7 Effective Field Theory of the Interacting Electron Gas

As an exercise in using the functional integral in the coherent state representation it will be useful to consider the case of an interacting electron gas. This will allow us to sample the techniques that will be needed to describe superconductivity, among other systems. We will integrate in degrees of freedom that may or may not become dynamical, depending of the applications. We start with the action in the Matsubara representation

$$\begin{aligned} S[\bar{\psi},\psi] &= \sum_p \bar{\psi}_{p,\sigma}\left(-i\omega_n + \frac{p^2}{2m} - \mu\right)\psi_{p,\sigma} \ , \\ &\quad + \frac{1}{2\beta L^3}\sum_{p,p',q} \bar{\psi}_{p+q,\sigma}\bar{\psi}_{p'-q,\sigma'} V(\mathrm{q})\psi_{p',\sigma'}\psi_{p,\sigma} \ , \end{aligned} \tag{7.1}$$

where use the four-momentum $p = (\omega_n, \mathrm{p})$ for simplicity so the p sum includes the frequency sums, $\sigma = \pm 1$ is the spin projection, and the potential is the Fourier transform of the Coulomb potential given by

$$V(\mathrm{q}) = \frac{4\pi e^2}{|\mathrm{q}|^2} \ . \tag{7.2}$$

We want to decouple the interaction to obtain a form that is quadratic in the fermions fields so that we can perform the functional integral. For this purpose we integrate in a real scalar degree of freedom ϕ by making use of

$$\mathbb{1} \sim \int \mathcal{D}\phi \, e^{-\frac{e^2\beta}{2L^3}\sum_q \phi_q V^{-1}(\mathrm{q})\phi_{-q}} \ . \tag{7.3}$$

This Gaussian functional integral is proportional to the identity so we can introduce it in our funcional integral without loss of generality. But before we do that we will shift the ϕ field by a constant (with respect to the ϕ functional integral). The shift we will choose is

$$\phi_q \to \phi_q + \frac{i}{e\beta} V(\mathrm{q})\rho_q \ , \tag{7.4}$$

where we have defined

$$\rho_q \equiv \sum_p \bar{\psi}_{p,\sigma}\psi_{p+q,\sigma} \ . \tag{7.5}$$

Making this shift in ϕ_q we can rewrite (7.3) as

$$\mathbb{1} \sim \int \mathcal{D}\phi \, e^{(1/L^3)\sum_q\left\{-\frac{e^2\beta}{2L^3}\phi_q V^{-1}(\mathrm{q})\phi_{-q} - ie\rho_q\phi_{-q} + (1/2\beta)\rho_q V(\mathrm{q})\rho_{-q}\right\}} \ . \tag{7.6}$$

Inserting (7.6) in

$$Z = \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\bar{\psi},\psi]} \ , \tag{7.7}$$

we can see that the last term in the exponential above will exactly cancel the interaction in (7.1). The resulting action is

$$S[\bar{\psi}, \psi, \phi] = \frac{\beta}{8\pi L^3}\sum_q \phi_q |\mathrm{q}|^2 \phi_{-q} + \sum_{p,p'} \bar{\psi}_{p,\sigma}\left[\left(-i\omega_n + \frac{p^2}{2m} - \mu\right)\delta_{pp'} + \frac{ie}{L^3}\phi_{p'-p}\right]\psi_{p',\sigma} \ . \tag{7.8}$$

So we decouple the purely fermionic interaction at the cost of introducing a new degree of freedom, the scalar field ϕ. This trick is called the Hubbard-Stratonovich transformation and it will be very useful in many other instances. We can go back to the τ representation using

$$\phi_q = \frac{1}{\sqrt{\beta}}\int_0^\beta d^3x \, e^{i(\omega_q\tau - \mathrm{q}\cdot\mathrm{x})} \, \phi(\tau, \mathrm{x}) \ , \tag{7.9}$$

and similarly for the Fourier transform from $\psi_{p,\sigma} \to \psi_\sigma(\tau, \mathrm{x})$ to get

$$S[\bar{\psi}, \psi, \phi] = \int d\tau \int d^3x \left\{\frac{1}{8\pi}\left(\partial_\mu\phi\right)^2 + \bar{\psi}_\sigma\left[\partial_\tau - \frac{\partial^2}{2m} - \mu + \frac{ie}{L^3}\phi\right]\psi_\sigma\right\} \ , \tag{7.10}$$

