8 Spontaneous Symmetry Breaking I: Superfluidity

The phenomenon of superfuluidity is observed at low temperatures in a variety of systems, for example ^{4}He below a critical temperature, and involves flow without dissipation. THe field theoretical description of this phenomenon is associated to that of a non-relativistic gas of interacting bosons. As we will see below, Bose-Einstein condensation in the presence of these interactions will lead to a phenomenon described as spontaneous symmetry breaking. This will be our first encounter with it: the coexistence of a continuous symmetry of a system with a ground state that is not invariant under it. We will first consider the question in second quantized language and then we will formulate a functional integral approach.

8.1 Bose-Einstein Condensation in Second Quantization

We start with the Hamiltonian for a non-relativistic gas of Bose particles. In position space this is written as

$$\begin{aligned} H &= \int d^3x\, \frac{\vec{\nabla}\phi^\dagger(\mathbf{x})\cdot\vec{\nabla}\phi(\mathbf{x})}{2m} \\ &\quad + \frac{g}{2}\int d^3x\, d^3y\, \phi^\dagger(\mathbf{x})\phi^\dagger(\mathbf{y})\phi(\mathbf{y})\phi(\mathbf{x})\,\delta^{(3)}(\mathbf{x}-\mathbf{y}) \ . \end{aligned} \tag{8.1}$$

As an aside, we comment that this Hamiltonian is just the non-relativistic limit of

$$\mathcal{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 \ , \tag{8.2}$$

where the relativistic field $\Phi(x)$ and the non-relativistic one $\phi(\mathbf{x})$ are related by

$$\Phi(x) = \frac{1}{\sqrt{2m}}e^{-imt}\phi(\mathbf{x},t) \ , \tag{8.3}$$

so that $\phi(\mathbf{x},t)$ is slowly varying, and $g=\sqrt{\lambda}/m$. Using the expansion of $\phi(\mathbf{x})$ in momentum space

$$\phi(\mathbf{x}) = \frac{1}{\sqrt{L^3}}\sum_{\mathbf{p}} a_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{x}} \ , \tag{8.4}$$

results in

$$H = \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} + \frac{g}{2L^3} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} a^\dagger_{\mathbf{p}-\mathbf{q}} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} a_{\mathbf{p}} \ . \tag{8.5}$$

We assume the temperature is small enough so that the $\mathbf{p} = 0$ state contains a macroscopically large number of particles N_0. Then, the ground state is $\Omega\rangle$, we have

$$a_{\mathbf{p}=0}|\Omega\rangle = \sqrt{N_0}|N_0 - 1\rangle \ , \tag{8.6}$$

where $|N_0 - 1\rangle$ is the ground state with one less particle in the $\mathbf{p} = 0$ state. Then we can approximate

$$a_{\mathbf{p}=0}|\Omega\rangle \simeq \sqrt{N_0}|\Omega\rangle \ , \tag{8.7}$$

and analogously for $a^\dagger_{\mathbf{p}=0}$. In sum, in this approximation where we assume that a very large number of particles is condensed in the $\mathbf{p} = 0$ state we will make the replacements

$$\begin{aligned} a_{\mathbf{p}=0} &\rightarrow \sqrt{N_0} \\ a^\dagger_{\mathbf{p}=0} &\rightarrow \sqrt{N_0} \ . \end{aligned} \tag{8.8}$$

Then, we can simplify the interaction term in (8.5) by making the replacements (8.8) each time an even number of operators has vanishing momenta. The cases are

$$\begin{array}{lll} \mathbf{p} = 0, \quad \mathbf{k} = 0 & \Rightarrow & N_0 a^\dagger_{-\mathbf{q}}\, a^\dagger_{\mathbf{q}} \\ \mathbf{p} = 0, \quad \mathbf{k} + \mathbf{q} = 0 & \Rightarrow & N_0 a^\dagger_{\mathbf{k}}\, a^\dagger_{\mathbf{k}} \\ \mathbf{p} = 0, \quad \mathbf{p} - \mathbf{q} = 0 & \Rightarrow & N_0 a^\dagger_{\mathbf{k}}\, a^\dagger_{\mathbf{k}} \\ \mathbf{k} = 0, \quad \mathbf{k} + \mathbf{q} = 0 & \Rightarrow & N_0 a^\dagger_{-\mathbf{p}}\, a^\dagger_{\mathbf{p}} \\ \mathbf{k} = 0, \quad \mathbf{p} - \mathbf{q} = 0 & \Rightarrow & N_0 a^\dagger_{\mathbf{p}}\, a^\dagger_{\mathbf{p}} \\ \mathbf{k} + \mathbf{q} = 0, \quad \mathbf{p} - \mathbf{q} = 0 & \Rightarrow & N_0 a^\dagger_{-\mathbf{q}}\, a^\dagger_{\mathbf{q}} \ , \end{array}$$

plus the case when all momentum indices vanish wich results in a factor of N_0^2. With these replacements, the interaction Hamiltonian is approximately given by

$$H_{\text{int}} \simeq \frac{g}{2L^3}\left[N_0^2 + 4N_0 \sum_{\mathbf{p}\neq 0} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + N_0 \sum_{\mathbf{p}\neq 0}\left(a^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}}\right)\right] \ . \tag{8.9}$$

