1 The Renormalization Group

We saw that the renormalization procedure results in renormalized parameters acquiring a momentum dependence. In particular, we saw that the electric charge, the QED coupling, becomes a function of the momentum transfer squared, q^2. This was a consequence of computing the vacuum polarization of the photon

$$\Pi(q^2) = \hat{\Pi}(q^2) - \Pi(0) \ , \tag{1.1}$$

where $\hat{\Pi}(q^2)$ is finite. The divergence in $\Pi(0)$ is absorbed by the coupling at $q^2 = 0$ as

$$e^2 \equiv \frac{e_0}{1 - \Pi(0)} \ , \tag{1.2}$$

where e_0 is the unrenormalized coupling. Thus, we obtained the q^2-dependent coupling

$$e^2(q^2) = \frac{e^2}{1 - \hat{\Pi}(q^2)} \ , \tag{1.3}$$

where the finite part of the vacuum polarization was computed to be

$$\hat{\Pi}(q^2) = -\frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x) \ln\left(\frac{m^2}{m^2 - x(1-x)}\right) \ , \tag{1.4}$$

with $\alpha = e^2/4\pi$. In the limit $-q^2 \gg m^2$ we have

$$\hat{\Pi}(q^2) \simeq \frac{\alpha}{3\pi} \left\{ \ln\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{-q^2}\right) \ \right\} \ , \tag{1.5}$$

resulting in

$$\boxed{e^2(q^2) = \frac{e^2}{1 - \frac{\alpha}{3\pi} \ln\left(\frac{-q^2}{m^2}\right) - \frac{5}{3}}} , \tag{1.6}$$

once again, valid for $-q^2 \gg m^2$. In principle, we can choose arbitrary renormalization conditions. For instance, we use $q^2 = \mu_R^2$ in order to obtain (as long as $\mu_R^2 \gg m^2$)

$$e^2(\mu_R^2) \simeq e^2 \left(1 + \frac{\alpha}{3\pi} \ln \frac{\mu_R^2}{m^2} - \frac{5}{3}\right) \ , \tag{1.7}$$

with μ_R^2 is some arbitrary renormalization scale. Of course, we could have chosen some other different scale $\mu_R^{'2}$. However, the relation between the coupling evaluated in both these scales is known

$$e^2(\mu_R^{'2}) = e^2(\mu_R^2) + \frac{e^4}{12\pi^2} \ln \frac{\mu_R^{'2}}{\mu_R^2} \ , \tag{1.8}$$

so that the functional dependence on the renormalization scale is a property of the interaction. We define the logarithmic derivative of the scale-dependent coupling as

$$\beta(e) \equiv \frac{\partial e}{\partial \ln \mu_R^{'}} = \frac{e^3}{12\pi^2} + \mathcal{O}(e^5) \ , \tag{1.9}$$

which is the QED one-loop beta function. It characterizes the dependence of the coupling with the choice of renormalization scale. In particular, in the case of QED we see that it is positive, meaning that the coupling growths with energy scale. This growth of the coupling with energy is characteristic of all abelian gauge theories (or $U(1)$ gauge theories) and it can be understood in terms of charge screening by vacuum polarization. Consider the schematic case of Figure 1.1: a positive charge at the center and a probe charge at a given distance R. As R growths the screening due to vacuum polarization is larger and the effective charge felt by q is smaller.

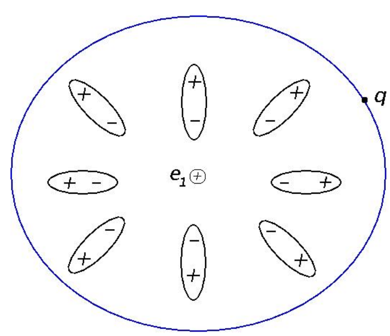

Figure 1.1: Charge screening by vacuum polarization. As we move away from the positive charge the effective coupling sensed by a probe q is smaller due to the effect of the polarization of the vacuum.

