Chapter 6

Relativistic Strings

We now begin our study of the classical relativistic string – a string that is, in many ways, much more elegant than the non-relativistic one considered before. Inspired by the point particle case, we focus our attention on the surface traced out by the string in spacetime. We use the proper area of this surface as the action, called the Nambu-Goto action. We study the reparameterization property of this action, identify the string tension, and find the equations of motion. For open strings, we focus on the motion of the endpoints, and introduce the concept of D-branes. Finally, we see that the only physical motion is transverse to the string.

6.1 Area functional for spatial surfaces

The action for a relativistic string must be a functional of the string trajectory. Just as a particle traces out a line in spacetime, a string traces out a surface. The line traced out by the particle in spacetime was called the world-line. The two-dimensional surface traced out by a string in spacetime will be called the world-sheet. A closed string, for example, will trace out a tube, while an open string will trace out a strip. These two-dimensional world-sheets are shown in the spacetime diagram of Figure 6.1. The lines of constant $x^0$ in these surfaces are the strings. These are the objects an observer sees at the fixed time $x^0$. They are open curves for the surface describing the open string evolution (left side), and they are closed curves for the surface describing the closed string evolution (right side).

In Chapter 5 we learned that the point particle action was given by the
proper time associated to the point particle world-line. The proper time, multiplied by $c$, is an invariant “length” associated to the world-line. For strings we will define the Lorentz invariant “proper area” of a world-sheet. The relativistic string action will be proportional to this proper area, and is called the Nambu-Goto action.

Figure 6.1: The world-sheets traced out by an open string and by a closed string.

Area functionals are useful in other applications: a soap film held between two rings, for example, automatically constructs the surface of minimal area which joins one ring to the other, as in Figure 6.2. The string world-sheet and the soap bubble between two rings are very different types of surfaces. At any given instant of time a Lorentz observer will see the full two-dimensional surface of the soap film, but he or she can only see one string from the two-dimensional world-sheet. Imagine the soap film is static in some Lorentz frame. In this case, time is not relevant to the description of the film, and we think of the film as a spatial surface, namely, a surface that extends along two spatial dimensions. The surface exists in its entirety at any instant of time. We will first study these familiar surfaces, and then we will apply our experience to the case of surfaces in spacetime.

A line in space can be parameterized using only one parameter. A surface in space is two-dimensional, so it requires two parameters $\xi^1$ and $\xi^2$. Given a parameterized surface, we can draw on that surface the lines of constant $\xi^1$ and the lines of constant $\xi^2$. These lines cover the surface with a grid. We call target space the world where the two-dimensional surface lives. In the case of a soap bubble in three dimensions, the target space is the three
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dimensional space $x^1$, $x^2$ and $x^3$. The parameterized surface is described by the collection of functions

$$\vec{x}(\xi^1, \xi^2) = \left( x^1(\xi^1, \xi^2), \ x^2(\xi^1, \xi^2), \ x^3(\xi^1, \xi^2) \right).$$

The parameter space is defined by the ranges of the parameters $\xi^1$ and $\xi^2$. It may be a square, for example, if we use parameters $\xi^1, \xi^2 \in [0, \pi]$. The real surface is the image of the parameter space under the map $\vec{x}(\xi^1, \xi^2)$. The physical surface is a surface in target space. Alternative, we can view the parameters $\xi^1$ and $\xi^2$ as coordinates on the physical surface, at least locally. The map inverse to $\vec{x}$ takes the surface to the parameter space. Locally this map is one-to-one and it assigns to each point on the surface two coordinates: the values of the parameters $\xi^1$ and $\xi^2$.

![Figure 6.2: A spatial surface stretching between two rings. If the surface is a soap film, it would be a minimal area surface.](image)

We want to calculate the area of a small element of the target space surface. Let's start by looking at an infinitesimal rectangle on the parameter space. Denote the sides of the square by $d\xi^1$ and $d\xi^2$. We want to find $dA$, the area of the image of this little rectangle in the target space. As shown in Figure 6.3, this is the area of the actual piece of surface that corresponds to the infinitesimal square on parameter space.