The expression above for the interaction Hamiltonian shows that in this approximation we have broken the $U(1)$ global symmetry

$$\phi(\mathbf{x}) \to e^{i\alpha}\phi(\mathbf{x}) \ , \tag{8.10}$$

where translates in momentum space to

$$a_{\mathbf{p}} \to e^{i\alpha} a_{\mathbf{p}} \ , \tag{8.11}$$

where α is a real constant. This global symmetry, respected by H in (8.5), corresponds to particle number conservation (boson number conservation). In particular, we see that the last two terms in (8.9) break this symmetry. We can rewrite (8.9) by defining the number density

$$n = \frac{N}{L^3} \ , \tag{8.12}$$

where

$$N = N_0 + \sum_{\mathbf{p}\neq 0} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \ , \tag{8.13}$$

is the total number of particles. Using thes definitions to solve in favor of n we obtain

$$H_{\rm int} \simeq gn \sum_{\mathbf{p}\neq 0} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2} gn \sum_{\mathbf{p}\neq 0} \left(a^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} \right) \ , \tag{8.14}$$

where we used that $N - N_0 \ll N_0$, so we neglected second order terms is the sum for $\not= 0$ in(8.13). The complete Hamiltonian can now be written as

$$\boxed{H = \sum_{\mathbf{p}\neq 0} \epsilon_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2} gn \sum_{\mathbf{p}\neq 0} \left(a^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} \right)} \ , \tag{8.15}$$

where we defined

$$\epsilon_{\mathbf{p}} \equiv \frac{\mathbf{p}^2}{2m} + ng \ . \tag{8.16}$$

We see from (8.15) that the Hamiltonian is not diagonal in the creation and annihilation operators $a_{\mathbf{p}}^\dagger$ and $a_{\mathbf{p}}$. We want to define new creation and annihilation operators that would diagonalize H.

Bogoliuvov Transformations

We define the operators $b_{\mathbf{p}}$ and $b_{\mathbf{p}}^\dagger$ through the relation

$$\begin{pmatrix} a_{\mathbf{p}} \\ a_{-\mathbf{p}}^\dagger \end{pmatrix} = \begin{pmatrix} u_{\mathbf{p}} & -v_{\mathbf{p}} \\ -v_{\mathbf{p}} & u_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{p}} \\ b_{-\mathbf{p}}^\dagger \end{pmatrix} , \tag{8.17}$$

where the coefficients $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ are real and satisfy

$$u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2 = 1 , \tag{8.18}$$

which means that the transformation (8.17) has a unit determinant. Then writing the Hamiltonian (8.15) in matrix form as

$$H = \sum_{\mathbf{p}} \begin{pmatrix} a_{\mathbf{p}}^\dagger & a_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} \epsilon_{\mathbf{p}} & \frac{1}{2}ng \\ \frac{1}{2}ng & 0 \end{pmatrix} \begin{pmatrix} a_{\mathbf{p}} \\ a_{-\mathbf{p}}^\dagger \end{pmatrix} , \tag{8.19}$$

and using the rotation (8.17) we arrive at the diagonal form

$$H = \sum_{\mathbf{p}} E_{\mathbf{p}} b_{\mathbf{p}}^\dagger b_{\mathbf{p}} , \tag{8.20}$$

where the energy eigenvalues are

$$\begin{aligned} E_{\mathbf{p}} &= \sqrt{\epsilon_{\mathbf{p}} \left(\epsilon_{\mathbf{p}} + 2ng \right)} \\ &= \sqrt{\frac{\mathbf{p}^2}{2m} \left(\frac{\mathbf{p}^2}{2m} + 2ng \right)} . \end{aligned} \tag{8.21}$$

From (8.21) we see that large values of $\mathbf{p}$ the dispersion is the usual one

$$E_{\mathbf{p}} \simeq \frac{\mathbf{p}^2}{2m} . \tag{8.22}$$

However, at small momenta the dispersion relation becomes linear. That is for $\mathbf{p}^2/2m \ll ng$, we have

$$E_{\mathbf{p}} \simeq \left(\frac{ng}{m}\right)^{1/2} |\mathbf{p}| \ , \tag{8.23}$$

which is the dispersion relation characteristic of a wavelike massless particle. This signals the presence of *gapless* (massless) excitations. As we will see below, these excitations correspond to the Nambu-Goldstone massless states that accompany the spontaneous breaking of a continuous symmetry. Below , we will describe the system in field theoretical terms and will find that there is such a degree of freedom with this dispersion in the low energy theory.

