Lecture 5

Solutions to the Dirac Equation

As a first step before we quantize Dirac fields, we need to study the solutions to the Dirac equation in momentum space. We start from the equation in position space

$$(i\gamma^\mu\partial_\mu - m)\,\psi(x) = 0\ , \tag{5.1}$$

If we act on the left with the conjugate of the Dirac operator

$$(-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m)\,\psi(x) = 0\ , \tag{5.2}$$

and use the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{5.3}$$

to prove that

$$\partial_\nu\partial_\mu\gamma^\mu\gamma^\nu = \partial^2\ , \tag{5.4}$$

we arrive at

$$(\partial^2 + m^2)\,\psi(x) = 0\ . \tag{5.5}$$

This expression means that each of the spinorial components of $\psi_a(x)$ with $a = 1, 2, 3, 4$ the spinorial index, obeys the Klein-Gordon equation. A a result it is clear that in moementum space solutions will be of the form

$$e^{\pm ip\cdot x}\ , \tag{5.6}$$

multiplied by a spinor that depends on the momentum $\mathbf{p}$. Here we use covanriant notation for $p{\cdot}x = p_0 x_0 - \mathbf{p}{\cdot}\mathbf{x}$. The solutions with positive energy (the minus sign in the exponential) can be written as (up to a normalization we will worry about later)

$$\psi^+(x) \sim \int \frac{d^3p}{(2\pi)^3}\, u(\mathbf{p})\, e^{-ip\cdot x} \;, \tag{5.7}$$

where clearly the momentum-dependent coefficient carries spinorial indices (i.e. is a spinor) and, when applying (5.1) obeys

$$(\not{p} - m) u(\mathbf{p}) = 0 \;. \tag{5.8}$$

On the other hand, the negative energy solutions (corresponding to the plus sign in the exponential in (5.6)) are defined in momentum space as

$$\psi^-(x) \sim \int \frac{d^3p}{(2\pi)^3}\, v(\mathbf{p})\, e^{+ip\cdot x} \;, \tag{5.9}$$

which results in

$$(\not{p} + m) v(\mathbf{p}) = 0 \;. \tag{5.10}$$

In order to understand better how to build these solutions, let us step back to the original form of the Dirac equation in terms of the α_i matrices, with $i = 1, 2, 3, 4$. The Dirac equation in momentum space is

$$\aleph\, u(\mathbf{p}) = (\alpha \cdot \mathbf{p} + \alpha_4 m)\, u(\mathbf{p}) = E\, u(\mathbf{p}) \;, \tag{5.11}$$

where E is the energy eigenvalue. We will first consider the positive-energy solutions in momentum space for a fermion at rest, i.e. $\mathbf{p} = 0$. In this limit and using the standard representation

$$\alpha_4 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \;, \qquad \alpha_i = \begin{pmatrix} \mathbf{0} & \sigma_{\mathbf{i}} \\ \sigma_{\mathbf{i}} & \mathbf{0} \end{pmatrix} \;, \tag{5.12}$$

we have that

$$\aleph\, u(\mathbf{0}) = \begin{pmatrix} m\,\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -m\,\mathbf{1} \end{pmatrix} u(\mathbf{0}) \;, \tag{5.13}$$

where the masses in the matrix are multiplied by the 2×2 identity matrix. The eigenvalues are $\{m, m, -m, -m\}$, corresponding to the eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{5.14}$$

We see clearly, in this example at $\mathbf{p} = 0$, that the two first eigenvectors correspond to positive energy eigenvalues ($+m$) whereas the last two eigenvectors are those associated with negative energy eigenvalues (in this case $-m$). We can write this in compact form as

$$u^s(\mathbf{0}) = \begin{pmatrix} \chi^s \\ \mathbf{0} \end{pmatrix} , \qquad v^s(\mathbf{0}) = \begin{pmatrix} \mathbf{0} \\ \chi^s \end{pmatrix} , \tag{5.15}$$

where $s = 1, 2$ is a two-component spinor index and we have defined the two-component spinors

$$\chi^1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi^2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.16}$$

