10 Spontaneous Symmetry Breaking II: Superconductivity

We have seen in a second quantization exercise that the interactions of electrons and phonons can lead to an effective *attractive* interaction between electrons in a material, as long as the energies of the participating electrons are close to the Fermi surface. By close we mean within $\pm\omega_D$, with ω_D the Debye energy which is the maximum frequency for phonons. We can express this as an effective attractive electron potential given by

$$V_{\text{eff.}}(\mathbf{q}) \simeq \frac{g}{\omega^2 - \omega_{\mathbf{q}}^2} \ , \tag{10.1}$$

where g is a positive constant and $\omega_{vfq} < \omega_D$. Under the right circumstances, this attractive interactions will lead to the formation of Cooper pairs, which in a superconductor will dominate the ground state. In what follows we will consider a phenomenological model of this interaction that was first proposed by Bardeen, Cooper and Schrieffer (BCS). Just as for the case of the superfluid, we first consider the second quantization formulation and then we go no to the functional integral treatment.

10.1 Superconductivity in Second Quantization

The approximation proposed in the BCS formulation involves assuming that the interaction takes place (predominantly) among electrons of the same energy and opposite momentum. This is a good approximation since the energy difference needs to be smaller than ω_D, which is much smaller than the typical electron energy. The BCS Hamiltonian is then given by

$$H = \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}}\, c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma} - g \sum_{\mathbf{p},\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \ , \tag{10.2}$$

where $c_{\mathbf{p}\sigma}$ and $c^\dagger_{\mathbf{p}\sigma}$ are fermion annihilation and creation operators, and g is a positive constant. The opposite spins in the interaction term are a reflection of the fact that it is supposed to come from integrating out a scalar field (phonons) interacting with electrons. To solve the Hamiltonian in (10.2) we consider a coherent state $|\Psi_{\text{BCS}}\rangle$ where the zero-momentum mode is macroscopically occupied when Cooper pairs (pairs of electrons with opposite spins) condense to form a superconducting ground state. We define an operator that creates Cooper pairs out of the non-interacting ground state $|0\rangle$ as

$$P^\dagger_{\mathbf{p}} \equiv c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow} \;, \tag{10.3}$$

which we then use to build the BCS coherent state

$$|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{p}} A_{\mathbf{p}} e^{\alpha_{\mathbf{p}} P^\dagger_{\mathbf{p}}} |0\rangle \;, \tag{10.4}$$

where $A_{\mathbf{p}}$ and α_{vfp} are complex coefficients. First, we notice that $P_{\mathbf{p}}$ and $P^\dagger_{\mathbf{p}}$ obey the commutation rules

$$[P_{\mathbf{p}}, P_{\mathbf{p}}] = 0, \qquad [P^\dagger_{\mathbf{p}}, P^\dagger_{\mathbf{p}}] = 0, \;, \tag{10.5}$$

which might tempt us to claim that Cooper pairs are just bosons. However,

$$[P_{\mathbf{p}}, P^\dagger_{\mathbf{q}}] = \delta_{\mathbf{pq}} \left(1 - N_{\mathbf{p}\uparrow} - N_{-\mathbf{p}\downarrow}\right) \;, \tag{10.6}$$

with $N_{\mathbf{p}\sigma} = c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma}$, shows that their properties as not the usual ones for bosons. In particular, it is straightforward to show that (exercise)

$$P^\dagger_{\mathbf{p}} P^\dagger_{\mathbf{p}} = 0 \;, \tag{10.7}$$

which betrays the fermionic character of the Cooper pair constituents. Using this property the coherent BCS state (10.4) can be rewritten as

$$|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{p}} A_{\mathbf{p}} \left(1 + \alpha_{\mathbf{p}} P^\dagger_{\mathbf{p}}\right) |0\rangle \;. \tag{10.8}$$

Furthermore, if we impose the normalization to be unity, we can determine the $A_{\mathbf{p}}$'s. This is

$$\begin{aligned} \langle \Psi_{\rm BCS} | \Psi_{\rm BCS} \rangle &= 1 \\ &= |A_{\mathbf{p}}|^2 \langle 0 | \left(1 + \alpha^*_{\mathbf{p}} P_{\mathbf{p}}\right) \left(1 + \alpha_{\mathbf{p}} P^\dagger_{\mathbf{p}}\right) |0\rangle \\ &= |A_{\mathbf{p}}|^2 \left(1 + |\alpha_{\mathbf{p}}|^2 \langle 0 | P_{\mathbf{p}} P^\dagger_{\mathbf{p}} |0\rangle\right) \;. \end{aligned} \tag{10.9}$$

