Quantum Field Theory II
Homework 6
Due 15/12/2017

1. Consider the integral

$$\mathcal{I}[g] = \frac{1}{24\pi^2}\int d\theta_1 d\theta_2 d\theta_3\, \epsilon^{ijk}\,\mathrm{Tr}\left[g(\theta)\partial_i g(\theta)^{-1}\, g(\theta)\partial_j g(\theta)^{-1}\, g(\theta)\partial_k g(\theta)^{-1}\right]$$

where the g's are elements of $SU(2)$. The integral is on S_3, the four-dimensional sphere.

   (a) Show that $\mathcal{I}[g]$ is invariant under continuous deformations.

   (b) Compute $\mathcal{I}[g]$ for

$$g(\theta) = \theta_4 + i\vec{\sigma}\cdot\vec{\theta}$$

2. **Sine-Gordon Equation**: Consider the Lagrangian in (1+1) dimensions:

$$\mathcal{L} = (\partial_\mu\phi)\,\partial^\mu\phi \;- V(\phi),$$

with

$$V(\phi) = \frac{a}{b}\,(1-\cos(b\phi))$$

with a and b real parameters.

   (a) Verify that the field configuration

$$\phi(t,x) = \frac{4}{b}\,\arctan\left\{\exp\left[\pm\sqrt{ab}\,\frac{(x-vt)}{\sqrt{1-v^2}}\right]\right\}\,,$$

   with v an arbitrary constant, is a solution of the equations of motion.

   (b) Define a conserved current J^μ, such the associated charge

$$Q = \int_{-\infty}^{+\infty} dx J^0\,,$$

   is the winding number of the solution.

   (c) Show that the energy carried by this field configuration at $t=0$ is

$$E = 8\sqrt{\frac{a}{b^3}}\,.$$

   (d) Expanding the Lagrangean in powers of ϕ, find the mass and the quartic coupling constant in terms of a and b. Express the energy in terms of the mass and coupling constants.