Quantum Field Theory II

Homework 2

Due 22/09/2017

1. Starting from the action

$$S[\bar{\phi},\phi] = \int_0^\beta d\tau \left\{ \bar{\phi}\partial_\tau \phi + H(\bar{\phi},\phi) - \mu N(\bar{\phi},\phi) \right\},$$

and using the Fourier transform into the Matsubara representation defined by

$$\phi(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} \phi_n \, e^{-i\omega_n \tau}, \qquad \phi_n = \frac{1}{\sqrt{\beta}} \int_0^\beta d\tau \, \phi(\tau) \, e^{-i\omega_n \tau},$$

with $\omega_n = 2\pi n T$ and n integers, show that it can be written as

$$S[\bar{\phi},\phi] = \sum_{ij,n} \bar{\phi}_{i,n} \left[(-i\omega_n - \mu)\, \delta_{ij} + h_{ij} \right] \phi_{i,n} + \frac{1}{\beta} \sum_{ijk\ell,n_i} \bar{\phi}_{i,n_1} \bar{\phi}_{j,n_2} \phi_{k,n_3} \phi_{\ell,n_4} \, \delta_{n_1+n_2,n_3+n_4}$$

2. Electron-Phonon Coupling (Altland-Simons pg.187):
Work through the derivation of the attractive electron interaction resulting from the electron-phonon coupling. The phonon Hamiltonian is schematically written as

$$H_{\text{ph.}} = \sum_{\mathbf{q},j} \omega_q \, a^\dagger_{\mathbf{q},j} \, a_{\mathbf{q},j}$$

where $\mathbf{q}$ is the 3-momentum, the energies ω_q depend only on the momentum's abolute value, and $j = 1, 2, 3$ are spatial indices indicatings that phonons can oscillate independently in all three directions. Phonons track the displacement of ions in the lattice. This displacement (in momentum space) is given by

$$\mathbf{u}_\mathbf{q} = \frac{a_{\mathbf{q},\mathbf{j}} + a^\dagger_{-\mathbf{q},j}}{(2m\omega_q)^{1/2}} \, \hat{e}_j \ ,$$

where $\hat{e}_j$ is the unit vector in the j direction. These displacements induce a polarization $\mathbf{P} \sim \mathbf{u}$ resulting in an induced charge $\rho_{\text{ind.}} \sim \nabla \cdot \mathbf{P}$ that couples to electrons. Then we can write the electron-phonon coupling hamiltonian as

$$H_{\text{el.-phi.}} = \gamma \int d^d r \, \hat{n}(\mathbf{r}) \, \nabla \cdot \mathbf{u}(\mathbf{r})$$

where $\hat{n}(\mathbf{r}) = a^\dagger(\mathbf{r}) \, a(\mathbf{r})$ is the electronic density.

(a) Show that the electron-phonon hamiltonian can be written as

$$H_{\text{el.-phi.}} = \gamma \sum_{\mathbf{q},j} \frac{i q_j}{(2m\omega_q)^{1/2}} \, \hat{n}_{\mathbf{q}} (a_{\mathbf{q},j} + a^\dagger_{-\mathbf{q},j})$$

with

$$\hat{n}_{\mathbf{q}} = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} \ .$$

(b) Write the coherent-state partition function as

$$Z = \int \mathcal{D}(\bar{\psi}, \psi) \int \mathcal{D}(\bar{\phi}, \phi) \, e^{-S_{\text{el.}}[\bar{\psi},\psi] - S_{\text{ph.}}[\bar{\phi},\phi] - S_{\text{el.-ph.}}[\bar{\psi},\psi,\bar{\phi},\phi]}$$

in terms of the fermionic and bosonic coherent-state variables ψ and ϕ. Show that

$$S_{\text{ph.}}[\bar{\phi}, \phi] = \sum_{q,j} \bar{\phi}_{q,j} (-i\omega_n + \omega_q) \phi_{q,j}$$

and

$$S_{\text{el.-ph.}}[\bar{\psi}, \psi, \bar{\phi}, \phi] = \gamma \sum_{q,j} \frac{i q_j}{(2m\omega_q)^{1/2}} \, \rho_q \, (\phi_{q,j} + \bar{\phi}_{-q,j})$$

where

$$\rho_q = \sum_k \bar{\psi}_{k+q} \psi_k \ .$$

(c) Integrate out the phonon fields and show that an attractive interaction between electrons is generated. (Remember that we need to shift the phonon fields to get rid of the electron-phonon interaction!)

Note: Show all your calculations in detail.

3. Consider the lagrangian for a complex scalar field $\phi(x)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - \frac{1}{2}\mu^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2 ,$$

(a) Obtain the value $|\phi_0(x)|^2$ that minimizes the potential energy. Under what circumstances it is non-vanishing ?

(b) Expand the theory around this minimum. That is write the field as

$$\phi(x) = \phi_0 + \eta(x) + i\xi(x) ,$$

where ϕ_0 is in general a complex constant and $\eta(x)$ and $\xi(x)$ are real scalar fields that vanish at the minimum of the potential. Show that, independently of the choice of ϕ_0 and as long as $|\phi_0|^2$ is the value tha minimizes the potential, there is always a massless field as well as a massive one. Compute the mass of the latter.

(c) Repeat point b) but now for the expansion around the minimum use

$$\phi(x) = (v + \rho(x))\ e^{i\pi(x)/v} ,$$

where $v \equiv |\phi_0|$ is real, and $\rho(x)$ and $\pi(x)$ are real scalar fields that vanish at the minimim of the potential. Compare your results with point b).