However, using (10.6), it is easy to see that

$$\langle 0 | P_{\mathbf{p}} P^\dagger{}_{\mathbf{p}} |0\rangle = \langle 0 | 1 - N_{\mathbf{p}\uparrow} - N_{-\mathbf{p}\downarrow} |0\rangle = 1 \;, \tag{10.10}$$

which results in

$$A_{\mathbf{p}} = \frac{1}{\sqrt{1+|\alpha_{\mathbf{p}}|^2}} \ , \tag{10.11}$$

where we ignored the irrelevant overall phase of the state $|\Psi_{\rm BCS}\rangle$. Then, defining

$$u_{\mathbf{p}} \equiv \frac{1}{\sqrt{1+|\alpha_{\mathbf{p}}|^2}}, \qquad\qquad v_{\mathbf{p}} \equiv \frac{\alpha_{\mathbf{p}}}{\sqrt{1+|\alpha_{\mathbf{p}}|^2}} \ , \tag{10.12}$$

the BCS ground state can be written as

$$\boxed{|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{p}} \left(u_{\mathbf{p}} + v_{\mathbf{p}} P_{\mathbf{p}}^\dagger\right)|0\rangle} \ . \tag{10.13}$$

where the coefficients defined in (10.12) satisfy

$$|u_{\mathbf{p}}|^2 + |v_{\mathbf{p}}|^2 = 1 \ . \tag{10.14}$$

We can interpret the coefficients $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ by noticing that

$$\begin{aligned} \langle|\Psi_{\rm BCS}|N_{\mathbf{p}\uparrow}|\Psi_{\rm BCS}\rangle &= \prod_{\mathbf{k},\mathbf{q}} \langle 0|\left(u_{\mathbf{k}}^* + v_{\mathbf{k}}^* P_{\mathbf{k}}\right) c_{\mathbf{p}\uparrow}^\dagger c_{\mathbf{p}\uparrow} \left(u_{\mathbf{q}} + v_{\mathbf{q}} P_{\mathbf{q}}^\dagger\right)|0\rangle \\ &= \prod_{\mathbf{k}} v_{\mathbf{k}}^* v_{\mathbf{p}} \langle 0|P_{\mathbf{k}} P_{\mathbf{p}}^\dagger|0\rangle = |v_{\mathbf{p}}|^2 \ , \end{aligned} \tag{10.15}$$

and similarly for $N_{\mathbf{p}\downarrow}$. Then, noticing that

$$N = \sum_{\mathbf{p},\sigma} c_{\mathbf{p}\sigma}^\dagger c_{\mathbf{p}\sigma} = \sum_{\mathbf{p}} \left(N_{\mathbf{p}\uparrow} + N_{\mathbf{p}\downarrow}\right) \ , \tag{10.16}$$

we obtain

$$\langle\Psi_{\rm BCS}|N|\Psi_{\rm BCS}\rangle = 2|v_{\mathbf{p}}|^2 \ . \tag{10.17}$$

Then, $2|v_{\mathbf{p}}|^2$ gives the number of occupied states with momentum $\mathbf{p}$. Through (10.14) we can see that $|u_{\mathbf{p}}|^2$ gives us the number of unoccupied states.

Next, we are interested in obtaining the ground state energy, i.e. using (10.2) we want to compute

$$\begin{aligned} E &= \langle \Psi_{\mathrm{BCS}} | H | \Psi_{\mathrm{BCS}} \rangle \\ &= 2 \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} |v_{\mathbf{p}}|^2 - g \sum_{\mathbf{p},\mathbf{k}} v_{\mathbf{p}}^* v_{\mathbf{k}} u_{\mathbf{k}}^* u_{\mathbf{p}} \ , \end{aligned} \tag{10.18}$$

where in the second line we used the properties of $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ (check!), and the 2 in the first terms there comes from summing over spins. The expression above can be used to obtain the ground state energy. The solution must minimize the energy subject to the constraints

$$\begin{aligned} &N = 2 \sum_{\mathbf{p}} |v_{\mathbf{p}}|^2 \\ &|u_{\mathbf{p}}|^2 + |v_{\mathbf{p}}|^2 = 1 \ . \end{aligned} \tag{10.19}$$

Then, the function to be extremized is

$$f \equiv E - \mu N + \sum_{\mathbf{p}} E_{\mathbf{p}} \left(|u_{\mathbf{p}}|^2 + |v_{\mathbf{p}}|^2 - 1 \right) , \tag{10.20}$$

where μ and $E_{\mathbf{p}}$ are the Lagrange multipliers. We need to solve

$$\begin{aligned} \frac{\partial f}{\partial u_{\mathbf{p}}} &= \frac{\partial E}{\partial u_{\mathbf{p}}} - \mu \frac{\partial N}{\partial u_{\mathbf{p}}} + E_{\mathbf{p}} u_{\mathbf{p}}^* = 0 \\ \frac{\partial f}{\partial v_{\mathbf{p}}} &= \frac{\partial E}{\partial v_{\mathbf{p}}} - \mu \frac{\partial N}{\partial v_{\mathbf{p}}} + E_{\mathbf{p}} v_{\mathbf{p}}^* = 0 \ . \end{aligned} \tag{10.21}$$