8.2 Superfluidity in the Functional Integral

We start with the quantum partition function for an interacting Bose gas written in the functional formalism for coherent states. This is

$$Z = \int \mathcal{D}(\bar{\phi}, \phi) e^{-S[\bar{\phi},\phi]} \ , \tag{8.24}$$

where the action is given by

$$S[\bar{\phi}, \phi] = \int_0^\beta d\tau \int d^d r \left\{ \bar{\phi}(\mathbf{r}, \tau) \left(\partial_\tau + H_0 - \mu\right) \phi(\mathbf{r}, \tau) + \frac{g}{2} \left(\bar{\phi}(\mathbf{r}, \tau)\phi(\mathbf{r}, \tau)\right)^2 \right\} \ , \tag{8.25}$$

where H_0 is the free Hamiltonian (i.e. the kinetic term) and the interaction term is the coherent state formulation equivalent of that of (8.1) in the previous section. The interaction is assumed to be repulsive, i.e. $g > 0$.

We know that there is a macroscopically large number of particles condensed in the $\mathbf{p} = 0$ state, N_0. Then we are going to treat this as a classical or mean field solution, and then treat the fluctuations around it as perturbations. We will assume that associated field configuration dominating the ground state ϕ_0 (meaning $\mathbf{p} = 0$) is spatially and homogeneous and τ-independent. Then its contribution to the action can be written as

$$S[\bar{\phi}_0, \phi_0] = -\beta\mu \, \bar{\phi}_0\phi_0 + \frac{g}{2}\frac{\beta}{L^d} \left(\bar{\phi}_0\phi_0\right)^2 \ . \tag{8.26}$$

In the first and second terms in (8.26), the factor of β is the result of the integral over τ in (8.25). In the second term the factor of the volume, L^d, appears as a result of using the momentum representation of ϕ, for $\mathbf{p}=0$. In order to find the solution ϕ_0 that dominates $S[\bar{\phi},\phi$ (at low temperatures), we look for the saddle-point solution that must satisfy

$$\frac{\delta S[\bar{\phi}_0,\phi_0]}{\delta\phi_0}=0\ , \tag{8.27}$$

and similarly for $\phi_0\to\bar{\phi}_0$. Then, we obtain the condition

$$\bar{\phi}_0\left(-\mu+\frac{g}{2}\frac{2}{L^d}\,\bar{\phi}_0\phi_0\right)\ , \tag{8.28}$$

Since $g>0$, then if $\mu\leq 0$ the only possible solution of (8.28) is

$$\phi_0=0\ ,\qquad \bar{\phi}_0=0\ . \tag{8.29}$$

On the other hand, for $\mu>0$ there is a non-trivial solution given by

$$\boxed{\bar{\phi}_0\phi_0=\frac{\mu L^d}{g}}\equiv v^2\ , \tag{8.30}$$

so that the ground state is dominated by a non-zero value of ϕ_0 satisfying (8.30). First, we notice that $\bar{\phi}_0\phi_0$ is proportional to the space volume L^d. This makes sense since in fact we will have that this gives N_0, the number of particles in the ground state. More importantly though, there is a continuous degeneracy of the solution since we do not know the phase of ϕ_0. We can express this by writing

$$\phi_0=|\phi_0|\,e^{i\theta}=\sqrt{\frac{\mu L^d}{g}}\,e^{i\theta}\ , \tag{8.31}$$

where θ is an arbitrary phase. However, in order to expand about the ground state, we need to choose a value for the phase θ. This arbitrary choice will result in what is called *spontaneous symmetry breaking.* Here, the symmetry is the global $U(1)$ respected by the action in (8.25) when we transform as

$$\phi(\mathbf{r},\tau)\to e^{i\theta}\phi(\mathbf{r},\tau)\ ,\qquad \bar{\phi}(\mathbf{r},\tau)\to e^{-i\theta}\bar{\phi}(\mathbf{r},\tau)\ . \tag{8.32}$$

In Figure 8.1 we sketch the form of the potential energy of the saddle-point solution. The red circle represents the continuously degenerate minimum defined by (8.30). The

