3 The Callan-Symanzik Equation

After we established the behavior of quantum field theories with scalings, we are now in a position to make contact with what we learned earlier about renormailization. Then we had seen that the renormalization conditions define the correlation functions of the theory at arbitrary input scales, which defines a renormlization scale μ. The variation of this scale will result in a renormalization flow of the renormalization parameters similar to the one derived in the previous section, but now cast as a renormalization scale flow.

To start making contact with the renormalization procedure, we recall the relation between renormalized and non-renormalized correlation functions as given by

$$\langle T\phi(x_1)\ldots\phi(x_n)\rangle = Z^{-n/2}\,\langle T\phi_0(x_1)\ldots\phi_0(x_n)\rangle\ , \tag{3.1}$$

where the field renormalization is defined by

$$\phi(x) = Z^{-1/2}\,\phi_0(x)\ , \tag{3.2}$$

We are going to concentrate on connected correlation functions, which we will denote without the c usbscript $G^n(x_1\ldots x_n)$. The non-renormalized correlation functions depend on the un-renormalized parameters $(\phi_0, \lambda_0, m_0, \ldots)$ as well as on the cutoff Λ. On the other hand, the renormalized correlation functions depend on the renormalized parameters $(\phi, \lambda, m, \ldots)$ as well as on the renormalization scale μ. Then, if we consider an infinitesimal variation of the renormalization scale $\delta\mu$ the un-renormalized correlation function must obeyed

$$\frac{dG_0^{(n)}}{d\mu} = 0\ , \tag{3.3}$$

since $G_0^{(n)}$ does not depend on μ. On the other hand, when $\mu \to \mu + \delta\mu$ the renormalized parameters shift as

$$\begin{aligned} \lambda &\to \lambda + \delta\lambda \\ \phi &\to \phi + \delta\phi \equiv (1+\delta\eta)\phi, \end{aligned} \tag{3.4}$$

where in the last line we defined a dimensionless shift of the field $\delta\eta$. This means that

$$\frac{d}{d\mu}\, Z^{n/2}\, G^{(n)} = \frac{\partial G^{(n)}}{\partial \mu} + \frac{\partial G^{(n)}}{\partial \mu}\,\frac{\partial \lambda}{\partial \mu} - n\frac{\partial \eta}{\partial \mu} = 0\ , \tag{3.5}$$

where we used that

$$Z^{1/2} = 1 - \delta\eta\ , \tag{3.6}$$

since $\phi = Z^{-1/2}\phi_0$. We can rewrite (3.5) as

$$\left(\mu\,\frac{\partial}{\partial \mu} + \mu\,\frac{\partial \lambda}{\partial \mu}\,\frac{\partial}{\partial \lambda} - n\,\mu\,\frac{\partial \eta}{\partial \mu}\right) G^{(n)} = 0\ . \tag{3.7}$$

Defining

$$\beta \equiv \mu\,\frac{\partial \lambda}{\partial \mu} \tag{3.8}$$

$$\gamma \equiv -\mu\,\frac{\partial \eta}{\partial \mu}\ , \tag{3.9}$$

the Callan-Symanzik equation is given by

$$\boxed{\left(\mu\,\frac{\partial}{\partial \mu} + \beta\,\frac{\partial}{\partial \lambda} + n\,\gamma\right) G^{(n)}(x_1, \ldots, x_n; \mu, \lambda) = 0\ .} \tag{3.10}$$

Since $G^{(n)}$ does not depend on the cutoff Λ neither do β and γ. Furthermore, (3.8) and (3.9) define β and γ as dimensionless. Thus, since the only scale in the problem is μ, they do not depend on it. They only depend on the renormalized coupling λ. The beta function β defined in (3.8) measures the coupling dependence on the renormalization scale μ, whereas γ encodes the μ dependence of the field renormalization. In the context of the program of renormalization by counterterms, the mu dependence of the correlation functions is due to the introduction of counterterms introduced to cancel divergencies. In fact, as we will see below, β and γ are directly related to the counterterms and in particular to the coefficient of the divergencies.