Of course, there is no reason why that infinitesimal area element in target space should be a rectangle. In general, it is a parallelogram. Let's call the
Figure 6.3: Left side: the parameter space, with a little square selected. The target space surface with the image of the little square: a parallelogram whose sides are the vectors \( d\vec{v}_1 \) and \( d\vec{v}_2 \) (shown magnified at the end of the wiggly arrow).

sides of this parallelogram \( d\vec{v}_1 \) and \( d\vec{v}_2 \). They are the images under the map \( \bar{x} \) of the vectors \( (d\xi^1, 0) \) and \( (0, d\xi^2) \), respectively. We can write them as

\[
d\vec{v}_1 = \frac{\partial \bar{x}}{\partial \xi^1} d\xi^1, \quad d\vec{v}_2 = \frac{\partial \bar{x}}{\partial \xi^2} d\xi^2.
\]

(6.1.2)

This makes sense: \( \frac{\partial \bar{x}}{\partial \xi^1} \), for example, represents the rate of variation of the space coordinates with respect to \( \xi^1 \). Multiplying this rate by the length \( d\xi^1 \) of the horizontal side of the tiny parameter-space rectangle, gives us the vector \( d\vec{v}_1 \) representing this side in the target space. Now let us calculate the area \( dA \). Using the formula for the area of a parallelogram,

\[
dA = |d\vec{v}_1||d\vec{v}_2||\sin \theta| = \sqrt{1 - \cos^2 \theta} |d\vec{v}_2|^2 \sin \theta
\]

(6.1.3)

where \( \theta \) is the angle between the vectors \( d\vec{v}_1 \) and \( d\vec{v}_2 \). In terms of spatial dot products, we have

\[
dA = \sqrt{(d\vec{v}_1 \cdot d\vec{v}_1)(d\vec{v}_2 \cdot d\vec{v}_2) - (d\vec{v}_1 \cdot d\vec{v}_2)^2}.
\]

(6.1.4)

Finally, using (6.1.2),

\[
dA = d\xi^1 d\xi^2 \sqrt{\left( \frac{\partial \bar{x}}{\partial \xi^1} \cdot \frac{\partial \bar{x}}{\partial \xi^1} \right) \left( \frac{\partial \bar{x}}{\partial \xi^2} \cdot \frac{\partial \bar{x}}{\partial \xi^2} \right) - \left( \frac{\partial \bar{x}}{\partial \xi^1} \cdot \frac{\partial \bar{x}}{\partial \xi^2} \right)^2}.
\]

(6.1.5)
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This is the general expression for the area element of a parameterized spatial surface. The full area functional $A$ is given by

$$A = \int d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right) \left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right) - \left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right)^2}.$$  \hspace{1cm} (6.1.6)

The integral extends over the relevant ranges of the parameters $\xi^1$ and $\xi^2$. The solution of a minimal area problem for a spatial surface is the function $\vec{x}(\xi^1, \xi^2)$ that minimizes the functional $A$.

### 6.2 Reparameterization invariance of the area

As we have seen, the parameterization of a surface allows us to write the area element in an explicit form. The area of the surface, or even more, the area of any piece of the surface, should be independent of the parameterization chosen to calculate it. This is what we mean when we say that the area must be reparameterization invariant.

Because we will soon equate the relativistic string action to some notion of proper area, it, too, will be reparameterization invariant. This means that we will be free to choose the most useful parameterization without changing the underlying physics. A good choice of parameterization will enable us to solve the equations of motion of the relativistic string in an elegant way.

Reparameterization invariance is thus an important concept so it should be understood thoroughly. To this end we will try to make it manifest in our formulae. The aim of the following analysis is to show how this can be done.

Let’s begin by asking: is the area functional $A$ in (6.1.6) reparameterization invariant? We would certainly hope it is. In fact, at first glance it appears to be manifestly reparameterization invariant. After all, if one reparameterizes the surface with $\tilde{\xi}^1(\xi^1)$ and $\tilde{\xi}^2(\xi^2)$, then all of the derivatives introduced by the chain rule cancel appropriately.

**Quick Calculation 6.1.** Verify the above statement. That is, show that (6.1.6), written fully with tilde parameters $(\tilde{\xi}^1, \tilde{\xi}^2)$ equals (6.1.6) when $\tilde{\xi}^1 = \xi^1(\xi^1)$ and $\tilde{\xi}^2 = \xi^2(\xi^2)$.

The above reparameterization, however, is not completely general for it fails to mix the $\xi^1$ and $\xi^2$ coordinates. Suppose, instead, that we make a reparameterization $\tilde{\xi}^1(\xi^1, \xi^2)$ and $\tilde{\xi}^2(\xi^1, \xi^2)$. This time we can verify, using
a somewhat laborious computation, that (6.1.6) is invariant under such a
reparameterization. But the invariance is no longer intuitively clear. To
make the reparameterization invariance of (6.1.6) manifest we will have to
rewrite the area functional in a different way.