The Hubbard-Stratonovich procedure introduced here is exact. We have not made any approximations yet. However, the way we decouple the fermion interaction is not unique. There are other possible choices, all of them valid. The choice we made is called *direct channel* since it couples $\bar{\psi}_\sigma(\tau, \mathrm{x})\psi_\sigma(\tau, \mathrm{x})$ with the scalar field. A second choice is called *exchange channel*, and it couples $\bar{\psi}_\sigma(\tau, \mathrm{x})\psi_{\sigma'}(\tau, \mathrm{x'})$. Finally, there is a third choice coupling

$\psi_\sigma(\tau, \mathrm{x})\psi_{\sigma'}(\tau, \mathrm{x'})$. This is the *Cooper channel* and we can see that it and its conjugate carry fermion number ± 2, whereas the other two cases carry zero fermion number. Which channel is chosen for the decoupling depends on what systems is being described, which determines what approximation needs to be done. We will carry on this example in the direct channel. For describing a superconductor we will make use of the Cooper channel.

Going back to the action in (7.8), we see that since it is now quadratic in the fermion fields we can integrate them out. The result of this procedure is

$$Z = \int \mathcal{D}\phi \, e^{-(\beta/8\pi L^3)\sum_q \phi_q |\mathrm{q}|^2 \phi_{-q}} \det\left[-i\hat{\omega} + \frac{\hat{\mathrm{p}}^2}{2m} - \mu + \frac{ie}{L^3}\hat{\phi}\right] , \tag{7.11}$$

As we have seen many times before, the fermionic determinant can be exponentiated resulting in the effective action for ϕ given by

$$S_{\text{eff}}[\phi] = \frac{\beta}{8\pi L^3}\sum_q \phi_q |\mathrm{q}|^2 \phi_{-q} - \mathrm{Tr}\left[\ln\left(-\mathrm{i}\hat{\omega} + \frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} - \mu + \frac{\mathrm{ie}}{\mathrm{L}^3}\hat{\phi}\right)\right] , \tag{7.12}$$

To go further we need to make some approximation. First, we consider the mean field approximation which results from the saddle point approximation of the functional integral in (7.11). Then, we will consider the fluctuations around the mean field solution.

7.1 Mean Field Solution

To start we consider the saddle point approximation and the simplest solution satisfying it: a constant (classical) background field ϕ. The saddle point solution must satisfy

$$\frac{\delta S_{\text{eff.}}[\phi]}{\delta \phi_q} = 0, \tag{7.13}$$

for all $q \neq 0$, since $\mathrm{q} = 0$ has no propagation. We define the operator

$$\hat{G}^{-1}[\phi] \equiv i\hat{\omega} - \frac{\hat{p}^2}{2m} + \mu - \frac{ie}{L^3}\hat{\phi} \, . \tag{7.14}$$

Then, to obtain an equation for the mean field solution we need to compute

$$\frac{\delta \mathrm{Tr}\left[\ln \hat{\mathrm{G}}^{-1}[\phi]\right]}{\delta \phi_q} \tag{7.15}$$

The question is how to perform the derivative of the trace. It turns out that we can just apply the usual rules of derivation as if the trace was not there. To see this let us consider a function $f(\hat{A})$ where $\hat{A}$ is some operator (for us is the $\ln \hat{G}^{-1}$). Then the derivative of the trace of $f(\hat{A})$ is

$$\partial \mathrm{Tr}\left[\mathrm{f}(\hat{\mathrm{A}})\right] = \partial \mathrm{Tr}\left[\sum_{\mathrm{n}} \frac{\mathrm{f}^{(\mathrm{n})}(0)}{\mathrm{n}!} \hat{\mathrm{A}}^{\mathrm{n}}\right] , \tag{7.16}$$

