2 Renormalization Flow

In the previous lecture we saw that integrating out high energy modes results in an effective action for the low energy fields that contains modifications of the existing operators as well as new operators. These modifications will appear as changes in the parameters of the action (couplings, masses, fields). Our next step is to understand how these respond to continuous changes in the rescaling responsible for the split between high and low energy. To start, we define the momentum and length rescaling in terms of the parameter b as

$$k' = \frac{k}{b}, \qquad x' = bx, \tag{2.1}$$

where just as before $0 < b < 1$. We write the effective action for the low energy modes $\phi_\ell \to \phi$ as

$$\begin{aligned} S[\phi] &= \int d^dx \left\{ \frac{1}{2}(1+\Delta Z)\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 \right. \\ &\quad \left. +\Delta C\left(\partial_\mu\phi\partial_\mu\phi\right)^2 + \Delta D\phi^6 + \ldots \right\} , \end{aligned} \tag{2.2}$$

where we now denote the low energy fields as just ϕ. In (2.2) the effects of integrating out the high energy modes ϕ_h are encoded in the parameters $\Delta Z, \Delta m^2, \Delta\lambda, \Delta C, \Delta D, \ldots$ etc. In the case of the higher dimensional operators, they actually are induced by this procedure, i.e. even if the original (classical) theory is renormalizable ($C = D = 0$) these operators are generated by integrating out ϕ_h, as we discussed in the previous lecture. But in addition to these shifts and the presence of the new operators, the parameters of the action change with the rescaling (2.1). Focusing on the length rescaling we have

$$d^dx = b^{-d}\, d^dx', \qquad \frac{\partial}{\partial x_\mu} = b\,\frac{\partial}{\partial x'_\mu} \equiv b\,\partial'_\mu\ . \tag{2.3}$$

Replacing these relations in (2.2) we obtain

$$\begin{aligned} S[\phi] &= \int d^dx'\, b^{-d} \left\{ \frac{1}{2}(1+\Delta Z)\, b^2\, \partial'_\mu\phi\partial'_\mu\phi + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 \right. \\ &\quad \left. +\Delta C\, b^4\, (\partial'_\mu\phi)^4 + \Delta D\phi^6 + \ldots \right\} , \end{aligned} \tag{2.4}$$

In order to have a canonically normalized kinetic term in (2.4) we define

$$\phi' \equiv \left[b^{2-d}\,(1+\Delta Z)\right]^{1/2}\,\phi\;. \tag{2.5}$$

Writing the action in terms of ϕ' and x' we have

$$S[\phi'] = \int d^dx'\,\left\{\frac{1}{2}\partial'_\mu\phi'\partial'_\mu\phi' + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4!}\lambda'\phi'^4 + C'\left(\partial'_\mu\phi'\right)^4 + D'\phi'^6 + \ldots\right\}\;, \tag{2.6}$$

where we defined

$$\begin{aligned}
m'^2 &\equiv (m^2+\Delta m^2)\,(1+\Delta Z)^{-1}\,b^{-2},\\
\lambda' &\equiv (\lambda+\Delta\lambda)\,(1+\Delta Z)^{-2}\,b^{d-4},\\
C' &\equiv \Delta C\,(1+\Delta Z)^{-2}\,b^{d}\\
D' &\equiv \Delta D\,(1+\Delta Z)^{-3}\,b^{2d-6},
\end{aligned} \tag{2.7}$$

which we can repeat for the coefficients of all other higher dimensional operators.

The first important lesson from (2.7) is that, even when absent in the initial lagrangian, coefficients like C' and D' corresponding to higher dimensional opeartors (non-renormalizable operators) are generated by the rescaling procedure. Thus, only considering renormalizable operators is not consistent with rescaling. The lagrangian must always contain an infinite tower of higher dimensional operators. However, as we will soon see, these will be suppressed by inverse powers of the cutoff Λ. Then, when we write down only renormalizable operators we are implicitely assuming that the cutoff of the theory is much higher than the typical energy scales in which the theory is being used. The combination of integrating out high energy degrees of freedom together with the rescaling of momenta and lengths results in a transformation of the lagrangian. If we now consider small rescalings with b just infinitesimally below 1, we can obtain a continous transformation. The differential equation (or equations) resulting from this process is referred to as the renormalization group.

We can compare the rescaling procedure with the renormalization by couterterms we used previously. In that case the UV behavior of the theory appear through loop effects which resulted in quantum corrections. The divergences were absorbed by the couterterms. In the renormalization group approach we obtain an effective lagrangian by integrating out the UV degrees of freedom. Their effect appears as redefinitions or shifts in the parameters of the theory, as in m', λ', etc. The effects are small as long as the couplings are perturbative.

