

Functional Integral for the Quantum

(L7)

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Partition Function

Let us consider the partition function (Quantum)

$$Z = \text{Tr} e^{-\beta(H - \mu N)}$$

$$Z = \sum_m \langle m | e^{-\beta(H - \mu N)} | m \rangle$$

We now insert the resolution of the identity operator for coherent states in Fock space

$$\int d(\bar{\phi}, \phi) e^{-\sum_k \bar{\phi}_k \phi_k} |\phi\rangle \langle \phi| = \mathbb{1}_F$$

$$\Rightarrow Z = \int d(\bar{\phi}, \phi) e^{-\sum_k \bar{\phi}_k \phi_k} \sum_m \langle m | \phi \rangle \langle \phi | e^{-\beta(H - \mu N)} | m \rangle$$

using $\sum_m |m\rangle \langle m| = \mathbb{1}$

$$Z = \int d(\bar{\phi}, \phi) e^{-\sum_k \bar{\phi}_k \phi_k} \langle \phi | e^{-\beta(H - \mu N)} | \phi \rangle$$

We assume that H contains 1-body and 2-body operators: generally written as

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$$H = \sum_{ij} h_{ij} a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l$$

We now do the "path integral" thing but using β as it .



defining an infinitesimal time interval

$$\Delta\beta = \frac{\beta}{N}$$

We insert the coherent state resolution of the identity in all intervals. Eg.

$$\int d(\bar{\phi}_m, \phi_m) e^{-\sum_k \bar{\phi}_m^k \phi_m^k} |\phi_m\rangle \langle \phi_m|$$

This,

$$\int d(\bar{\phi}_m, \phi_m) e^{-\sum_k \bar{\phi}_m^k \phi_m^k} \langle \phi_m | e^{-\Delta\beta (H - \mu N)} | \phi_{m+1} \rangle$$

We divide the $(0, \beta)$ interval and insert $\mathbb{1}_E$ (16)

Eq:

$$\sum_m \langle m | e^{-\Delta\beta(H-\mu N)} e^{-\Delta\beta(H-\mu N)} \dots e^{-\Delta\beta(H-\mu N)} e^{-\Delta\beta(H-\mu N)} | m \rangle$$

$$= \sum_m \langle m | \phi_1 \rangle \langle \phi_1 | e^{-\bar{\phi}_1 \phi_1} e^{-\Delta\beta(H-\mu N)} | \phi_2 \rangle \langle \phi_2 | e^{-\bar{\phi}_2 \phi_2} e^{-\Delta\beta(H-\mu N)} | \phi_3 \rangle \langle \phi_3 | e^{-\bar{\phi}_3 \phi_3} e^{\Delta\beta(H-\mu N)} \underbrace{\langle \phi_m | e^{-\bar{\phi}_m \phi_m} e^{-\Delta\beta(H-\mu N)} | \phi_{m+1} \rangle}_{\otimes} \langle \phi_m | m \rangle$$

Here $\bar{\phi}_m \phi_m = \sum_k \bar{\phi}_{m,k} \phi_{m,k}$

and $\int d(\bar{\phi}_m, \phi_m)$ is understood

$$\Rightarrow \otimes = \int d(\bar{\phi}_m, \phi_m) \langle \phi_m | e^{-\frac{\Delta\beta}{\Delta\beta} (\bar{\phi}_m \phi_m + H(a^\dagger, a) - \mu N(a^\dagger, a))} | \phi_{m+1} \rangle$$

But $e^{-\Delta\beta H} | \phi_{m+1} \rangle = e^{-\Delta\beta \sum_{ij} h_{ij} a_i^\dagger a_j} | \phi_{m+1} \rangle$

$$= e^{-\Delta\beta \sum_{ij} h_{ij} a_i^\dagger \phi_j^{m+1}} | \phi_{m+1} \rangle$$

$$\Rightarrow \langle \phi_m | e^{\Delta\beta H} | \phi_{m+1} \rangle = e^{-\Delta\beta \sum_{ij} h_{ij} \bar{\phi}_m^i \phi_{m+1}^j} \langle \phi_m | \phi_{m+1} \rangle$$

but also

$$\langle \Phi_m | \Phi_{m+1} \rangle = e^{\bar{\Phi}_m \Phi_{m+1}} \quad (\text{seems understood})$$