where ∂ refers to derivation with respect to the variable $\hat{A}$ depends on, $f^{(n)}(0)$ is the n-th derivative of f and we expanded around $\hat{A} = 0$. Then we have

$$\begin{aligned}
\partial \mathrm{Tr}\left[\mathrm{f}(\hat{\mathrm{A}})\right] &= \partial \sum_n \frac{f^{(n)}(0)}{n!} \mathrm{Tr}\left[\hat{\mathrm{A}}^{\mathrm{n}}\right] \\
&= \sum_n \frac{f^{(n)}(0)}{n!} \partial \mathrm{Tr}\left[\hat{\mathrm{A}}^{\mathrm{n}}\right] \\
&= \sum_n \frac{f^{(n)}(0)}{n!} \mathrm{Tr}\left[\partial\hat{\mathrm{A}}\hat{\mathrm{A}}^{\mathrm{n}-1} + \hat{\mathrm{A}}\partial\hat{\mathrm{A}}\hat{\mathrm{A}}^{\mathrm{n}-2} + \cdots + \hat{\mathrm{A}}^{\mathrm{n}-1}\partial\hat{\mathrm{A}}\right] \\
&= \sum_n \frac{f^{(n)}(0)}{n!} n \,\mathrm{Tr}\left[\hat{\mathrm{A}}^{\mathrm{n}-1}\partial\hat{\mathrm{A}}\right] \\
&= \mathrm{Tr}\left[\mathrm{f}'(\hat{\mathrm{A}})\,\partial\hat{\mathrm{A}}\right] ,
\end{aligned} \tag{7.17}$$

which proves that the derivative of the trace is just the trace of the derivative. Going back to (7.15), applying (7.17), we now have

$$\begin{aligned}
\frac{\delta \mathrm{Tr}\left[\ln \hat{\mathrm{G}}^{-1}[\phi]\right]}{\delta \phi_q} &= \mathrm{Tr}\left[\hat{\mathrm{G}}\, \frac{\delta \hat{\mathrm{G}}^{-1}}{\delta \phi_{\mathrm{q}}}\right] \\
&= 2 \sum_{q_1, q_2} \hat{G}_{q_1 q_2} \left(\frac{\delta \hat{G}^{-1}}{\delta \phi_q}\right)_{q_2 q_1} ,
\end{aligned} \tag{7.18}$$

where the factor of 2 comes from the summ over spins $\sigma = \pm 1$. From (7.8). we can read the matrix form for the operator $\hat{G}^{-1}$. This is

$$\hat{G}^{-1}_{q_1 q_2} = \left(-i\hat{\omega} - \frac{\hat{p}^2}{2m} + \mu\right) \delta_{q_1, q_2} - \frac{ie}{L^3} \phi_{q_2 - q_1} , \tag{7.19}$$

which means that the only non-diagonal term is the one depending on ϕ. In fact, $\hat{\phi}$ is *always* off-diagonal since for $q_1 = q_2$ we would have ϕ_0 which is zero due to charge neutrality. Then, we have

$$\left(\frac{\delta \hat{G}^{-1}}{\delta \phi_q} \right)_{q_2 q_!} = -\frac{ie}{L^3} \delta_{q_1 - q_2, q} \ , \tag{7.20}$$

Putting it all together, the saddle point condition (7.13) reads

$$\begin{aligned} \frac{\delta S_{\text{eff.}}[\phi]}{\delta \phi_q} &= \frac{\beta}{4\pi L^3} |\mathrm{q}|^2 \phi_{-q} + 2\frac{ie}{L^3} \sum_{q_1, q_2} \hat{G}_{q_1 q_2} \delta_{q_1 - q_2, q} \\ &= \frac{\beta}{4\pi L^3} |\mathrm{q}|^2 \phi_{-q} + 2\frac{ie}{L^3} \sum_{q_1} \hat{G}_{q_1 (q_1 - q)} = 0 \ . \end{aligned} \tag{7.21}$$