Defining

$$\Delta \equiv g \sum_{\mathbf{p}} u_{\mathbf{p}}^* v_{\mathbf{p}} \ , \tag{10.22}$$

the minimization equations (10.21) can be written in matrix form as

$$\begin{pmatrix} (\epsilon_{\mathbf{p}} - \mu) & \Delta \\ \bar{\Delta} & -(\epsilon_{\mathbf{p}} - \mu) \end{pmatrix} \begin{pmatrix} u^*_{\mathbf{p}} \\ v^*_{\mathbf{p}} \end{pmatrix} = E_{\mathbf{p}} \begin{pmatrix} u^*_{\mathbf{p}} \\ v^*_{\mathbf{p}} \end{pmatrix} , \tag{10.23}$$

which has eigenvalues

$$E_{\mathbf{p}} = \pm\sqrt{(\epsilon_{\mathbf{p}} - \mu)^2 + |\Delta|^2} \, . \tag{10.24}$$

Obtaining the eigenvectors (i.e. the solution for $u^*_{\mathbf{p}}$ and $v^*_{\mathbf{p}}$) can be made easier by rewriting (10.23) as

$$\begin{pmatrix} \cos 2\theta_{\mathbf{p}} & \sin 2\theta_{\mathbf{p}} \\ \sin 2\theta_{\mathbf{p}} & -\cos 2\theta_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u^*_{\mathbf{p}} \\ v^*_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} u^*_{\mathbf{p}} \\ v^*_{\mathbf{p}} \end{pmatrix} , \tag{10.25}$$

where we defined

$$\cos 2\theta_{\mathbf{p}} \equiv \frac{\epsilon_{\mathbf{p}} - \mu}{E_{\mathbf{p}}}, \qquad \sin 2\theta_{\mathbf{p}} \equiv \frac{\Delta}{E_{\mathbf{p}}} \, . \tag{10.26}$$

The solution for the eigenvectors of (10.25) results in

$$\boxed{u^*_{\mathbf{p}} = \cos\theta_{\mathbf{p}}, \qquad v^*_{\mathbf{p}} = \sin\theta_{\mathbf{p}}} \, . \tag{10.27}$$

Also worth noticing, we now have

$$u^*_{\mathbf{p}} v_{\mathbf{p}} = \cos\theta_{\mathbf{p}} \sin\theta_{\mathbf{p}} = \frac{1}{2}\sin 2\theta_{\mathbf{p}} = \frac{\Delta}{E_{\mathbf{p}}} \, . \tag{10.28}$$

This results in the transcendental equation

$$\Delta = g \sum_{\mathbf{p}} u^*_{\mathbf{p}} v_{\mathbf{p}} = g \sum_{\mathbf{p}} \frac{\Delta}{2E_{\mathbf{p}}} \, . \tag{10.29}$$

Then, the energy gap Δ must satisfy

$$\boxed{1 = g \sum_{\mathbf{p}} \frac{1}{2\sqrt{(\epsilon_{\mathbf{p}} - \mu)^2 + \Delta^2}}} \, . \tag{10.30}$$

This is the so-called the gap equation. Before we go on exploring it, we can now write the BCS ground state as

$$|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{p}} \left(\cos\theta_{\mathbf{p}} + \sin\theta_{\mathbf{p}} c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow} \right) |0\rangle \; . \tag{10.31}$$

Now examining (10.30), we must remember that in order for the Cooper pairs to form the electron-electron interaction must be attractive, which means that

$$|\epsilon_{\mathbf{p}} - \mu| < \omega_D \; . \tag{10.32}$$

Then, in replacing the sum in (10.30) by an integral we will have

$$\sum_{\mathbf{p}} \to \int_{-\omega_D}^{\omega_D} \nu(\epsilon) d\epsilon \; , \tag{10.33}$$

where $\nu(\epsilon)$ is the density of states. But since the integration will be in a small band around the Fermi surface, we will make the approximation

$$\nu(\epsilon) \simeq \nu_F = \text{ constant} \; . \tag{10.34}$$

Then the gap equation (10.30) can be written as

$$1 = g\nu_F \int_{-\omega_D}^{\omega_D} d\epsilon \frac{1}{2\sqrt{\epsilon^2 + \Delta^2}} \; , \tag{10.35}$$

where we defined $\epsilon = \epsilon_{\mathbf{p}} - \mu$. Since the integrand in (10.35) is even we arrive at

$$\begin{aligned} \frac{1}{g\nu_F} &= \int_0^{\omega_D} d\epsilon \frac{1}{\sqrt{\epsilon^2 + \Delta^2}} \\ &= \text{arcsinh}\left[\frac{\omega_D}{\Delta}\right] \; . \end{aligned} \tag{10.36}$$

Then we arrive at

$$\boxed{\Delta = \frac{2\omega_D}{e^{1/g\nu_F} - e^{-1/g\nu_F}}} \; . \tag{10.37}$$