Conversely, as R is smaller (or q^2 larger) the effective coupling growths. In the limit of $R \to 0$ the effective coupling diverges (i.e. $e_0 \to \infty$, and we see one more time that a diverging parameter of the theory is associated with taking a distance to zero or an energy/momentum to infinity. But the basic fact is that as we move from small to large scales the effective coupling of the theory changes due to the effects of quantum corrections. In the next section we explore this idea in a way pioneered by Kenneth Wilson.

1.1 Wilsonian Renormalization

The basic idea is very simple. We consider a quantum field theory and separate fields according to their momenta. "High energy" fields will then be integrated out in the function integral, leaving us with a low energy effective theory for the "low energy" fields. In order to implement this procedure we will formulate the theory in euclidean d-dimensional space. To illustrate the effect of this Wick rotation from four-dimensional Minkowski to euclidean space, let us consider a ϕ^4 theory with action for a real scalar field

$$S[\phi] = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right\} . \tag{1.10}$$

Defining the Wick rotation by the euclidean coordinate $x_4 \equiv ix_0$, we make the replacement

$$dx_0 = -idx_4 , \tag{1.11}$$

which results in $d^4x \to -id^4x$, where the latter is the four-dimensional euclidean differential volume. The action (1.10) now reads

$$S[\phi] = i\int d^4x \left\{ \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right\} \equiv iS_E[\phi] , \tag{1.12}$$

where in the first term both contracted indices are down to signal that this is just the euclidean four-dimensional gradient squared, and we defined the euclidean action such that the generating functional is given by

$$Z = \int \mathcal{D}\phi \, e^{-S_E[\phi]} . \tag{1.13}$$

This is not just the Wick-rotated euclidean action of the relativistic ϕ^4 theory. It is also the partition function of many condensed matter systems. In fact the euclidean action

$$S_E[\phi] = \int d^dx \left\{ \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right\} , \tag{1.14}$$

is just the hamiltonian of a d-dimensional system, where now the kinetic term turned into the energy associated with field gradients. This makes (1.13) the partition function of the system. We will see later how to derive the proper partition function for statistical systems using coherent states. But for now, we should know that (1.14) describes several condensed matter systems within some approximation. For instance, this is the Ginzburg-Landau action describing a d-dimensional superconductor, where ϕ is the order paramenter of the pase transition (the energy gap). It is also a long range description of the d-dimensional Ising model, which describes the interactions of spins in a magnetic system.

The Wilsonian formulation starts by considering and overall momentum cutoff for the field ϕ. That is, we start from the generating functional

$$Z = \int [\mathcal{D}\phi]_\Lambda \; e^{-\int d^dx \left\{ \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right\}} \;, \tag{1.15}$$

where

$$[\mathcal{D}\phi]_\Lambda = \prod_{|k|<\Lambda} d\phi(k) \;, \tag{1.16}$$

is the measure corresponding to the field modes with momenta below the cutoff Λ, i.e. $|k| < \Lambda$. The cutoff is arbitrary, but at least we must assume that is much larger that any dimensionfull parameter in the action. In this case this means $\Lambda \gg m$. Next, we separate high and low energy modes by writing

$$\phi(k) = \phi_\ell(k) + \phi_h(k) \;, \tag{1.17}$$

where we are using the momentum representation for the fields defined by the Fourier transformation

$$\phi(x) = \int \frac{d^dk}{(2\pi)^d}\, e^{-ik\cdot x}\phi(k). \tag{1.18}$$

Here we defined the high-momentum modes as

$$\phi_h(k) = \begin{cases} \phi(k) & \text{for} \quad b\Lambda \leq |k|\Lambda \\ \\ 0 & \text{for} \quad |k| < b\Lambda \end{cases} \tag{1.19}$$

whereas the low-momentum ones are defined as

$$\phi_\ell(k) = \begin{cases} \phi(k) & \text{for} \quad |k| < b\Lambda \\ 0 & \text{for} \quad |k| > b\Lambda \end{cases} \tag{1.20}$$

We illustrate this scale separation in Figure 1.2.