Armed with this notation, we can now obtain the general solutions of the Dirac equation in momentum space. Let us first consider the positive energy solutions $u^s(\mathbf{p})$. We can write them as

$$u(\mathbf{p}) = N_p\,(\not{p} + m)\,u(\mathbf{0}) \;, \tag{5.17}$$

where N_p is a momentum dependent normalization to be determined later. The reason why we can write the solution as in (5.17) is that it trivially satisfies (5.8) since

$$(\not{p} - m)\,u(\mathbf{p}) = (\not{p} - m)\,(\not{p} + m)\,u(\mathbf{0}) = 0 \;, \tag{5.18}$$

since $(\not{p} - m)(\not{p} + m) = p^\mu p_\mu - m^2 = 0$[1]. Explicitly, we have

$$\begin{aligned} \not{p} + m &= E\gamma^0 - \gamma\cdot\mathbf{p} + m \\ &= \begin{pmatrix} E + m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & m - E \end{pmatrix} , \end{aligned} \tag{5.19}$$

[1]Convince yourself of this by proving that $\not{p}\,\not{p} = p^\mu p_\mu = p^2$

where we use the standard representation for the γ matrices. Then, applying (5.19) to $u(\mathbf{0})$ gives

$$u^s(\mathbf{p}) = N_p\,(\not{p} + m)\begin{pmatrix}\chi^s\\ \mathbf{0}\end{pmatrix} = N_p\begin{pmatrix}(E+m)\chi^s\\ \sigma\cdot\mathbf{p}\chi^s\end{pmatrix} = N_p\,(E+m)\begin{pmatrix}\chi^s\\ \frac{\sigma\cdot\mathbf{p}}{E+m}\,\chi^s\end{pmatrix}\ . \tag{5.20}$$

Finally, in order to obtain the normalization of the spinors we will impose that factors of $1/\sqrt{2E}$ that we will have in the fermion field momentum expansion in terms of $u^s(\mathbf{p})$ and $v^s(\mathbf{p})$ are cancelled. This requires that

$$u^{s\dagger}(\mathbf{p})\,u^s(\mathbf{p}) = 2E\delta^{sr}\ , \tag{5.21}$$

which results in

$$N_p = \frac{1}{\sqrt{E+m}}\ . \tag{5.22}$$

Using (5.22) in (5.20) we obtain the final expression for the positive energy solutions,

$$\boxed{u^s(\mathbf{p}) = \sqrt{E+m}\begin{pmatrix}\chi^s\\ \frac{\sigma\cdot\mathbf{p}}{E+m}\,\chi^s\end{pmatrix}\ .} \tag{5.23}$$

Similarly, postulating that

$$v^s(\mathbf{p}) = N_p(\not{p} - m)\,v^s(\mathbf{0})\ , \tag{5.24}$$

which clearly satisfies

$$(\not{p} + m)\,v^s(\mathbf{p}) = 0\ , \tag{5.25}$$

and following the same procedure as for the $u^s(\mathbf{p})$ spinors, we obtain

$$\boxed{v^s(\mathbf{p}) = \sqrt{E+m}\begin{pmatrix}\frac{\sigma\cdot\mathbf{p}}{E+m}\,\chi^s\\ \chi^s\end{pmatrix}}. \tag{5.26}$$

Equations (5.23) and (5.26) are the spinors in momentum space that solve the Dirac equation, with the plus and minus signs in (5.6) respectively. Each of them consists actually of two solutions, one with spin "up" ($s = 1$) and the other with spin "down" ($s = 2$).

