Quantum Field Theory II

Homework 2

Due 28/09/2018

1. Starting from the action

$$S[\bar{\phi},\phi] = \int_0^\beta d\tau \left\{ \bar{\phi}\partial_\tau \phi + H(\bar{\phi},\phi) - \mu N(\bar{\phi},\phi) \right\},$$

   and using the Fourier transform into the Matsubara representation defined by

$$\phi(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} \phi_n \, e^{-i\omega_n \tau}, \qquad \phi_n = \frac{1}{\sqrt{\beta}} \int_0^\beta d\tau \, \phi(\tau) \, e^{-i\omega_n \tau},$$

   with $\omega_n = 2\pi n T$ and n integers, show that it can be written as

$$S[\bar{\phi},\phi] = \sum_{ij,n} \bar{\phi}_{i,n} \left[(-i\omega_n - \mu)\,\delta_{ij} + h_{ij} \right] \phi_{i,n} + \frac{1}{\beta} \sum_{ijk\ell,n_i} \bar{\phi}_{i,n_1} \bar{\phi}_{j,n_2} \phi_{k,n_3} \phi_{\ell,n_4} \, \delta_{n_1+n_2,n_3+n_4}$$

2. <u>Electron-Phonon Coupling</u> (Altland-Simons pg.187):
   Work through the derivation of the attractive electron interaction resulting from the electron-phonon coupling. The phonon Hamiltonian is schematically written as

$$H_{\text{ph.}} = \sum_{\mathbf{q},j} \omega_q \, a^\dagger_{\mathbf{q},j} \, a_{\mathbf{q},j}$$

   where $\mathbf{q}$ is the 3-momentum, the energies ω_q depend only on the momentum's abolute value, and $j = 1,2,3$ are spatial indices indicating that phonons can oscillate independently in all three directions. Phonons track the displacement of ions in the lattice. This displacement (in momentum space) is given by

$$\mathbf{u_q} = \frac{a_{\mathbf{q},\mathbf{j}} + a^\dagger_{-\mathbf{q},j}}{(2m\omega_q)^{1/2}} \, \hat{e}_j \ ,$$

   where $\hat{e}_j$ is the unit vector in the j direction. These displacements induce a polarization $\mathbf{P} \sim \mathbf{u}$ resulting in an induced charge $\rho_{\text{ind.}} \sim \nabla \cdot \mathbf{P}$ that couples to electrons. Then we can write the electron-phonon coupling hamiltonian as

$$H_{\text{el.-ph.}} = \gamma \int d^d r\, \hat{n}(\mathbf{r})\, \nabla \cdot \mathbf{u}(\mathbf{r})$$

where $\hat{n}(\mathbf{r}) = a^\dagger(\mathbf{r})\, a(\mathbf{r})$ is the electronic density.

(a) Show that the electron-phonon interaction hamiltonian can be written as

$$H_{\text{el.-ph.}} = \gamma \sum_{\mathbf{q},j} \frac{i q_j}{(2m\omega_q)^{1/2}}\, \hat{n}_{\mathbf{q}} (a_{\mathbf{q},j} + a^\dagger_{-\mathbf{q},j})$$

with

$$\hat{n}_{\mathbf{q}} = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}}$$

the density of electrons in terms of creation and annihilation operators.

(b) Write the coherent-state partition function as

$$Z = \int \mathcal{D}(\bar{\psi}, \psi) \int \mathcal{D}(\bar{\phi}, \phi)\, e^{-S_{\text{el.}}[\bar{\psi},\psi] - S_{\text{ph.}}[\bar{\phi},\phi] - S_{\text{el.-ph.}}[\bar{\psi},\psi,\bar{\phi},\phi]}$$

in terms of the fermionic and bosonic coherent-state variables ψ and ϕ. Show that

$$S_{\text{ph.}}[\bar{\phi}, \phi] = \sum_{q,j} \bar{\phi}_{q,j} (-i\omega_n + \omega_q) \phi_{q,j}$$

and

$$S_{\text{el.-ph.}}[\bar{\psi}, \psi, \bar{\phi}, \phi] = \gamma \sum_{q,j} \frac{i q_j}{(2m\omega_q)^{1/2}}\, \rho_q\, (\phi_{q,j} + \bar{\phi}_{-q,j})$$

where

$$\rho_q = \sum_k \bar{\psi}_{k+q} \psi_k \ .$$

(c) Integrate out the phonon fields and show that an attractive interaction between electrons is generated. (Remember that we need to shift the phonon fields to get rid of the electron-phonon interaction!)

<u>**Note**</u>: Show all your calculations in detail.