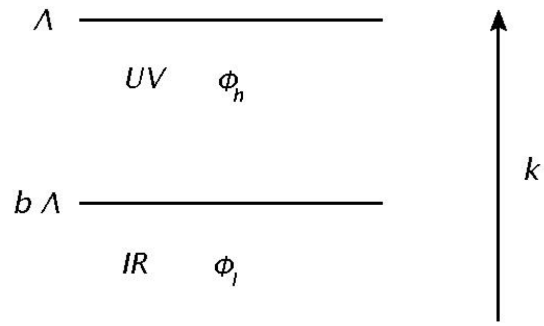

Figure 1.2: Separation of low and high energy modes, $\phi_\ell(k)$ and $\phi_h(k)$. The separation is made at the momentum $b\Lambda$, with $0 < b < 1$.

The idea is to split the integration in (1.15) and integrate out the $\phi_h(k)$ high-energy modes to obtain an effective theory for the low-energy modes $\phi_\ell(k)$. The generating functional in terms of the split fields is

$$\begin{aligned} Z &= \int \mathcal{D}\phi_\ell \int \mathcal{D}\phi_h \, e^{-\int d^dx \left\{ \frac{1}{2}(\partial_\mu\phi_\ell + \partial_\mu\phi_h)^2 + \frac{1}{2}m^2(\phi_\ell+\phi_h)^2 + \frac{\lambda}{4!}(\phi_\ell+\phi_h)^4 \right\}} \\ &= \int \mathcal{D}\phi_\ell \, e^{-S_E[\phi_\ell]} \int \mathcal{D}\phi_h \, e^{-\int d^dx \left\{ \frac{1}{2}(\partial_\mu\phi_h)^2 + \frac{1}{2}m^2\phi_h^2 + \lambda\left(\frac{1}{6}\phi_\ell^3\phi_h + \frac{1}{4}\phi_\ell^2\phi_h^2 + \frac{1}{6}\phi_\ell\phi_h^3 + \frac{1}{4!}\phi_h^4\right)\right\}} \ . \end{aligned} \tag{1.21}$$

Here, $S_E[\phi_\ell]$ is the euclidean action (1.14) but for the low-energy modes. In (1.21) we used the fact that mixed terms that are linear in both ϕ_ℓ and ϕ_h, such as $\partial_\mu\phi_\ell\partial_\mu\phi_h$ and $\phi_\ell\phi_h$, do not contribute to the action since the orthogonality conditions in momentum space apply to them and result in momentum delta functions that vanish ($\phi_\ell(k)$ and $\phi_h(k)$ cannot have the same momentum). On the other hand, non-linear mixed terms such as the ones in the last term in (1.21) are not affected by this and give non-vanishing contributions. Integrating out ϕ_h in (1.21) will result in an effective action for ϕ_ℓ, i.e.

$$Z = \int \mathcal{D}\phi_\ell \, e^{-S_E^{\text{eff.}}[\phi_\ell]} \ , \tag{1.22}$$

where

$$S_E^{\text{eff.}}[\phi_\ell] = \int d^dx \left\{ \frac{1}{2}\partial_\mu\phi_\ell\partial_\mu\phi_\ell + \frac{1}{2}m^2\phi_\ell^2 + \frac{\lambda}{4!}\phi_\ell^4 + \mathcal{O}(\lambda) \text{ corrections } \right\} , \tag{1.23}$$

where the $\mathcal{O}(\lambda)$ corrections come from integrating out ϕ_h since the last term in the action in (1.21) is the only one containing both ϕ_ℓ and ϕ_h (The quadratic terms in ϕ_h integrate to a ϕ_ℓ-independent determinant which is absorbed as normalization).