3.1 Example: Massless $\lambda\phi^4$ theory

We start with the 2-point function. The Callan-Symanzik (CS) equation is

$$\left(\mu, \frac{\partial}{\partial\mu} + \beta \frac{\partial}{\partial\lambda} + 2\gamma\right) G^{(2)}(p) = 0 \ . \tag{3.11}$$

The 2-point function to one loop order is depicted in Figure 3.1.

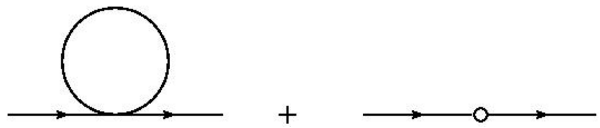

Figure 3.1: Corrections to $G^{(2)}$ to one loop order. The diagram on the right is the counterterm.

Since $m = 0$ there are no corrections to $G^{(2)}(p)$ at one loop since the renormalization condition forces the two diagrams to cancel. Thus, to this order we obtain

$$\boxed{\gamma = 0 \ .} \tag{3.12}$$

On the other hand, the one-loop 4-point function receives the contributions depicted in Figure 3.2.

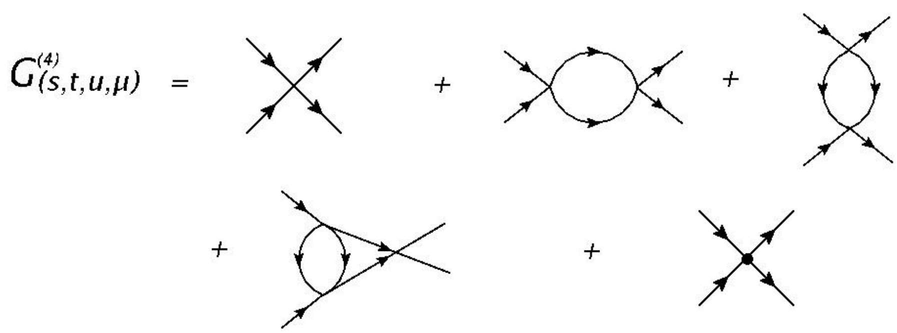

Figure 3.2: Contributions to $G^{(4)}$ to one loop order. The last diagram on the right is the vertex counterterm.

The contributions to corresponding to the renormalization of the external legs vanish since, as we saw below $\gamma = 0$. The connected 4-point function then is

$$G^{(4)}(s,t,u,\mu) = \left[-i\lambda + (-i\lambda)^2 \left\{\Gamma(s) + \Gamma(t) + \Gamma(u)\right\} - i\delta\lambda\right] \prod_{i=1}^{4} \frac{i}{p_i^2} \ , \tag{3.13}$$

where the last factor is the product of the four massless propagators. Imposing the renormalization condition at the space-like point $s = t = u = -\mu^2$ we obtain

$$G^{(4)}(-\mu^2, -\mu^2, -\mu^2) = -i\lambda \prod_{i=1}^{4} \frac{i}{p_i^2} \;, \tag{3.14}$$

which, after using dimensional regularization, fixes the vertex counterterm to be

$$\delta\lambda = \frac{3}{2(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2-d/2)}{[x(1-x)\mu^2]^{2-d/2}} \tag{3.15}$$

In terms of $\epsilon = 4 - d$ the counterterm is

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2} \left\{ \frac{2}{\epsilon} - \ln\mu^2 + \quad \text{finite terms} \right\} \;. \tag{3.16}$$

The CS equation for the 4-point function is then

$$\left(\mu \frac{\partial}{\partial\mu} + \beta \frac{\partial}{\partial\lambda} \right) G^{(4)}(s, t, u, \mu) = 0 \;. \tag{3.17}$$

Clearly, the only μ dependence is in $\delta\lambda$. Then, we have

$$\mu \frac{\partial G^{(4)}}{\partial\mu} = i \frac{3\lambda^2}{16\pi^2} \;. \tag{3.18}$$

On the other hand, to leading order in λ we have

$$\frac{\partial G^{(4)}}{\partial\lambda} = -i + \mathcal{O}(\lambda) \;. \tag{3.19}$$

Putting these into the CS equation (3.17) we obtain

$$\boxed{\beta = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \;.} \tag{3.20}$$

We can infer many interesting things from the *beta* function of the theory, even if it is just in perturbation theory. But we will first generalize our calculation of β and γ.