For instance, for $g\nu_F \ll 1$ we have that the gap obeys

$$\Delta \simeq 2\omega_D e^{-1/g\,\nu_F} \;, \tag{10.38}$$

which results in $\Delta \ll \omega_D$ as expected. We will later see that there is a crucial temperature dependence of the energy gap. But for now we see that essentially at zero temperature, or rather for $T \ll T_c$, the ground state has an energy gap Δ given by (10.37). The excitation spectrum has energies $E_{\mathbf{p}}$. The next question is: what are the exited states, or quasi-particles. One way to see this is the consider the Hamiltonian written as

$$H - \mu N = \sum_\sigma (\epsilon_{\mathbf{p}} - \mu) c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma} - g \sum_{\mathbf{p},\mathbf{k}} \left(\langle c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow} \rangle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle \right) \;, \tag{10.39}$$

where we averaged over pairs of operators in the interaction term. The expectation values in (10.39) are taken with respect to the BCS ground state, $|\Psi_{\rm BCS}\rangle$. Explicitly, we have

$$g \langle \Psi_{\rm BCS} | c_{\mathbf{p}\uparrow} c_{-\mathbf{p}\downarrow} | \Psi_{\rm BCS} \rangle = \Delta \;, \tag{10.40}$$

where we used that

$$\Delta = g \sum_{\mathbf{p}} u^*_{\mathbf{p}} v_{\mathbf{p}} \;. \tag{10.41}$$

Then, we can write (10.39) as

$$H - \mu N = \sum_\sigma (\epsilon_{\mathbf{p}} - \mu) c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma} - \sum_{\mathbf{p}} \left(\bar{\Delta}\, c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} + \Delta\, c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow} \right) \;. \tag{10.42}$$

The expression above can be written in matrix form as

$$H - \mu N = \sum_{\mathbf{p}} \begin{pmatrix} c^\dagger_{\mathbf{p}\uparrow} & c_{-\mathbf{p}\downarrow} \end{pmatrix} \begin{pmatrix} \epsilon_{\mathbf{p}} - \mu & -\Delta \\ -\bar{\Delta} & -(\epsilon_{\mathbf{p}} - \mu) \end{pmatrix} \begin{pmatrix} c_{\mathbf{p}\uparrow} \\ c^\dagger_{-\mathbf{p}\downarrow} \end{pmatrix} \;. \tag{10.43}$$

The diagonalization of this Hamiltonian now can proceed through Bogoliuvov's transformations defining the annihilation and creation operators of the eigenstates, $b_{\mathbf{p}\sigma}$ and $b^\dagger_{\mathbf{p}\sigma}$, and satisfying the usual anticommutation relations

$$\left\{b_{\mathbf{p}\sigma}, b^{\dagger}_{\mathbf{p}'\sigma'}\right\} = \delta_{\mathbf{p}\mathbf{p}'}\delta_{\sigma\sigma'} \ , \ldots \tag{10.44}$$

In terms of them, we can rewrite (10.43) as

$$H - \mu N = \sum_{\mathbf{p}} \begin{pmatrix} b^{\dagger}_{\mathbf{p}\uparrow} & b_{-\mathbf{p}\downarrow} \end{pmatrix} \begin{pmatrix} E_{\mathbf{p}} & 0 \\ 0 & -E_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{p}\uparrow} \\ b^{\dagger}_{-\mathbf{p}\downarrow} \end{pmatrix} \ . \tag{10.45}$$

where $E_{\mathbf{p}}$ is given by (10.24). Then, we arrive at

$$H - \mu N = \sum_{\mathbf{p},\sigma} E_{\mathbf{p}}\, b^{\dagger}_{\mathbf{p}\sigma} b_{\mathbf{p}\sigma} \ . \tag{10.46}$$

The states created and annihilated by $b^{\dagger}_{\mathbf{p}\sigma}$ and $b_{\mathbf{p}\sigma}$ are excitations, as can be seen by the fact that

$$b_{\mathbf{p}\sigma}|\Psi_{\rm BCS}\rangle = 0 \ , \tag{10.47}$$

meaning that the BCS ground state contains none of them.

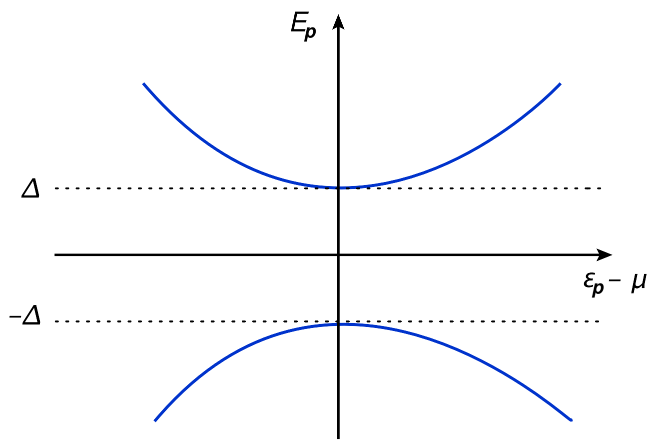

Figure 10.1: Dispersion for the excitations in a superconductor. They must have at least an energy Δ in order to occur,

In Figure 10.1 we show the excitation energy as a function of $\epsilon_{\mathbf{p}} - \mu$, where we can see that there is a gap defined by Δ where there can be no excitations.