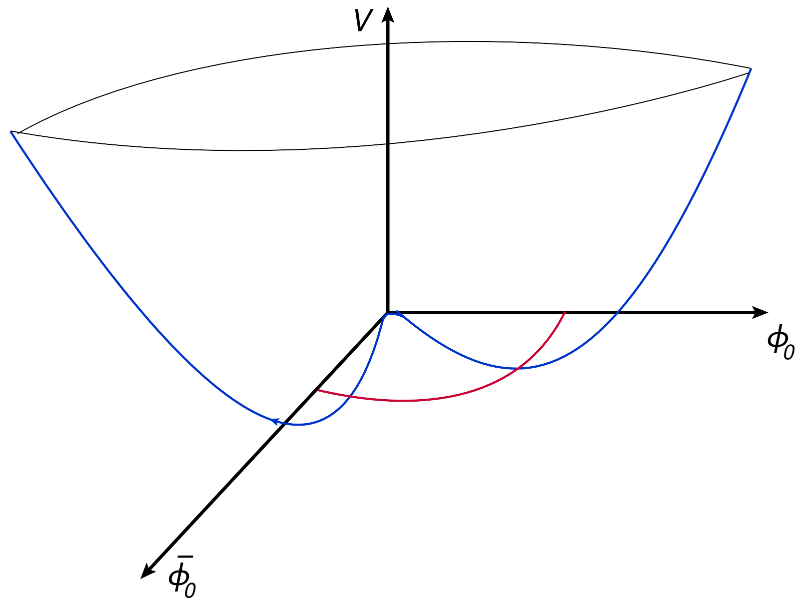

Figure 8.1: The potential of the ground state solution as a function of ϕ_0 and $\bar{\phi}_0$. The blue line is the potential energy of the saddle point solution for $\mu > 0$. Notice that it has rotational symmetry in the $(\phi_0, \bar{\phi}_0)$ plane. The red circle (only a quarter shown) is the locus of the minimum of $V(\phi_0, \bar{\phi}_0)$.

symmetry of the action (8.25) corresponds to this degeneracy. However, since we must choose one value of the phase η in order to expand the fluctuations about it, we will be force to choose a point, any point, in this circle. This can be more clearly seen in Figure 8.2. The ground state solution must satisfy $\bar{\phi}_0\phi_0 = v^2$. This means we must choose an arbitrary value of the phase θ

$$\phi_0 = v\, e^{i\theta_0} \ , \tag{8.33}$$

to expand around this value of ϕ_0. In order to better understand the expansion around the ground state, we are going to change variables, from $(\phi, \bar{\phi})$ to the absolute value and the phase defined by

$$\begin{aligned} \phi(\mathbf{r},\tau) &\equiv \sqrt{\rho(\mathbf{r},\tau)}\, e^{i\theta(\mathbf{r},\tau)} \\ \bar{\phi}(\mathbf{r},\tau) &\equiv \sqrt{\rho(\mathbf{r},\tau)}\, e^{-i\theta(\mathbf{r},\tau)} \ , \end{aligned} \tag{8.34}$$

where $\bar{\phi}(\mathbf{r},\tau)\phi(\mathbf{r},\tau) = \rho(\mathbf{r},\tau)$ is the local density. Then, the mean field solution can be written as

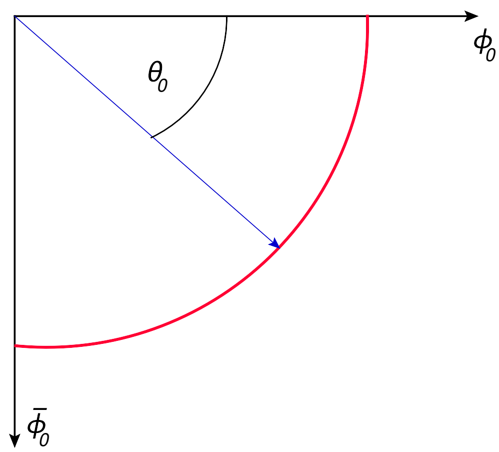

Figure 8.2: The continuously degenerate minimum of the potential energy in the $(\phi_0, \bar{\phi}_0)$ plane. The potential energy axis is coming out of the page. We will have to expand around an arbitrary value of the phase, e.g. θ_0 in order to consider fluctuations above the mean-field solution.