3.2 General Treatment

Our aim is to use the CS equation in order to obtain a general expression for the β function of a theory in terms of the μ derivatives of the counterterms. As a first step, we are going to obtain an expression for γ in terms of μ derivatives of the field renormalization Z. Starting from

$$\phi(\mu) = Z^{-1/2}(\mu)\phi_0 \ , \tag{3.21}$$

we consider the change under the shift in renormalization scale $\mu \to \mu + \delta\mu$. Then we have

$$\phi + \delta\phi = Z^{-1/2}(\mu + \delta\mu)\, \phi_0 \ . \tag{3.22}$$

Dividing (3.22) by (3.21) we have

$$1 + \frac{\delta\phi}{\phi} = \frac{Z^{-1/2}(\mu + \delta\mu)}{Z^{-1/2}(\mu)} \ . \tag{3.23}$$

Remembering that $\delta\eta = \delta\phi/\phi$ we arrive at

$$\delta\eta = \frac{Z^{-1/2}(\mu + \delta\mu)}{Z^{-1/2}(\mu)} - 1 = \frac{Z^{-1/2}(\mu + \delta\mu) - Z^{-1/2}(\mu)}{Z^{-1/2}(\mu)} \ . \tag{3.24}$$

Dividing this expression by $\delta\mu$ and taking the limit $\delta\mu \to 0$ we obtain

$$\frac{\partial\eta}{\partial\mu} = -\frac{1}{2}\,\frac{1}{Z}\,\frac{\partial Z}{\partial\mu} \ , \tag{3.25}$$

which recalling the definition of γ from (3.9) results in

$$\boxed{\gamma = \frac{1}{2}\,\frac{1}{Z}\,\mu\,\frac{\partial Z}{\partial\mu} \ .} \tag{3.26}$$

The above expression is exact. However, since we will be using perturbation theory, we can always write

$$Z = 1 + \delta Z \ , \tag{3.27}$$

with $\delta Z \ll 1$, which results in the approximate expression

$$\gamma \simeq \frac{1}{2}\,\mu\,\frac{\partial \delta Z}{\partial \mu} + \ldots\ , \tag{3.28}$$

where the dots denote higher orders in perturbation theory (or higher powers in δZ). We will use this expression in what follows.

We now consider a generic theory with a coupling constant g. A given n-point function can be schematically described as

$$\begin{aligned} G^{(n)} &= [-ig + 1PI \text{ loops} \quad + \quad \text{vertex counterterms} \\ &\quad + \quad \text{loops of external legs} \quad + \text{counterterms of external legs}] \prod_{i=1}^{n} \frac{i}{p_i^2}\ , \end{aligned} \tag{3.29}$$

In general we can have different external fields each with their δZ_i, we can write the above expression as

$$G^{(n)} = \left[-ig + 1PI \text{ loops} - i\delta g + (-ig) \sum_{i=1}^{n} (\text{loops of ext. legs} - \delta Z_i) \right] \prod_{i=1}^{n} \frac{i}{p_i^2}\ . \tag{3.30}$$

Then, the CS equation becomes

$$\mu\,\frac{\partial}{\partial \mu}\left(\delta g - g \sum_i \delta Z_i \right) + \beta(g) + g \sum_i \gamma_i = 0\ . \tag{3.31}$$

The last term in the expression above replaces $n\gamma G^{(n)}$ since here we consider the possibility of having different fields (e.g. in QED having photons and charged fermions). Also in the last terms in (3.31) we have taken $G^{(n)} \simeq -ig$ as we are working in perturbation theory to leading order in the coupling constant g. As a result we can obtain an expression for β as a function of the coupling g and valid up to a given order in g. This is

$$\boxed{\beta(g) = \mu\,\frac{\partial}{\partial \mu}\left(\frac{1}{2}\,g \sum_i \delta Z_i - \delta g \right)\ ,} \tag{3.32}$$

where we have used (3.28) for the γ_i's. The expression in (3.32) allows us to compute the beta function of a theory once we know the counterterms. In particular, all we need to know is the renormalization scale μ dependence n the counterterms. This, as we saw in the example of $\lambda\phi^4$ theory, is basically given by the coefficient of the divergences (up to a signa and a factor of two).

