a compact $K \subset \mathbb{R}^n$ such that

$$\exists N \in \mathbb{N}, \operatorname{supp} g_n \subseteq K \, \forall n \geq N. \tag{B.4}$$

2

$$\lim_{n\to\infty} \sup_{x\in K} \left\{ |D^\beta (g_n - g)(x)| \right\} = 0. \tag{B.5}$$

Note that convergence in $\mathcal{D}(\mathbb{R}^n)$ implies convergence in $\mathcal{J}(\mathbb{R}^n)$ but the opposite is not true. As a counter example, let $g_n \in \mathcal{D}(\mathbb{R})$ be given by [2]:

$$g_n(x) = \begin{cases} e^{-n^2} \exp\left(-\frac{1}{1-(\frac{x}{n})^2}\right), & x \in (-n, n) \\ 0 & x \notin (-n, n) \end{cases}. \tag{B.6}$$

g_n converges in $\mathcal{J}(\mathbb{R})$ but does not converges in $\mathcal{D}(\mathbb{R})$ (the condition 1 is not satisfied).

The spaces above are important due to the nice properties we can impose on then. First of all, we can define **linear differential operators** acting on $\mathcal{J}(\mathbb{R}^n)$: Given $a_k(x)$ functions limited by polynomial[1] grow and $\alpha^k = (\alpha_1^k, ..., \alpha_n^k)$ multi-indexes, then $\mathcal{L}f \in \mathcal{J}(\mathbb{R}^n)$:

$$(\mathcal{L}f)(x) := \sum_{k=1}^{N} a_k(x) D^{\alpha^k} f(x) \equiv \sum_{k=1}^{N} a_k(x) \frac{\partial^{|\alpha^k|} f}{\partial^{\alpha_1^k} x_1 ... \partial^{\alpha_n^k} x_n}. \tag{B.7}$$

defines a differential linear operator acting on $\mathcal{J}$.

We can define (at least) two products in $\mathcal{J}(\mathbb{R}^n)$: The usual pointwise product $(fg)(x) := f(x)g(x)$ and the convolution product:

$$(f * g)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int dy \, f(x-y) g(y). \tag{B.8}$$

For our porpoises, the most important feature of $\mathcal{J}(\mathbb{R}^n)$ is that the Fourier transform $\mathcal{F}(f)(k)$ (3.28), its complex conjugate $\mathcal{F}^c(f)(-k)$ are bijective maps $\mathcal{F} : \mathcal{J} \to \mathcal{J}$ with $\mathcal{F}^{-1} = \mathcal{F}^c$.

[1] Functions limited by polynomials is to be understood as smooth functions g such that for all multi-index α there is a $C_\alpha \in \mathbb{R}$ and $m_\alpha \in \mathbb{N}$ such that $|D^\alpha g(x)| \leq C_\alpha (1 + \|x\|)^{m_\alpha}$.

B.3 Distributions and tempered distributions

As mentioned in the beginning of the dissertation, the distributions are continuous linear functionals $l : V \to \mathbb{C}$ where V is a complex vector space. Here we have defined two vector spaces, the Schwartz vector space $\mathcal{J}$ and the vector space of the test functions $\mathcal{D}$. We call $T \in \mathcal{D}'(\mathbb{R}^n)$ a distribution and $S \in \mathcal{J}'(\mathbb{R}^n)$ a temperate distribution. Note that since $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{J}(\mathbb{R}^n) \Rightarrow \mathcal{J}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$ and for that reason, we usually call the elements of $\mathcal{J}'(\mathbb{R}^n)$ distributions also. In both cases, the notion of continuity is given by:

$$T \text{ continuous} \iff \forall \|g_k - g\| \to 0 \Rightarrow |T(g_k - g)| \to 0. \tag{B.9}$$

where the norm $\|\cdot\|$ was introduced in the previous section. We emphasize that $T \in \mathcal{D}'(\mathbb{R}^n)$ does not imply $T \in \mathcal{J}'(\mathbb{R}^n)$. An example is the distribution $T \in \mathcal{D}'(\mathbb{R})$ given by

$$T(g(x)) = \int dx\, e^{x^4} g(x). \tag{B.10}$$

Since $g \in \mathcal{D}(\mathbb{R})$ has compact support, the integral above is finite. But that is not true for functions that decay faster than any polynomial, for example $f(x) = e^{-x^2}$.

B.3.1 Divergent integrals

A recurrent problem when working with distributions is divergent integrals. As mentioned in (8.45), usually we need to "integrate" over domains in which the "integral" diverges. In this subsection, we want to explain briefly how one can still make sense of such expressions. To do it, we introduce as an example two distributions: The Cauchy's principal value distribution and the finite part of Hadamard [2]. For simplicity, we restrain ourselves to the one-dimensional case.