5.1 Some Properties of the Solutions

It is useful to have the follow identities:

$$\begin{aligned} \bar{u}^s(\mathbf{p})\, u^r(\mathbf{p}) &= 2m\,\delta^{sr} \\ \bar{v}^s(\mathbf{p})\, v^r(\mathbf{p}) &= -2m\,\delta^{sr} \ , \end{aligned} \tag{5.27}$$

where, as usual, we have $\bar{u}^s = u^{s\dagger}\gamma^0$, and analogously for $\bar{v}^s$. To prove (5.27) we just use (5.23) to write

$$\begin{aligned} \bar{u}^{s\dagger}(\mathbf{p})\, u^r(\mathbf{p}) &= (E+m)\begin{pmatrix} \chi^{s\dagger} & \frac{\sigma\cdot\mathbf{p}}{E+m}\chi^{s\dagger} \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}\begin{pmatrix} \chi^r \\ \frac{\sigma\cdot\mathbf{p}}{E+m}\chi^r \end{pmatrix} , \\ &= (E+m)\begin{pmatrix} \chi^{s\dagger} & \frac{\sigma\cdot\mathbf{p}}{E+m}\chi^{s\dagger} \end{pmatrix}\begin{pmatrix} \chi^r \\ -\frac{\sigma\cdot\mathbf{p}}{E+m}\chi^r \end{pmatrix} , \\ &= (E+m)\left(1 - \frac{(\sigma\cdot\mathbf{p})^2}{((E+m)^2}\right) = E+m-(E-m) \ , \\ &= 2m \ . \end{aligned} \tag{5.28}$$

Proving the second equation in (5.27) is very similar. Finally, it is strightforward to see explicitely that the $u^s(\mathbf{p})$ and $v^r(\mathbf{p})$ solutions are orthogonal.
Also very useful, are the following identities, which will call polarization (or helicity) sums.

$$\begin{aligned} \sum_{s=1,2} u^s(\mathbf{p})\,\bar{u}^s(\mathbf{p}) &= \not{p} + m \ , \\ \sum_{s=1,2} v^s(\mathbf{p})\,\bar{v}^s(\mathbf{p}) &= \not{p} - m \ , \ . \end{aligned} \tag{5.29}$$

To prove the first identity in (5.29) we start from

$$\sum_{s=1,2} u^s(\mathbf{p})\,\bar{u}^s(\mathbf{p}) = (E+m)\sum_s \begin{pmatrix} \chi^s \\ \frac{\sigma\cdot\mathbf{p}}{E+m}\chi^s \end{pmatrix}\begin{pmatrix} \chi^{s\dagger} & -\frac{\sigma\cdot\mathbf{p}}{E+m}\chi^{s\dagger} \end{pmatrix} \ . \tag{5.30}$$

Here, it would be useful to do the two-component spin sums carefully. The expression (5.30) is a 4×4 matrix. At ever 2×2 block there is a spin sum resulting in

$$\sum_s \chi^s\,\chi^{s\dagger} = \mathbf{1} \ , \tag{5.31}$$

i.e. in the 2×2 identity matrix. Then we have

$$\sum_{s=1,2} u^s(\mathbf{p})\,\bar{u}^s(\mathbf{p}) \quad = \quad (E+m)\begin{pmatrix} \mathbf{1} & -\frac{\sigma\cdot\mathbf{p}}{E+m} \\ & \\ \frac{\sigma\cdot\mathbf{p}}{E+m} & -\frac{(\sigma\cdot\mathbf{p})^2}{(E+m)^2} \end{pmatrix} . \tag{5.32}$$

Using that $(\sigma\cdot\mathbf{p})^2 = (\mathbf{p})^2$ (prove it!) and writing $(\mathbf{p})^2 = (E-m)(E+m)$ in the expression above, we arrive at

$$\sum_{s=1,2} u^s(\mathbf{p})\,\bar{u}^s(\mathbf{p}) = \begin{pmatrix} E+m & -\sigma\cdot\mathbf{p} \\ & \\ \sigma\cdot\mathbf{p} & m-E \end{pmatrix} = \not{p} + m , \tag{5.33}$$

where the last equality comes from using (5.19). Deriving the second equality in (5.29) is very similar and is left as an exercise.

Additional suggested readings

- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Chapter 2.