3.3 Example: the QED Beta Function

We will compute the QED beta function to one loop accuracy. The counterterms that contribute to it are determined by the diagrams shown in Figure 3.3.

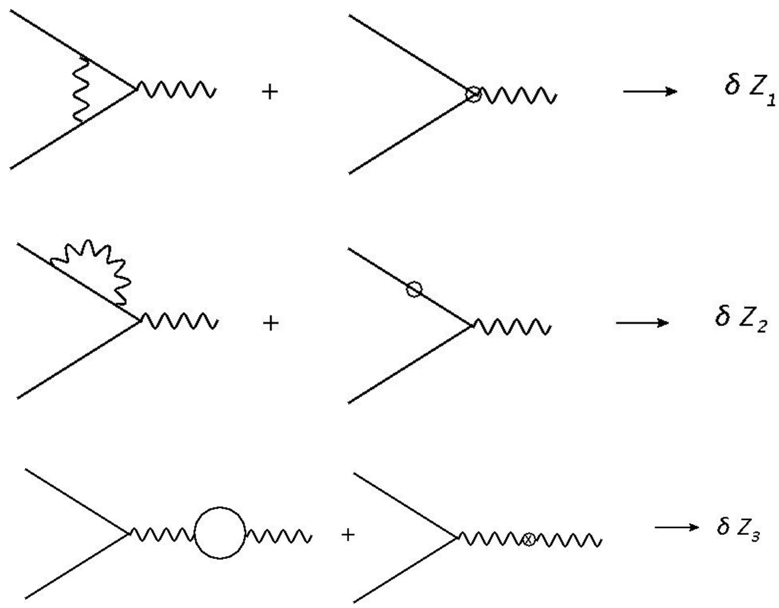

Figure 3.3: Contributions to the QED beta function to one loop order. For the case of δZ_2 it is understood that there is another similar diagram for the lower fermion leg.

Using (3.32) for QED we have

$$\beta(e) = \mu \frac{\partial}{\partial \mu} \left(\frac{1}{2} e \delta Z_2 + \frac{1}{2} e \delta Z_2 + \frac{1}{2} e \delta Z_3 - \delta Z_1 \right) , \tag{3.33}$$

where the first two terms reflect the renormalization of the two external fermion legs. Making use of the Ward identity relation

$$\delta Z_1 = e \, \delta Z_2 , \tag{3.34}$$

we arrive at

$$\beta(e) = \frac{1}{2} e \, \mu \frac{\partial \delta Z_3}{\partial \mu} , \tag{3.35}$$

which is consistent with the fact that the result is determined by the photon renormalization only, i.e. gauge invariance guarantees that the result is independent of the identity of the external fermions. Finally, making use of our previous result

$$\delta Z_3 = -\frac{e^2}{16\pi^2}\,\frac{4}{3}\,\frac{\Gamma(\epsilon/2)}{(\mu^2)^{\epsilon/2}} = -\frac{e^2}{12\pi^2}\,\left(\frac{2}{\epsilon} - \ln\mu^2 + \mu\text{−independent terms}\right) \;, \tag{3.36}$$

where we used $\epsilon = 4 - d$ and d is the number of dimensions we use (3.37) to obtain

$$\beta(e) = \frac{e^3}{12\pi^2} + \mathcal{O}(e^4) \;. \tag{3.37}$$

We then see that –up to a sign– the beta function is determined by the coefficient of the divergence in the counterterm δZ_3. This is a generic feature: in order to compute the beta function of a given coupling we need only determine the divergent parts of the relevant counterterms.