$$\Rightarrow \textcircled{*} = \int \prod_{m=1}^N d(\bar{\Phi}_m, \Phi_m) e^{-\Delta\beta \left(\frac{\bar{\Phi}_m \Phi_m - \bar{\Phi}_m \Phi_{m+1}}{\Delta\beta} + H(\Phi_m, \Phi_{m+1}) - \mu N(\Phi_m, \Phi_{m+1}) \right)}$$

$$= \int \prod_{m=1}^N d(\bar{\Phi}_m, \Phi_m) e^{-\Delta\beta \left(\bar{\Phi}_m \frac{(\Phi_m - \Phi_{m+1})}{\Delta\beta} + H(\Phi_m, \Phi_{m+1}) - \mu N(\Phi_m, \Phi_{m+1}) \right)}$$

$$\Delta S = \sum_{m=1}^N \Delta\tau \left(\bar{\Phi}_m \frac{(\Phi_m - \Phi_{m+1})}{\Delta\tau} + H(\Phi_m, \Phi_{m+1}) - \mu N(\Phi_m, \Phi_{m+1}) \right)$$

with

$$H(\Phi_m, \Phi_{m+1}) = \sum_{ij} h_{ij} \bar{\Phi}_m^i \Phi_{m+1}^j + \sum_{ijkl} V_{ijkl} \bar{\Phi}_m^i \bar{\Phi}_m^j \Phi_{m+1}^k \Phi_{m+1}^l$$

and same for

$$N(\Phi_m, \Phi_{m+1}) = \sum_i \bar{\Phi}_m^i \Phi_{m+1}^i$$

For $\Delta\beta = \Delta\tau \rightarrow 0$ and $N \rightarrow \infty$

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$$Z = \int \mathcal{D}(\bar{\phi}, \phi) e^{-S[\bar{\phi}, \phi]}$$

with

$$S[\bar{\phi}, \phi] \equiv \int_0^\beta d\tau \left\{ \bar{\phi} \partial_\tau \phi + H(\bar{\phi}, \phi) - \mu N(\bar{\phi}, \phi) \right\}$$

and the measure is defined by

$$\mathcal{D}(\bar{\phi}, \phi) = \lim_{N \rightarrow \infty} \prod_{m=1}^N d(\bar{\phi}_m, \phi_m)$$

Finally, the integral requires the boundary conditions

$$\phi(0) = \phi(\beta) \quad ; \quad \text{and} \quad \bar{\phi}(0) = \bar{\phi}(\beta)$$

→ periodic (remember it comes from the trace!)
"Initial" state = "final" state

Comment

This form of the "action" can be thought of (19)

$$\int d\tau (P \dot{\Phi} - H) \text{ with } \dot{\Phi} = P$$

This is the "time" representation (or temperature) of the functional integral

We must understand $\phi(\tau)$ and $\Phi(\tau)$ energy where

Frequency Representation:

The periodic boundary conditions \Rightarrow we can write the Fourier transformation as series:

$$\phi(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_m} \phi_m e^{-i\omega_m \tau}$$

and inversely

$$\phi_m = \frac{1}{\sqrt{\beta}} \int_0^{\beta} d\tau \phi(\tau) e^{i\omega_m \tau}$$

with $\omega_m = 2\pi m T$ (for bosons)

are the Matsubara frequencies

In this representation, the action can be written as $\textcircled{20}$

$$S[\bar{\Phi}, \Phi] = \sum_{ijm} \bar{\Phi}_{im} [(-i\omega_m - \mu)\delta_{ij} + h_{ij}] \Phi_{jm}$$

$$+ \frac{1}{\beta} \sum_{ijkl, m_i} \bar{\Phi}_{im_i} \bar{\Phi}_{jm_j} \Phi_{km_k} \Phi_{lm_l} \delta_{m_1+m_2, m_3+m_4}$$

(use $\int_0^\beta d\tau e^{i\omega_m \tau} = \beta \delta_{\omega_m, 0}$, etc)

Example Non-Interacting Gas

$$H = H_0(\bar{\Phi}, \Phi) = \sum_{ij} \bar{\Phi}_i H_{0ij} \Phi_j$$

Go to a diagonal H_{0ij} basis (unitary transf.)

$$H_0(\bar{\Phi}, \Phi) = \sum_a \bar{\Phi}_a(\tau) \Phi_a(\tau) \epsilon_a$$

ϵ_a eigenvalues

$$\Rightarrow S = \sum_a \sum_{\omega_m} \bar{\Phi}_{am} (-i\omega_m + \epsilon_a - \mu) \Phi_{am}$$