2.1 Gaussian Fixed Point

The rescaling procedure defined in (2.1) and (2.3) leave the kinetic term unchanged. Thus, if we consider all operators other than this as perturbations, including the mass term ($m^2 \gg \Lambda^2$), we can say that the free massless theory

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \ , \tag{2.8}$$

is unchanged by these rescalings. A theory that behaves in this way is called a fixed point. In particular, the theory in (2.8) is called a Gaussian fixed point. In the vicinity of a Gaussian fixed point, theories are very simple since only a few operators are generated as perturbations, those that are renormalizable and not suppressed by the cutoff Λ. We will see next how to make this statement more precise.

2.2 Renormalization Flow and Renormalizability

Let us consider a generic action written as

$$S[\phi] = \int d^dx \sum_j c_j \, \mathcal{O}_j(\phi) \ , \tag{2.9}$$

where ϕ is a generic field or set of fields. We will ignore the quantum corrections obtained from integrating out high energy fields (i.e. the Δ's in (2.2)) and only consider the effects of the rescaling by $b < 1$. Usisng (2.1), (2.3) and (2.5) the action is now

$$S[\phi'] = \int d^dx' \, b^{-d} \sum_j c_j b^{N(d/2-1)+M} \, \mathcal{O}(\phi') \ , \tag{2.10}$$

where N is the numbers of fields ϕ in $\mathcal{O}(\phi)$ and M is the number of derivatives in it. Then we can define the rescaled coefficients of the operators as

$$c'_j = b^{N(d/2-1)+M-d} \, c_j \ . \tag{2.11}$$

Noticing that

$$d_{\mathcal{O}} \equiv N(d/2-1) + M \ , \tag{2.12}$$

is the canonical dimension of the operator $\mathcal{O}$ we can write

$$c_j' = b^{d_{\mathcal{O}}-d}\, c_j \ . \tag{2.13}$$

We can now establish the differential change of the coefficients of operators as a differential equation, and use it to classify the different behaviors according to the rescaling flow. To do this we consider a value of b only infinitesimally below 1, i.e.

$$b \simeq 1 - \delta \ , \tag{2.14}$$

with $\delta \gg 1$ and positive. Then we have

$$x' = b\, x = (1-\delta)\, x = x - \delta x \equiv x - dx \ . \tag{2.15}$$

Then

$$c_j' = c_j + \delta c_j = (1-\delta)^{d_{\mathcal{O}}-d}\, c_j \tag{2.16}$$

resulting in

$$\delta c_j \simeq -\left(d_{\mathcal{O}} - d\right)\, \delta\, c_j \ . \tag{2.17}$$

Using $\delta = dx/x$, this can be rewritten as

$$\boxed{x\frac{dc_j}{dx} = -\left(d_{\mathcal{O}} - d\right)\, c_j} \ . \tag{2.18}$$

We see that there are three possibilities.

- $d_{\mathcal{O}} = d$: Marginal or Renormalizable Operator

  This corresponds to dimensionless coefficients, such as λ in the ϕ^4 theory in 4D. We see from (2.18) that rescalings in length do not shift this coefficients. Remember this is just the behavior under rescalings. This changes when considering quantum corrections from the UV (i.e. the Δ's). What happens, as we will see later, is that $\mathcal{O}$ acquires an anomalous dimension due to quantum corrections that would change the behavior of the coefficient into one of the two other cases.

- $d_{\mathcal{O}} < d$: Relevant or Super-renormalizable Operator

  This case corresponds to coefficients that have positive mass dimensions, such as the mass term in ϕ^4 theory in 4D. From (2.18) we see that the derivative of the coefficient is positive meaning that it grows as x does. In other words, the mass term grows in the IR, so it is a relevant operator in the effective low energy theory.

- $d_{\mathcal{O}} > d$: Irrelevant or Non-renormalizable Operator

  Finally, we see that this case corresponds to non-renormalizable interactions (higher dimensional operators). For this coefficients (2.18) implies that they become smaller towards larger scales. This is why they are called irrelevant. This means, for instance, that the higher the dimension of the operator the faster it becomes small at large distances. Also, if the cutoff is large the larger will be the evolution of the coefficient. Then, as anticipated, in a theory with a larger cutoff the coefficients of non-renormalizable operators become irrelevant faster.

  The analysis performed above is very intuitive and gives us a clear picture of what is renormalization flow. In the next lecture we will develop a formalism that will allow us to make contact with the renormalization program as presented in the first part of the course.