Finally, we comment on the fact that the non-trivial expectation value of the Cooper pair operator $P_{\mathbf{p}}$ given by (10.40) implies spontaneous symmetry breaking in a form analogous to the case of a superfluid, since in general Δ has a phase, which we here chose to be a constant equal to zero. However, unlike the case of a superfluid, here the phase is a local function, i.e. $\theta(x)$, with $x = (x_0, \mathbf{x})$ a spacetime point. The corresponding symmetry is a local $U(1)$ corresponding to the gauge symmetry of electromagnetism. A gauge transformation is given by the local shift

$$\theta(x) \to \theta(x) + \alpha(x) \ . \tag{10.48}$$

As we will see later, this gauge freedom will allow us to get rid of the phase field $\theta(x)$. The price will be to give mass to the photon inside the superconductor. This is called the Anderson-Higgs mechanism, and it translates into the screening of the electric field which leads to zero resistivity, as well as to the expulsion of magnetic field in what is known as the Meissner effect. But before we get into this, we will reformulate superconductivity through the functional integral and show that we will understand the gap Δ better by deriving its temperature dependence.

10.2 Superconductivity in the Functional Integral

We will start by writing the BCS Hamiltonian (10.2) in position space. Turning on the electromagnetic field, we have

$$\begin{aligned} H - \mu N &= \int d^d r \, c^\dagger_\sigma(\mathbf{r}) \left[\frac{1}{2m} \left(-i\vec{\nabla} - e\mathbf{A} \right)^2 + e\phi - \mu \right] c_\sigma(\mathbf{r}) \\ &- g \int d^d r \, c^\dagger_\uparrow(\mathbf{r}) c^\dagger_\downarrow(\mathbf{r}) c_\downarrow(\mathbf{r}) c_\uparrow(\mathbf{r}) \ , \end{aligned} \tag{10.49}$$

where the field operators $c_\sigma(\mathbf{r})$ are the usual Fourier transforms of the momentum space operators $c_{\mathbf{p}\sigma}$, and ϕ and $\mathbf{A}$ are the scalar and vector potentials respectively. We can then write the partition function as a functional integral in the coherent state representation, as usual. This takes the form

$$Z = \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\bar{\psi},\psi]} \ , \tag{10.50}$$

with the action given by

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \int d^d r \left[\bar{\psi}_\sigma \left(\partial_\tau + ie\phi + \frac{1}{2m} \left(-i\vec{\nabla} - e\mathbf{A} \right)^2 \right) \psi_\sigma - g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right] \ , \tag{10.51}$$

where it is understood that $\psi_\sigma = \psi_\sigma(\tau, \mathbf{x})$, and that they satisfy anti-periodic boundary conditions

$$\psi_\sigma(0, \mathbf{x}) = -\psi_\sigma(\beta, \mathbf{x}) \ , \tag{10.52}$$

given that these come from fermion fields. In going from the Hamiltonian in (10.49) to (10.51), we notice that the scalar potential term picks up a factor of i. This comes from the imaginary time τ formulation of the function integral and can be understood from the fact that ϕ transform as the time component of a four-vector: $A^\mu = (\phi, \mathbf{A})$. The action (10.51) is invariant under the local transformations

$$\begin{cases} \psi_\sigma \to e^{i\theta(\tau,\mathbf{x})}\psi_\sigma \ , & \bar{\psi}_\sigma \to e^{-i\theta(\tau,\mathbf{x})}\bar{\psi}_\sigma \\ \phi \to \phi - \frac{1}{e}\partial_\tau \theta(\tau, \mathbf{x}) \ , & \mathbf{A} \to \mathbf{A} + \frac{1}{e}\vec{\nabla}\theta(\tau, \mathbf{x}) \end{cases} \tag{10.53}$$

The next step is to use a Hubbard-Stratonovich transformation to decouple the quartic interaction in he action (10.51). In particular, we will do this in the Cooper channel, that us we will introduce the complex scalar field Δ so that it couples to Cooper pairs as

$$\Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow \ , \tag{10.54}$$

as well as its complex conjugate $\bar{\Delta}$, coupled as

$$\bar{\Delta} \psi_\downarrow \psi_\uparrow \ . \tag{10.55}$$

This means that we will replace the quartic interaction in (10.51) by *integrating in* Δ using

$$e^{g\int d\tau d^d r \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow} = \int \mathcal{D}(\Delta, \bar{\Delta}) e^{-\int d\tau d^d r \left\{(1/g)\|\Delta|^2 - \left(\bar{\Delta}\psi_\downarrow\psi_\uparrow + \Delta\bar{\psi}_\downarrow\bar{\psi}_\uparrow\right)\right\}} \ , \tag{10.56}$$

where the field Δ is a boson and therefore satisfies periodic boundary conditions:

