Chern-Simons Theories : (I)

L29

(1)

QED (Maxwell's theory)

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

The "matter" current is conserved, as in

$$\partial_\mu J^\mu = 0$$

$\mathcal{L}_M$ is gauge invariant. I.e. under

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \qquad \forall\, \alpha(x)$$

The equations of motion are

$$\partial_\mu F^{\mu\nu} = J^\nu$$

(note: anti-symmetry of $F^{\mu\nu}$ $\Leftrightarrow$ current conservation)

This theory can be defined in any number of space-time dimensions d. I.e. for

$$A_\mu \quad \text{with} \quad \mu = 0, 1, 2, \ldots, (d-1)$$

Independently of d, still valid

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{etc}$$

But the number of independent components of $F_{\mu\nu}$ is $\frac{1}{2} d(d-1)$ (anti-symmetric matrix)

$\Rightarrow$ For $d=4$ we have 6: $3\vec{E}$ and $3\vec{B}$.

In particular, we are interested in $2+1=3$ d

In $d=3$, $\vec{A}$ is a 2 dim vector

And so is the electric field

$$\vec{E} = -\vec{\nabla} A_0 - \partial_0 \vec{A} \Rightarrow \left\{ \begin{array}{l} 2 \text{ components} \\ \text{of } F_{\mu\nu} \end{array} \right\}$$

But $B \equiv \varepsilon^{ij} \partial_i A_j$ is a (pseudo) scalar

(in $d=3+1$, we have $\varepsilon^{ijk} \partial_j A_k = B_i \Rightarrow \vec{B}$ vector is $\vec{\nabla} \times \vec{A}$)

$$\Rightarrow \left\{ \begin{array}{l} F_{\mu\nu} \text{ is } 3\times 3 \text{ anti-symmetric matrix with} \\ 3 \text{ components: } 2 \text{ for } \vec{E},\ 1 \text{ for } B \end{array} \right)$$

The New thing in 2+1 dimensions is that there is a new term we can write. It is like a new type of gauge theory (instead of Maxwell's)

$$\mathcal{L}_{CS} = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - A_\mu J^\mu$$

Is this gauge invariant?

$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \alpha(x)$

$\Rightarrow \epsilon^{\mu\nu\rho}(A_\mu + \partial_\mu \alpha) \partial_\nu (A_\rho + \partial_\rho \alpha)$

$= \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \epsilon^{\mu\nu\rho}(A_\mu + \partial_\mu \alpha) \underbrace{\partial_\nu \partial_\rho \alpha}_{0}$

$+ \epsilon^{\mu\nu\rho} \partial_\mu \alpha \partial_\nu A_\rho$

$\overset{IbP}{=} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \partial_\mu (\epsilon^{\mu\nu\rho} \alpha(x) \partial_\nu A_\rho) - \underbrace{\epsilon^{\mu\nu\rho} \alpha \partial_\mu \partial_\nu A_\rho}_{0}$

$\Rightarrow \mathcal{L}_{CS} \longrightarrow \mathcal{L}_{CS} + \underbrace{\frac{\kappa}{2} \partial_\mu (\epsilon^{\mu\nu\rho} \alpha \partial_\nu A_\rho)}_{\text{total derivative}}$

Equation of motion

$$\mathcal{L} = \frac{k}{2}\,\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - A_\mu J^\mu$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{k}{2}\,\varepsilon^{\mu\nu\rho} \partial_\nu A_\rho - J^\mu$$

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\rho)} = \frac{k}{2}\,\varepsilon^{\mu\nu\rho} \partial_\nu A_\mu = -\frac{k}{2}\,\varepsilon^{\mu\nu\rho} \partial_\nu A_\rho$$

$$= \frac{k}{2}\,\varepsilon^{\mu\nu\rho} \partial_\rho A_\nu$$

$$\Rightarrow \frac{k}{2}\,\varepsilon^{\mu\nu\rho} \partial_\rho A_\nu - \frac{k}{2}\,\varepsilon^{\mu\nu\rho} \partial_\nu A_\rho = J^\mu$$

$$\Rightarrow \boxed{\frac{k}{2}\,\varepsilon^{\mu\nu\rho} F_{\rho\nu} = J^\mu}$$

$\Rightarrow$ If we can neglect boundary terms (not always)

$\Rightarrow$ $S_{CS} = \int d^3x \, \mathcal{L}_{CS}$ is gauge invariant

This can be confirmed by the fact that the equations of motion are

$$\frac{k}{2} \varepsilon^{\mu\nu\rho} F_{\nu\rho} = J^{\mu}$$

Secondly: Can't have this term in 3+1 dims

$$\varepsilon^{\mu\nu\rho\sigma} A_{\mu} \partial_{\nu} A_{\rho} A_{\sigma} = 0$$

$$\varepsilon^{\mu\nu\rho\sigma} \partial_{\mu} A_{\nu} \partial_{\rho} A_{\sigma} = 0$$

only $\neq 0$ in odd number of dimensions

Eg $d = 4+1$

$$\varepsilon^{\mu\nu\rho\sigma\tau} A_{\mu} \partial_{\nu} A_{\rho} \partial_{\sigma} A_{\tau}$$

But it is not quadratic in A_{μ}. Only in 2+1 is.

A priori CS theory in 2+1 d look boring. Eq of motion in vacuum is

$$F_{\mu\nu} = 0$$

(in 3+1 is $\partial_\mu F^{\mu\nu} = 0 \Rightarrow$ plane-electromagnetic waves!)

But, interesting when (among other things)

- Coupled to charged matter fields
- Coupled to Maxwell term
- Embedded in non-trivial topology
- Non-Abelian gauge fields