From the last line in (7.21) we can see that for $\mathrm{q} \neq 0$, $\hat{\phi} = 0$ is a solution. This is the case since for $\mathrm{q} \neq 0$ the non-diagonal term $\hat{G}_{q_1(q_1-q)}$ only vanishes if $\phi_q = 0$. Then we conclude that $\phi_q = 0$ for $\mathrm{q} \neq 0$ is a solution of (7.21) and therefore a mean field solution. On the other hand, we can see that this is the only solution that is homogeneous, i.e. a constant independent on n and q since there is a momentum dependence in the first term of (7.21). Then, we conclude that our classical background solution is $\phi_q = 0$ for all $q \neq 0$, Below, we will consider the quantum fluctuations around this background.

7.2 Fluctuations

We expand around the mean field solution $\hat{\phi}_{\text{MF}} = 0$. Thus

$$\hat{\phi} = \hat{\phi}_{\text{MF}} + \delta \hat{\phi} \ , \tag{7.22}$$

where $\delta \hat{\phi}$ represents the small fluctuation. Of course, since the mean field solution vanishes, we can simply use $\hat{\phi} = \delta \hat{\phi}$. We define the mean field value of the operator $\hat{G}^{-1}$ as

$$\hat{G}_0^{-1} \equiv i\hat{\omega} - \frac{\hat{p}^2}{2m} + \mu \ , \tag{7.23}$$

so that to expand around the mean field solution we need to expand

$$\mathrm{Tr}\left[\ln \hat{\mathrm{G}}^{-1} \right] = \mathrm{Tr}\left[\ln \left(\hat{\mathrm{G}}_0^{-1} - \frac{\mathrm{ie}}{\mathrm{L}^3} \hat{\phi} \right) \right] \ , \tag{7.24}$$

where we are already using $\hat{\phi} = \delta\hat{\phi}$ in the right hand side. We now Taylor expand (7.24) around zero. This gives, to second order in $\hat{\phi}$ (i.e. in $\delta\hat{\phi}$)

$$\mathrm{Tr}\left[\ln \hat{\mathrm{G}}^{-1}\right] = \mathrm{Tr}\left[\ln \hat{\mathrm{G}}_0^{-1}\right] - \frac{\mathrm{ie}}{\mathrm{L}^3}\mathrm{Tr}\left[\hat{\mathrm{G}}_0\,\hat{\phi}\right] + \frac{1}{2}\left(\frac{\mathrm{e}}{\mathrm{L}^3}\right)^2 \mathrm{Tr}\left[\hat{\mathrm{G}}_0\,\hat{\phi}\,\hat{\mathrm{G}}_0\,\hat{\phi}\right] + \dots \;, \tag{7.25}$$

The first term in (7.25) is $\hat{\phi}$ independent, so it can be taken out of the functional integral. It gives the partition function for the non-interacting electron gas

$$Z_0 = e^{\mathrm{Tr}\left[\ln \hat{\mathrm{G}}_0^{-1}\right]} = \det\left[\hat{G}_0^{-1}\right] \;. \tag{7.26}$$

The second term must vanish since we are expanding around the mean field solution $\hat{\phi}_{\mathrm{MF}}$ which satisfies

$$\frac{\delta S_{\mathrm{eff.}}[\phi]}{\delta\hat{\phi}} = 0 \;, \tag{7.27}$$

but this term must be proportional to the first derivative evaluated at $hat\phi_{\mathrm{MF}}$.

Then the first contribution beyond the mean field approximations comes from the third term in (7.25). We need to compute

$$\mathrm{Tr}\left[\hat{\mathrm{G}}_0\,\hat{\phi}\,\hat{\mathrm{G}}_0\,\hat{\phi}\right] = 2\sum_{\mathrm{q}_1,\mathrm{q}_2,\mathrm{q}_3,\mathrm{q}_4}\left(\hat{\mathrm{G}}_0\right)_{\mathrm{q}_1\mathrm{q}_2}\hat{\phi}_{\mathrm{q}_3-\mathrm{q}_2}\left(\hat{\mathrm{G}}_0\right)_{\mathrm{q}_3\mathrm{q}_4}\hat{\phi}_{\mathrm{q}_1-\mathrm{q}_4} \;, \tag{7.28}$$