Cauchy Principal value

The **Cauchy's principal value distribution** in $\mathcal{D}'(\mathbb{R})$ is defined as the limit:

$$\langle PV_{x_0}, g\rangle := \lim_{r\to 0}\left(\int_{-\infty}^{x_0 - r} dx\, \frac{g(x)}{x - x^0} + \int_{x_0+r}^{\infty} dx\, \frac{g(x)}{x - x_0}\right). \tag{B.11}$$

We will show that this distribution is well defined (the proof originally found at [2]). Without loss of generality, we set $x_0 = 0$. To prove the finitude of the integral, let $\operatorname{supp} g \subset [-A, A]$ for A be sufficiently big. We compute:

$$
\begin{aligned}
&\int_{-A}^{-r} dx\,\frac{g(x)}{x} + \int_{r}^{A} dx\,\frac{g(x)}{x} \\
&= \int_{-A}^{-r} dx\,\frac{g(x)}{x} + \int_{r}^{A} dx\,\frac{g(x)}{x} + \int_{-A}^{-r} dx\,\frac{g(0)-g(0)}{x} + \int_{r}^{A} dx\,\frac{g(0)-g(0)}{x} \\
&= \int_{-A}^{-r} dx\,\frac{g(x)-g(0)}{x} + \int_{r}^{A} dx\,\frac{g(x)-g(0)}{x} + \underbrace{\int_{-A}^{-r} dx\,\frac{g(0)}{x} + \int_{r}^{A} dx\,\frac{g(0)}{x}}_{=0}. \qquad \text{(B.12)}
\end{aligned}
$$

Note that $g(x)-g(0)$ can be written as $xf(x)$, $f(x) \in \mathcal{D}(\mathbb{R})$ [2] prop 39.2. Hence:

$$
\int_{-A}^{-r} dx\,\frac{g(x)}{x} + \int_{r}^{A} dx\,\frac{g(x)}{x} = \int_{-A}^{-r} dx\,\frac{xf(x)}{x} + \int_{r}^{A} dx\,\frac{xf(x)}{x} = \int_{-A}^{A} dx\,f(x) < \infty. \qquad \text{(B.13)}
$$

The continuity of PV_0 can be proved directly by the definition of the distribution. Let $g_k(x) \in \mathcal{D}'(\mathbb{R})$ be a test function that converges to $g(x) = 0\,\forall x \in \mathbb{R}$ [2]. Then:

$$
\begin{aligned}
|\langle PV_0, g_k\rangle| &= \left| \int_{-A}^{-r} dx\,\frac{g_k(x)}{x} + \int_{r}^{A} \frac{g_k(x)}{x} \right| \\
&\leq \int_{-A}^{-r} dx \left| \frac{xf_k(x)}{x} \right| + \left| \int_{r}^{A} \frac{xf_k(x)}{x} \right| \leq 2\sup|f_k(x)|(A-r). \qquad \text{(B.14)}
\end{aligned}
$$

In the above equation, we have used the same notation and trick as in (B.13). Now we have to show that $\sup(f_k(x)) \to 0$ as $k \to \infty$. To do it, let us first consider $x \in (r, A)$:

$$
\begin{aligned}
|xf_k(x)| = x|f_k(x)| = |g_k(x) - g(0)| &= \left| \int_0^x dx\, g_k'(x) \right| \\
&\leq \int_0^x dx\,|\sup(g_k'(x))| = x\|g_k'\|_{0,1} \\
\Rightarrow |f_k(x)| \leq \|g_k'\|_{0,1} &= \sup(g_k'(x)) \to 0. \qquad \text{(B.15)}
\end{aligned}
$$

Hence, $\sup(f_k) \to 0$ and $|\langle PV_0, g_k\rangle| \to 0$ prove that it is a continuous functional and, therefore, a distribution.

If we have to compute higher powers of PV_0 we can simply use the derivative:

$$
\frac{1}{x^{n+1}} = \frac{(-1)^n}{(n-1)!}\frac{d^n}{dx^n}\frac{1}{x}. \qquad \text{(B.16)}
$$

Thus,

[2] that is equivalent to defining $g_k(x) := h(x) - h_k(x)$ where $h_k \to h$ in $\mathcal{D}(\mathbb{R})$

$$
\begin{aligned}
\lim_{r\to 0}\int_{\mathbb{R}\setminus[-r,r]} dx\, \frac{1}{x^{n+1}} g(x) &= \frac{(-1)^n}{(n-1)!}\lim_{r\to 0}\int_{\mathbb{R}\setminus[-r,r]} dx\, \left(\frac{d^n}{dx^n}\frac{1}{x}\right) g(x) \\
&= \frac{(-1)^{2n}}{(n-1)!}\lim_{r\to 0}\int_{\mathbb{R}\setminus[-r,r]} dx\, \frac{1}{x}\left(\frac{d^n}{dx^n} g(x)\right) \\
&= \frac{1}{(n-1)!}\langle PV_0, g^n\rangle. \qquad \text{(B.17)}
\end{aligned}
$$