$$\bar{\phi}_0 \phi_0 = \rho_0 \, L^d \ , \tag{8.35}$$

where the volume factor again appears since $\bar{\phi}_0 \phi_0$ is the integral over the whole volume. We will then expand around the ground state density ρ_0. We have

$$\rho(\mathbf{r}, \tau) = \rho_0 + h(\mathbf{r}, \tau) \ , \tag{8.36}$$

where $h(\mathbf{r}, \tau)$ is a small fluctuation in the density around its ground state value ρ_0. Then we can expand

$$\sqrt{\rho(\mathbf{r}, \tau)} \simeq \sqrt{\rho_0} \left(1 + \frac{h(\mathbf{r}, \tau)}{2\rho_0} \right) \ , \tag{8.37}$$

and use this plus (8.32) to write the action (8.25) as

$$S[h, \theta] = \int d\tau \, d^d r \, \left\{ \left(\rho_0 + \frac{h}{2} \right) e^{-i\theta} \left(\partial_\tau + \frac{p^2}{2m} - \mu \right) \left(\rho_0 + \frac{h}{2} \right) e^{i\theta} + \frac{g}{2} \left(\rho_0 + h \right)^2 \right\} \ , \tag{8.38}$$

where for simplicity we omitted the $\mathbf{r}$ and τ dependence in h and θ. Making use of the relation $\rho_0 = \mu/g$, we arrive at

$$S[h,\theta] = \int d\tau\, d^d r \left\{ hi\partial_\tau\theta + \frac{\rho_0}{2m}\left(\vec{\nabla}\theta\right)^2 + \frac{g}{2}h^2 \right.$$

$$\left. + \text{ higher order interaction terms} \right\} . \tag{8.39}$$

We can see from the expression above that the fluctuation in the density, i.e. h, have an energy cost. That is, they correspond to a *massive* field, since there is a quadratic term in h. On the other hand, we see from (8.39) that θ corresponds to a *massless* excitation. It cost no energy to vary it by a constant. This is called *shift* symmetry

$$\theta \to \theta + c \;, \tag{8.40}$$

with c a constant is a symmetry of the action above. θ is a Nambu-Goldstone boson (NGB), a massless state associated with the spontaneous breaking of a continuous symmetry. We will talk about them in more detail. But for now we can just notice that the phase field $\theta(\mathbf{r},\tau)$ shifts by a constant when we travel along the minimum of the potential in Figures 8.1 and 8.2. So choosing an arbitrary point to expand around results in a continuum of states that can be reached from the arbitrarily chosen ground state with phase field value θ_0 by $U(1)$ symmetry transformations without any energy cost. The NGB are these states.

If we integrate out the massive h field we obtain the effective low energy action for the NGB. This is (exercise)

$$\boxed{S_{\text{eff.}}[\theta] = \int d\tau\, d^d r \left\{ \frac{1}{2g}\left(\partial_\tau\theta\right)^2 + \frac{\rho_0}{2m}\left(\vec{\nabla}\theta\right)^2 \right\}} \;. \tag{8.41}$$

We can see the similarity with a harmonic oscillator. This results in a dispersion relation that is linear, the telltale sign of a massless excitation. That is

$$E_{\mathbf{p}} = \sqrt{\frac{g\rho_0}{m}}\,|\mathbf{p}| \;, \tag{8.42}$$

just as we obtained in (8.23) in the previous section using second quantization methods. Finally, we can compute the current

$$\mathbf{j}(\mathbf{r},\tau) = \frac{i}{2m}\left\{ \vec{\nabla}\bar{\phi}(\mathbf{r},\tau)\phi(\mathbf{r},\tau) - \bar{\phi}(\mathbf{r},\tau)\vec{\nabla}\phi(\mathbf{r},\tau) \right\} \;, \tag{8.43}$$

which, neglecting terms such as $\vec{\nabla}h(\mathbf{r},\tau)$, can be rewritten as

$$\boxed{\mathbf{j}(\mathbf{r},\tau)=\frac{\rho_0}{m}\vec{\bigtriangledown}\theta(\mathbf{r},\tau)}\ . \tag{8.44}$$

Then, using the equations of motion for h and θ as derived from (8.39), one can see that there is a static solution for θ, i.e. $\partial_\tau\theta=0$, which implies $h=0$ and results in (exercise)

$$\vec{\bigtriangledown}\cdot\mathbf{j}(\mathbf{r},\tau)=0\ , \tag{8.45}$$

meaning the fluid is incompressible. Thus, there is a stable solution for θ with zero energy cost (since $h=0$) that has a divergenceless current. This is the supercurrent in a superfluid. This is closely related to the fact that the phase θ is a massless excitation. At low energy it is possible to have dissipationless flow.

Additional suggested readings

- *Condensed Matter Field Theory*, Altland and Simons, Section 6.3.
- *Quantum Theory of Many Particle Systems*, A. L. Fetter and J. D. Walecka, Chapter 14.