1.2 Integrating Out High Energy Modes

In addition to considering the $\mathcal{O}(\lambda)$ terms as perturbations, we will also treat the mass term as such, since as we pointed out earlier $m^2 \ll \Lambda^2$. Then, the leading order term depending on the high-energy modes is

$$\begin{aligned} \int d^dx \frac{1}{2}\partial_\mu\phi_h\partial_\mu\phi_h &= \frac{1}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\phi_h(k)\,(-ik)\,(-ik')\phi_h(k')e^{i(k+k')\cdot x} \\ &= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\phi_h(k)\,k^2\,\phi_h(-k)\ . \end{aligned} \tag{1.24}$$

From here we can obtain the momentum-space propagator by noticing that, up to normalization, the free theory for ϕ_h is defined by

$$Z \sim \int \mathcal{D}\phi_h e^{-\int \frac{d^dk}{(2\pi)^d}\left\{\frac{1}{2}\phi_h(k)k^2\phi_h(-k)+J(k)\phi_h(k)\right\}} , \tag{1.25}$$

where we included the Fourier transform of an external source $J(x)$. Demanding that the operator

$$O_k = \frac{k^2}{(2\pi)^d} \tag{1.26}$$

is the momentum-space inverse propagator, we have

$$D_F(k+p) = \frac{(2\pi)^d}{k^2}\,\delta^{(d)}(k+p)\ . \tag{1.27}$$

Now we are in a position to integrate out ϕ_h. We start with the term

$$-\frac{\lambda}{4}\int d^dx\,\phi_\ell^2(x)\,\phi_h(x)\phi_h(x)\ , \tag{1.28}$$

which we get from expanding the exponential to leading order in λ. Since the high-momentum fields $\phi_h(x)$ are integrated out, they result in

$$\int \mathcal{D}\phi_h e^{-\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\left\{\phi_h(k)k^2\phi_h(-k)\right\}} \phi_h(q)\phi_h(p) \to D_F(q+p) \tag{1.29}$$

where the equality is obtained with the appropriate normalization. Then integrating out ϕ results in contracting the two high-momentum fields in (1.27) into the momentum propagator, which gives

$$-\frac{\lambda}{4}\int^{b\Lambda} \frac{d^dk_1}{(2\pi)^d}\phi(k_1)\phi(-k_1) \int_{b\Lambda}^{\Lambda} \frac{d^dk_2}{(2\pi)^d}\frac{1}{k_2^2} \tag{1.30}$$

Thus, if we define

$$\mu \equiv \frac{\lambda}{2}\int_{b\Lambda}^{\Lambda} \frac{d^dk_2}{(2\pi)^d}\frac{1}{k_2^2} \ , \tag{1.31}$$

we can interpret it as a shift on the mass m^2 since it is equivalent to a term in the action as

$$-\frac{\mu}{2}\int d^dx\phi_\ell^2(x) \ . \tag{1.32}$$

The integral in (1.31) results in

$$\mu = \lambda\frac{1}{(4\pi)^d\Gamma(d/2)}\frac{\left(1-b^{d-2}\right)\,\Lambda^{d-2}}{d-2} \ , \tag{1.33}$$

which clearly shows the cutoff dependence of this mass shift. In terms of Feynman diagrams the interaction (1.28) is represented by the diagram of Figure 1.3, with the high-momentum fields ϕ_h represented by the double lines.

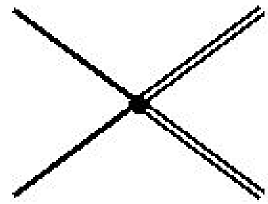

Figure 1.3: The interaction (1.28) between two low-momentum and two high-momentum fields. The latter are represented by double lines.

The mass shift defined by (1.31) is represented by the ϕ_h loop in Figure 1.4.