Note that the same arguments hold for the definition of $PV \in \mathcal{J}'(\mathbb{R})$. As an example of a calculation, we can compute $\langle \frac{1}{x^2}, e^{-x^2}\rangle$:

$$
\begin{aligned}
\langle \frac{1}{x^2}, e^{-x^2}\rangle &= \lim_{r\to 0}\int_{\mathbb{R}\setminus[-r,r]} dx\, \frac{1}{x^2} e^{-x^2} = \lim_{r\to 0}\int_{\mathbb{R}\setminus[-r,r]} dx\, \left(\frac{d}{dx}\frac{-1}{x}\right) e^{-x^2} \\
&= \lim_{r\to 0}\int_{\mathbb{R}\setminus[-r,r]} dx\, \frac{1}{x}\left(\frac{d}{dx} e^{-x^2}\right) = \lim_{r\to 0}\int_{\mathbb{R}\setminus[-r,r]} dx\, \frac{1}{x}\left(-2xe^{-x^2}\right) \\
&= -2\sqrt{\pi}. \qquad \text{(B.18)}
\end{aligned}
$$

Hadamard finite part

Another distribution that "controls" bad-behaved integrals is called **Hadamard finite part**. That distribution is closely related to the regularization scheme in quantum field theory. Before introducing the distribution, we need an auxiliary result ([2]):

Let $(a_k, b_k), k = 1, ..., n$ be a finite collection of complex numbers with $Re(a_k)$, $Re(b_k) \geq 0$ but not simultaneously zero. Then if there are $c_1, ..., c_n \in \mathbb{C}$ such that:

$$
\lim_{x\to 0^+}\left(\sum_{k=1}^{n} c_k \frac{(\ln x)^{b_k}}{x^{a_k}}\right) = \alpha \in \mathbb{C},\ |\alpha| < \infty. \qquad \text{(B.19)}
$$

Then $\alpha = c_1 = ... = c_n = 0$. In addition, we can prove that if $f = s + h$ with $s = \left(\sum_{k=1}^{n} c_k \frac{(\ln x)^{b_k}}{x^{a_k}}\right)$ and $\lim_{x\to 0} h(x) = L$, $|L| < \infty$, then the decomposition of f is unique, i.e, for another combination $f = s' + h'$, $s' = \left(\sum_{k=1}^{n} c_k \frac{(\ln x)^{b_k}}{x^{a_k}}\right)$ we have $s' = s, h' = h$.

The proof of these results can be found in [2] section 39.3.

Equipped with the above results, we define the **Hadamard finite part** of $\int dx\, f(x)$ if for all $r > 0$[3] $\int dx\, f(x)$ can be written as:

$$
\int_{\mathbb{R}\setminus[-r,r]} dx\, f(x) = F(r) + D(r). \qquad \text{(B.20)}
$$

[3] We assume f is singular only in 0

with

$$D(r) = \left(\sum_{k=1}^{n} c_k \frac{(\ln x)^{b_k}}{x^{a_k}} \right) \text{ and } \lim_{x \to 0} F(r) = L, \; |L| < \infty. \tag{B.21}$$

as $\lim_{r\to 0} F(r)$. Due to the results in the beginning of the section, that limit is unique. We denote the Hadamard finite part by $FP \int dx\, f(x)$. Essentially, what we have done is to separate the divergent part in an expression containing only powers of $\frac{1}{x}$ and $\ln(x)$ and take as the result of the integral what is left.

We define the **Hadamard finite part distribution** as:

$$\left\langle FP \left(\frac{1}{(x - x_0)^m} \right), g \right\rangle := FP \int dx, \frac{g(x)}{(x - x_0)^m}. \tag{B.22}$$

The distribution above is closely related to the Cauchy principal value distribution, since:

$$FP \int dx \, \frac{g(x)}{x} = PV \int dx \, \frac{g(x)}{x}. \tag{B.23}$$

To see it, we write $g(x) = g(0) + xf(x)$ and perform the integral over $\operatorname{supp} g \setminus [-r, r]$. For higher orders, we just need to repeat the trick using the derivatives and the construction of $FP \left(\frac{1}{(x - x_0)^m} \right)$.

With both methods mentioned above, we hope that the divergent integrals appearing in quantum field theory are now less tricky.