$$\Delta(0, \mathbf{x}) = +\Delta(\beta, \mathbf{x}) \ . \tag{10.57}$$

Then, we can replace the quartic interaction term in (10.51) in favor of a purely quadratic (in terms of the fermion fields) scalar-fermion interaction. Before we rewrite the action in this form, we will define a notation that will simplify the expressions. We define the so-called Nambu spinors as

$$\bar{\Psi} \equiv \left(\begin{array}{cc} \bar{\psi}_\uparrow & \psi_\downarrow \end{array} \right) \ , \qquad \Psi \equiv \left(\begin{array}{c} \psi_\uparrow \\ \\ \bar{\psi}_\downarrow \end{array} \right) \ , \tag{10.58}$$

which is analogous to what we did with $c_{\mathbf{p}\sigma}$ in the second quantization formulation earlier. Then the partition function can be written as

$$Z = \int \mathcal{D}(\psi, \bar{\psi}) \, \mathcal{D}(\Delta, \bar{\Delta}) e^{-\int d\tau d^d r \left\{ (1/g)|\Delta|^2 - \bar{\Psi}\mathcal{O}^{-1}\Psi \right\}} \ , \tag{10.59}$$

where we defined the operator

$$\mathcal{O}^{-1} \equiv \left(\begin{array}{cc} \left(G_0^{(p)}\right)^{-1} & \Delta \\ \\ \bar{\Delta} & \left(G_0^{(h)}\right)^{-1} \end{array} \right) \ , \tag{10.60}$$

and we also defined

$$\left(G_0^{(p)}\right)^{-1} \equiv - \left(\partial_\tau + ie\phi + \frac{1}{2m} \left(-i\vec{\nabla} - e\mathbf{A} \right)^2 - \mu \right) \ , \tag{10.61}$$

and

$$\left(G_0^{(h)}\right)^{-1} \equiv - \left(\partial_\tau - ie\phi - \frac{1}{2m} \left(i\vec{\nabla} - e\mathbf{A} \right)^2 + \mu \right) \ . \tag{10.62}$$

This form can be checke by using the anti-commuting rules for ψ_σ and $\bar{\psi}_\sigma$, as well as integration by parts in ∂_τ and $\vec{\nabla}$. Since the action in the exponent in (10.59) is quadratic in the fermion fields, we can integrate them out, resulting in

$$Z = \int \mathcal{D}(\Delta, \bar{\Delta}) e^{-(1/g)\int d\tau d^d r \, |\Delta|^2 + \mathrm{Tr} \ln\left[\mathcal{O}^{-1}\right]} \ , \tag{10.63}$$

Then, the effective action for the Δ fields is then

$$S[\Delta, \bar{\Delta}] = \frac{1}{g} \int d\tau d^d r \, \bar{\Delta}\Delta - \mathrm{Tr} \ln \left[\mathcal{O}^{-1} \right] \ , \tag{10.64}$$

Next, we will proceed as usually. First we obtain the mean field solution, and then consider fluctuations.

Mean Field Theory:

We are after the saddle point solution for $\Delta(\tau, \mathbf{x})$, i.e. the solution of

$$\frac{\delta S[\Delta, \bar{\Delta}]}{\delta \Delta} = 0 \ . \tag{10.65}$$

We will assume that the mean field solution is a constant in τ and $\mathbf{x}$, i.e. $\Delta(\tau, \mathbf{x}) = \Delta_0$. Thus, the mean field equation (10.65) becomes

$$\frac{1}{g}\bar{\Delta}_0 = \mathrm{Tr}\left[\mathcal{O}\,\frac{\delta \mathcal{O}^{-1}}{\delta \Delta}\right]_{\Delta=\Delta_0} \ , \tag{10.66}$$

Where $\mathcal{O}$ is the inverse of $\mathcal{O}^{-1}$. Ignoring the electromagnetic fields for now, and using (10.60), we have

$$\frac{\delta \mathcal{O}^{-1}}{\delta \Delta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ . \tag{10.67}$$

The operator $\mathcal{O}$ is given by

$$\mathcal{O} = \frac{1}{\det \mathcal{O}^{-1}} \begin{pmatrix} -\partial_\tau - \frac{\nabla^2}{2m} - \mu & -\Delta \\ -\bar{\Delta} & -\partial_\tau + \frac{\nabla^2}{2m} + \mu \end{pmatrix} \ . \tag{10.68}$$

Since we need to compute the trace, it is convenient to go to the Matsubara and momentum representations. Defining

$$\xi_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \mu \ , \tag{10.69}$$

we can rewrite (10.60) as

$$\mathcal{O}^{-1} = \frac{1}{\beta L^d} \begin{pmatrix} i\omega_n - \xi_{\mathbf{p}} & \Delta_0 \\ \bar{\Delta}_0 & i\omega_n + \xi_{\mathbf{p}} \end{pmatrix} \ , \tag{10.70}$$

with ω_n the Matsubara frequencies. The determinant in (10.68) can then be written as

$$\det \mathcal{O}^{-1} = \frac{1}{\beta L^d} \sum_{n,\mathbf{p}} \left(|\Delta_0|^2 + \omega_n^2 + \xi_{\mathbf{p}}^2\right) \ . \tag{10.71}$$