where again the factor of 2 comes from the spins and we have used that

$$\hat{\phi}_{qp} = \hat{\phi}_{p-q} \;. \tag{7.29}$$

Remembering that

$$\left(\hat{G}_0^{-1}\right)_{q_1q_2} = \left(i\hat{\omega} - \frac{\hat{p}^2}{2m} + \mu\right)\delta_{q_1,q_2} \;, \tag{7.30}$$

we obtain

$$\mathrm{Tr}\left[\hat{\mathrm{G}}_0\,\hat{\phi}\,\hat{\mathrm{G}}_0\,\hat{\phi}\right] = 2\sum_{\mathrm{q}_1,\mathrm{q}_3}\hat{\mathrm{G}}_{0,\mathrm{q}_1}\,\hat{\phi}_{\mathrm{q}_3-\mathrm{q}_1}\,\hat{\mathrm{G}}_{0,\mathrm{q}_3}\,\hat{\phi}_{\mathrm{q}_1-\mathrm{q}_3} \;. \tag{7.31}$$

It will be convenient to relabel the indexes as

$$q_1 \to p \ , \qquad q_3 - q_1 \to q \ , \qquad q_3 \to p + q \ , \tag{7.32}$$

which results in

$$\begin{aligned} \mathrm{Tr}\left[\hat{\mathrm{G}}_0\,\hat{\phi}\,\hat{\mathrm{G}}_0\,\hat{\phi}\right] &= 2\sum_{q,p} G_{0,p}\,\phi_q\,G_{0,p+q}\,\phi_{-q} \\ &= \frac{L^3}{T}\sum_q \Pi_q\,\phi_q\,\phi_{-q} \ , \end{aligned} \tag{7.33}$$

where in the last line we have defined

$$\Pi_q \equiv 2\frac{T}{L^3}\sum_q G_{0,p}G_{0,p+q} \ . \tag{7.34}$$

In this way, the effective action of the scalar field ϕ to quadratic order is given by

$$S_{\mathrm{eff.}}[\phi] = \frac{\beta}{8\pi L^3}\sum_q \phi_q|\mathrm{q}|^2\phi_{-q} - \frac{e^2}{2TL^3}\sum_q \Pi_q\phi_q\phi_{-q} \ , \tag{7.35}$$

which can be rewritten as

$$\boxed{S_{\mathrm{eff.}}[\phi] = \frac{1}{2TL^3}\sum_q \phi_q\left(\frac{|\mathrm{q}|^2}{4\pi} - e^2\Pi_q\right)\phi_{-q}} \ . \tag{7.36}$$

We can understand this expression as a first term for the free propagation of the field ϕ and a *self-energy* of ϕ resulting from integrating out the fermions. This expression results in a quadratic functional integral for ϕ itself, which then can be performed. As an application, let us compute the contribution of the self-energy (or the contributions from virtual fermions) to the free energy. We need to compute

$$\Delta F = -T\left(\ln Z - \ln Z_0\right) \ , \tag{7.37}$$

which is

$$\Delta F = \frac{T}{2} \sum_q \ln \left(1 - \frac{4\pi e^2}{|\mathrm{q}|^2} \Pi_q \right) \ . \tag{7.38}$$

The result above could have been obtained by using diagrammatic techniques in perturbation theory plus resummation. They correspond to the contribution to the free energy coming from the creation and annihilation of electrons above the Fermi energy (holes below the Fermi energy) to second order in the interaction. The use of the functional integral techniques above allows a rather simple way of accounting for these contributions. The result in perturbation theory is obtained by using the so-called random phase approximation. The functional integral has allowed us to account for all these diagrams and summed them in a very straightforward way.

Additional suggested readings

- *Condensed Matter Field Theory*, Altland and Simons, Section 4.2.
- *Quantum Theory of Many Particle Systems*, A. L. Fetter and J. D. Walecka, Chapters 1 and 2.