

Meson Field Theory and Broken Symmetry (Lec. 8) ①

Introduce a field that describes the degrees of freedom of the low energy theory.

The resulting theory \rightarrow SEff that minimizes SEff (Stationary phase). This gives the ground state of the low energy theory. Then we can consider the excitations (collective) around the ground state expanding the functional integral around it.

Let's introduce an example of this procedure in plasma theory.

Interacting Electron Gas

$$S[\psi, \bar{\psi}] = \sum_p \bar{\psi}_p \left(-i\omega_m + \frac{\vec{p}^2}{2m} - \mu \right) \psi_p$$

$$+ \frac{T}{2L^3} \sum_{pp'q} \bar{\psi}_{p+q, \sigma} \bar{\psi}_{p'-q, \sigma'} V(q) \psi_{p, \sigma} \psi_{p', \sigma'}$$

and $V(q) = \frac{4\pi e^2}{|q|^2}$ is the Fourier transform of

the Coulomb potential

The factor of L^3 in the denominator comes from the Fourier transform of $V(\vec{r}) \rightarrow V(\vec{q})$ (2)

$$V(\vec{q}) = \int_0^L d^3x \frac{4\pi e^2}{r} e^{-\vec{r} \cdot \vec{q}} \Rightarrow \frac{1}{L^3} = \frac{1}{V}$$

here the labels include summing over m and $p \equiv (\omega_m, \vec{p})$ is a "four-momentum"

We want to decouple the interaction to obtain a form that is quadratic in the fermion fields so we can do the functional integral.

For this purpose we integrate in a scalar degree of freedom ϕ such that

$$1 \equiv \int \mathcal{D}\phi e^{-\frac{e^2 \beta}{2L^3} \sum_{\vec{q}} \phi_{\vec{q}} V^{-1}(\vec{q}) \phi_{-\vec{q}}}$$

with $\phi = (\omega_m, \vec{q}) \Rightarrow \sum_{\vec{q}} = \sum_{m, \vec{q}_1, \vec{q}_2}$

Before we insert this we can make the shift

(1)

$$\phi_{\vec{q}} \rightarrow \phi_{\vec{q}} + \frac{i}{e\beta} V(\vec{q}) \rho_{\vec{q}}$$

where we defined

$$\rho_{\vec{q}} \equiv \sum_{\vec{p}} \bar{\Psi}_{\vec{p}\sigma} \Psi_{\vec{p}+\vec{q},\sigma}$$

$$\Rightarrow \mathbb{1} = \int \mathcal{D}\phi e^{\frac{1}{L^3} \sum_{\vec{q}} \left\{ -\frac{e^2\beta}{2} \phi_{\vec{q}} V(\vec{q}) \phi_{-\vec{q}} + i e \rho_{\vec{q}} \phi_{-\vec{q}} + \frac{1}{2\beta} \rho_{\vec{q}} V(\vec{q}) \rho_{-\vec{q}} \right\}}$$

↓ This will cancel out quartic interaction

$$\Rightarrow S[\phi, \Psi, \bar{\Psi}] = \frac{\beta}{8\pi L^3} \sum_{\vec{q}} \phi_{\vec{q}} \vec{q}^2 \phi_{-\vec{q}}$$

$$\sum_{\vec{p}, \vec{p}'} \bar{\Psi}_{\vec{p}\sigma} \left[(-i\omega_m + \frac{p^2}{2m} - \mu) \delta_{\vec{p}\vec{p}'} + \frac{ie}{L^3} \phi_{\vec{p}-\vec{p}} \right] \Psi_{\vec{p}'\sigma}$$

→ no fermion quartic interaction!

Transforming back to "space-time":

(4)

$$\phi_{\vec{q}} = \frac{1}{\beta} \int_0^\beta d\tau \int d^d x e^{i\omega\tau - i\vec{q}\cdot\vec{x}} \phi(\tau, \vec{x})$$

and similarly for $\psi_{\vec{p}} \rightarrow \psi_0(\tau, \vec{x})$
we obtain (check)

$$S[\phi, \psi, \bar{\psi}] = \int d\tau \int d^d x \left\{ \frac{1}{2\pi} (\partial_\mu \phi)^2 + \bar{\psi} \left[\partial_\tau - \frac{\partial^2}{2m} - \mu + \frac{ie\phi}{L^d} \right] \psi \right\}$$

\Rightarrow scalar field ϕ is massless and coupled to fermions with a Yukawa coupling $\frac{ie}{L^d}$

This trick (which we already used before!!) is called a Hubbard-Stratonovich transformation

Comment:

(5)

The HS transformation is exact. However the way we introduced the auxiliary field ϕ is not unique. If we generally write the interaction

as

$$V_{\alpha\beta\gamma\delta} \bar{\Psi}_\alpha \Psi_\beta \bar{\Psi}_\gamma \Psi_\delta$$

we can define the auxiliary field in one of 3 ways:

1) Direct channel

The choice we made corresponds to

$$\alpha = \beta = (\sigma, \vec{r})$$

$$\gamma = \delta = (\sigma', \vec{r}')$$

in

$$\frac{1}{2} \int d\vec{r}' \int d\vec{r} \underbrace{\Psi_\sigma(\vec{r}, \tau) \bar{\Psi}_{\sigma'}(\vec{r}', \tau)}_{\hat{P}_{\sigma\sigma'}} \underbrace{\Psi_{\sigma'}(\vec{r}', \tau) \Psi_\sigma(\vec{r}, \tau)}_{\hat{P}_{\sigma\sigma}}$$

2) Exchange channel

$$\hat{P}_{\sigma\sigma'} \hat{P}_{\sigma\sigma'} \quad \text{or} \quad \alpha = \delta \quad \text{But still}$$
$$\beta = \gamma \quad \text{fermion-anti-fermion}$$

3) Cooper channel

The decoupling is done in fermion pair and anti-fermion pair. E_F

$$\hat{\rho}_{\sigma\sigma'}, \hat{\rho}_{\sigma\sigma'}^\dagger$$

where the auxiliary fields will carry fermion number ± 2 !

This is the channel used to model the low energy behavior of a Superconductor.

The choice depends on the approximation that one needs to do afterwards. But any of the three is equally valid.