The mean field equation (10.65) then becomes

$$\frac{1}{g}\bar{\Delta}_0 - \frac{1}{\beta L^d}\sum_{n,\mathbf{p}}\frac{\bar{\Delta}_0}{\omega_n^2+\xi_{\mathbf{p}}^2+|\Delta_0|^2} = 0 \ . \tag{10.72}$$

We then arrive to the so-called gap equation in the mean field approximation

$$\frac{1}{g} = \frac{T}{L^d}\sum_{n,\mathbf{p}}\frac{1}{\omega_n^2+\xi_{\mathbf{p}}^2+|\Delta_0|^2} \ . \tag{10.73}$$

The Matsubara sum can be performed, resulting in

$$\frac{1}{g} = \frac{1}{L^d}\sum_{\mathbf{p}}\frac{1-2n_F(\lambda_{\mathbf{p}})}{2\lambda_{\mathbf{p}}} \ , \tag{10.74}$$

where we defined [1]

$$\lambda_{\mathbf{p}}^2 \equiv \xi_{\mathbf{p}}^2 + |\Delta_0|^2 \ , \tag{10.75}$$

and

$$n_F(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \ , \tag{10.76}$$

is the Fermi-Dirac distribution. We want to turn the sum over $\mathbf{p}$ into an integral. for this we need to remember that the BCS interaction will be attractive as long as $|\xi_{\mathbf{p}}| < \omega_D$, that is within a small slice of $2\omega_D$ around the Fermi surface. Then (10.74) can be written as

$$\frac{1}{g} = \int_{-\omega_D}^{\omega_D} d\xi \frac{1}{L^d}\sum_{\mathbf{p}}\delta(\xi-\xi_{\mathbf{p}})\frac{1-2n_F(\lambda(\xi))}{2\lambda(\xi)} \ . \tag{10.77}$$

We can identify the density of states in this integral as

$$\nu(\xi) = \frac{1}{L^d}\sum_{\mathbf{p}}\delta(\xi-\xi_{\mathbf{p}}) \ , \tag{10.78}$$

[1] Notice that $\lambda_{\mathbf{p}}$ coincides with the eigenvalues $E_{\mathbf{p}}$ we defined above up to the fact that now we take temperature effects into account.

and use the fact that

$$1 - 2n_F(\epsilon) = \tanh\left(\epsilon/2T\right) \ , \tag{10.79}$$

to obtain

$$\frac{1}{g} = \int_{-\omega_D}^{\omega_D} d\xi\, \nu(\xi)\, \frac{\tanh\left(\frac{\lambda(\xi)}{2T}\right)}{2\lambda(\xi)} \ . \tag{10.80}$$

Given that the energy range being integrated is very narrow and close to the Fermi surface we make the approximation that in the integral (10.80)

$$\nu(\xi) \simeq \nu_F \ , \tag{10.81}$$

with ν_F the density of states at the Fermi surface. Then we can write

$$\boxed{\frac{1}{g\,\nu_F} = \int_0^{\omega_D} d\xi\, \frac{\tanh\left(\frac{\lambda(\xi)}{2T}\right)}{2\lambda(\xi)}} \ , \tag{10.82}$$

where we used that the integrand now is even in ξ. This is the gap equation in the mean field approximation. Notice that to recover our previous result from (10.36) we need to take the $T \ll \Delta_0$ limit, for which we have $\tanh(\lambda(\xi)/2T) \simeq 1$ resulting in

$$\frac{1}{g\,\nu_F} \simeq \text{arcsinh}\left(\frac{\omega_D}{\Delta_0}\right) \ , \tag{10.83}$$

as before. But now (10.82) gives us the correct temperature dependence for the order parameter Δ_0. In particular we are interested in the behavior close to the critical temperature, $T \lesssim T_c$. By expanding (10.82) around this point we can obtain (exercise) that

$$\Delta_0(T) \simeq \text{const.}\, \sqrt{T_c(T_c - T)} \ , \tag{10.84}$$

which is illustrated in Figure 10.2. This behavior of the order parameter with temperature has been experimentally confirmed in the so-called s-wave superconductors in a variety of materials.

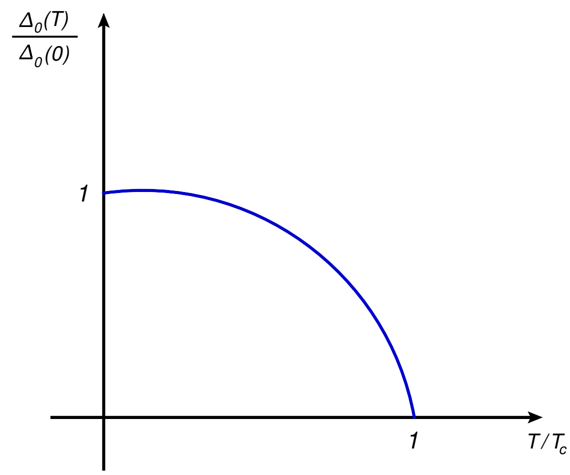

Figure 10.2: Behavior of the gap order parameter Δ_0 as a function of temperature.