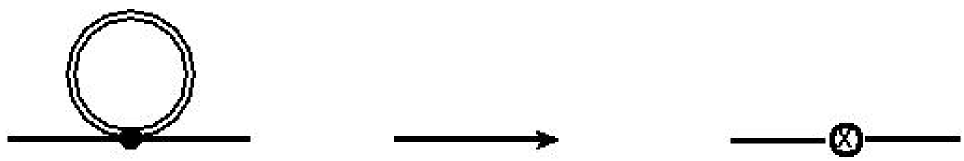

Figure 1.4: Mass shift from the integration of the high-momentum modes ϕ_h. The loop integration on the left results in a counter-term μ, represented on the right.

So we can interpret the integral over the high-momentum modes as generating a mass counter-term. From (1.33) we see that this is exactly what we obtain for $d = 4$ before: a quadratic dependence on the cutoff. The same interaction (1.28) leads to a shift in the ϕ_ℓ^4 interaction at $\mathcal{O}(\lambda^2)$, i.e. from the second term in the expansion of the exponential

$$\left(\frac{-\lambda}{4}\right)^2 \int d^dx\, \phi_\ell^2(x)\phi_h^2(x) \int d^dy\, \phi_\ell(y)^2\phi_h(y)^2 \ . \tag{1.34}$$

This results in

$$-\frac{\xi}{4!} \int d^dx\, \phi_\ell(x)^4 \ , \tag{1.35}$$

where we defined

$$\xi \equiv -4! \left(\frac{\lambda}{4}\right)^2 \int_{b\Lambda}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^2} \ . \tag{1.36}$$

Once again, the integral in (1.36) can be computed and gives

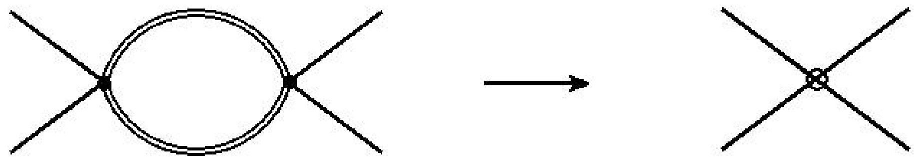

Figure 1.5: Shift in the ϕ_ℓ^4 interaction from the integration of the high-momentum modes ϕ_h. The loop integration on the left results in a counter-term ξ, represented on the right.

$$\xi = -\frac{3\lambda^2}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)} \frac{(1-b^{d-4})\Lambda^{d-4}}{d-4} \ . \tag{1.37}$$

In addition to the shifts of coefficients already existing in the original action, integrating out high-momentum modes also generates operators that were not there to begin with in the classical theory. For instance, the $\phi_\ell^3\phi_h$ term in (1.21) will result in a ϕ_ℓ^6 term once ϕ_h is integrated out. This can be seen schematically in Figure 1.6.

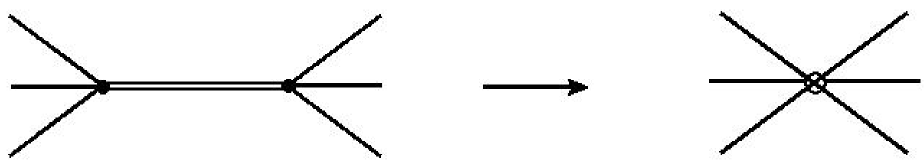

Figure 1.6: Generation of the higher-dimensional operator ϕ_ℓ^6 by integrating out ϕ_h.

From this last exercise it is clear that integrating out ϕ_f, weather at tree-level as in Figure 1.6 or in loops, we can generate an infinite tower of higher-dimensional operators. For instance, in $d = 4$ the ϕ_ℓ^6 operator is non-renormalizable. This appears surprising since we started with a renormalizable theory. We will address this point in detail in the next couple of lectures. But already we can see that when deciding to write only a finite set of operators in the action, say the ones that are renormalizable, we are neglecting operators that in principle should be there. Infinitely many of them. However, as we will see next, non-renormalizable operators are suppressed by the cutoff Λ. So it turns out what we call a renormalizable theory, is one with a cutoff that is high enough, for instance when compared with the typical energy of the process we are describing.