Below, we will consider an expansion around small values of Δ_0, i.e. not far from the critical temperature. This will allow us to study fluctuations around the mean field solution close to the phase transition.

Landau-Ginzburg Expansion:
Close to the phase transition for $T \lesssim T_c$ we have that

$$\frac{\Delta(T)}{\Delta(0)} \ll 1 \ , \tag{10.85}$$

where we dropped the subscript 0 for simplicity. Thus, we expand around $\Delta = 0$, the symmetric phase (since $\Delta \neq 0$ spontaneously breaks the $U(1)$ gauge symmetry). To prepare this expansion it is convenient to rewrite (10.60) as

$$\mathcal{O}^{-1} \equiv \begin{pmatrix} \left(G_0^{(p)}\right)^{-1} & 0 \\ & \\ 0 & \left(G_0^{(h)}\right)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \Delta \\ & \\ \bar{\Delta} & 0 \end{pmatrix} \ , \tag{10.86}$$

which defines the operator in the symmetric phase through

$$\mathcal{O}^{-1} = \mathcal{O}_0^{-1} + \hat{\Delta} \ , \tag{10.87}$$

with $\hat{\Delta}$ the second term in (10.87). Then the trace in (10.64) is

$$
\begin{aligned}
\operatorname{Tr}\ln\left[\mathcal{O}^{-1}\right] &= \operatorname{Tr}\ln\left[\mathcal{O}_0^{-1}\left(1+\mathcal{O}\hat{\Delta}\right)\right] \\
&= \operatorname{Tr}\ln\left[\mathcal{O}_0^{-1}\right] + \operatorname{Tr}\left[\mathcal{O}_0\hat{\Delta}\right] - \frac{1}{2}\operatorname{Tr}\left[\left(\mathcal{O}_0\hat{\Delta}\right)^2\right] + \dots\ , \qquad (10.88)
\end{aligned}
$$

In the last line of (10.88), the first term does not depend on Δ, so it will only generate an irrelevant overall factor in the functional integral, while the second term vanishes on inspection. Then we can approximate the trace by

$$
\begin{aligned}
\operatorname{Tr}\ln\left[\mathcal{O}^{-1}\right] &= -\frac{1}{2}\operatorname{Tr}\left[\left(\mathcal{O}_0\hat{\Delta}\right)^2\right] + \dots \\
&= -\operatorname{Tr}\left[\mathcal{O}_{011}\mathcal{O}_{022}\Delta\bar{\Delta}\right] \ + \dots\ , \qquad (10.89)
\end{aligned}
$$

When going to the momentum/Matsubara representation we can write

$$
\operatorname{Tr}\ln\left[\mathcal{O}^{-1}\right] = \frac{1}{\beta \mathrm{L^d}}\sum_{\mathrm{q}}\sum_{\mathrm{p}} \mathrm{G_{0p}G_{0-p+q}}\Delta(\mathrm{q})\bar{\Delta}(\mathrm{q})\ , \qquad (10.90)
$$

where G_{op} is the momentum/Matsubara representation free two-point function. This results in the action

$$
S[\Delta,\bar{\Delta}] = \sum_q \left(\frac{1}{g} - \frac{T}{L^d}\sum_p G_{0p}G_{0-p+q}\right)|\Delta(q)|^2\ . \qquad (10.91)
$$

If we now consider the $q \to 0$ component, i.e. the constant order parameter, we see that there is the possibility of a sign change. In fact the quadratic action is unstable below a critical temperatura. We must them consider higher dimensional terms in order to stabilize it. The first one corresponds to extend the expansion in (10.88) to quartic order. Writing the resulting action back in space and time coordinates we have

$$
S[\Delta,\bar{\Delta}] = \int d\tau d^d r \left\{r(T)|\Delta|^2 + \lambda|\Delta|^4 + \mathcal{O}(\partial_\tau\Delta, \partial\Delta, |\Delta|^6)\right\}\ , \qquad (10.92)
$$

where we know that

$$
r(T) \sim T - T_c\ , \qquad (10.93)
$$

would change sign for temperatures below T_c and result in a non-zero value for the order paramenter Δ in the ground state. In the action above we neglected derivatives and higher dimensional terms of Δ. However, in here we must consider the presence of the electromagnetic field through derivatives. This will result in the Landau-Ginzburg theory for a superconductor. The spontaneous breaking of the $U(1)$ symmetry will result in a massive photon inside the superconductor. We will discuss these issues further below in the context of relativistic theories, but we will come back to superconductivity and the effects associated with the Anderson-Higgs mechanism, namely the Meissner effect and the existence of supercurrents.