We begin by observing how the measure of integration transforms. The
change-of-variable theorem from calculus tells us that
\[ d\xi^1 d\xi^2 = \left| \text{det} \left( \frac{\partial \xi^i}{\partial \tilde{\xi}^j} \right) \right| d\tilde{\xi}^1 d\tilde{\xi}^2, \quad (6.2.1) \]
where \( M = [M_{ij}] \) is the matrix defined by \( M_{ij} = \partial \xi^i / \partial \tilde{\xi}^j \). Similarly,
\[ d\tilde{\xi}^1 d\tilde{\xi}^2 = \left| \text{det} \left( \frac{\partial \tilde{\xi}^i}{\partial \xi^j} \right) \right| d\xi^1 d\xi^2, \quad (6.2.2) \]
where \( \tilde{M} = [\tilde{M}_{ij}] \) is the matrix defined by \( \tilde{M}_{ij} = \partial \tilde{\xi}^i / \partial \xi^j \). Combining equations (6.2.1) and (6.2.2), we see that
\[ \left| \text{det} M \right| \left| \text{det} \tilde{M} \right| = 1. \quad (6.2.3) \]
Let us now consider a target space surface \( S \), described by the mapping
functions \( \vec{x}(\xi^1, \xi^2) \). Given a vector \( d\vec{x} \) tangent to the surface, let \( ds \) denote
its length. Then we can write
\[ ds^2 \equiv (ds)^2 = d\vec{x} \cdot d\vec{x} \quad (6.2.4) \]
For surfaces in space, as we are considering now, it is not customary to add
a minus sign in front of \( ds^2 \) (compare with (2.2.15)). The vector \( d\vec{x} \) can be
expressed in terms of partial derivatives and the differentials \( d\xi^1, d\xi^2 \):
\[ d\vec{x} = \frac{\partial \vec{x}}{\partial \xi^i} d\xi^1 + \frac{\partial \vec{x}}{\partial \xi^i} d\xi^2 = \frac{\partial \vec{x}}{\partial \xi^i} d\xi^i. \quad (6.2.5) \]
The repeated index \( i \) is summed over its possible values 1 and 2. Back in
(6.2.4)
\[ ds^2 = \left( \frac{\partial \vec{x}}{\partial \xi^i} d\xi^i \right) \cdot \left( \frac{\partial \vec{x}}{\partial \xi^j} d\xi^j \right) = \frac{\partial \vec{x}}{\partial \xi^i} \cdot \frac{\partial \vec{x}}{\partial \xi^j} d\xi^i d\xi^j. \quad (6.2.6) \]
This can be neatly summarized as
\[ ds^2 = g_{ij}(\xi) d\xi^i d\xi^j, \quad (6.2.7) \]
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where \( g_{ij}(\xi) \) is defined as

\[
g_{ij}(\xi) \equiv \frac{\partial \vec{x}}{\partial \xi^i} \cdot \frac{\partial \vec{x}}{\partial \xi^j}.
\]  

(6.2.8)

The quantity \( g_{ij}(\xi) \) is known as the induced metric on \( S \). It is called a metric because (6.2.7) takes, up to a sign, the form of equation (3.6.2), where we introduced the general concept of a metric. It is a metric on \( S \) because, with \( \xi^i \) playing the role of coordinates on \( S \), equation (6.2.7) determines distances on \( S \). It is said to be induced because it uses the metric on the ambient space in which \( S \) lives to determine distances on \( S \). Indeed, the dot product which appears in definition (6.2.8) is to be performed in the space where \( S \) lives and therefore presupposes that a metric exists on that space. We only have two parameters \( \xi^1 \) and \( \xi^2 \), so the full matrix \( g_{ij} \) takes the form:

\[
g_{ij} = \begin{pmatrix}
\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^1} & \frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2} \\
\frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^1} & \frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^2}
\end{pmatrix}.
\]  

(6.2.9)

Now we see something truly nice! The determinant of \( g_{ij} \) is precisely the quantity which appears under the square root in (6.1.6). Letting

\[
g \equiv \det(g_{ij}),
\]  

(6.2.10)

we can write

\[
A = \int d\xi^1 d\xi^2 \sqrt{g}.
\]  

(6.2.11)

This is an elegant formula for the area in terms of the determinant of the induced metric. Instead of trying to understand the reparameterization invariance of (6.1.6), we now focus on the equivalent but simpler expression (6.2.11).

We are now in position to understand the invariance of the area in terms of the transformation properties of the metric \( g_{ij} \). The key to this lies in equation (6.2.7). The length squared \( ds^2 \) is a geometrical property of the vector \( d\vec{x} \) that must not depend upon the particular parameterization used to calculate it. For another set of parameters \( \tilde{\xi} \) and metric \( \tilde{g}(\tilde{\xi}) \), the following equality must therefore hold:

\[
g_{ij}(\xi) d\xi^i d\xi^j = \tilde{g}_{pq}(\tilde{\xi}) d\tilde{\xi}^p d\tilde{\xi}^q.
\]  

(6.2.12)
Making use of the chain rule to express the differentials \( d\tilde{\xi} \) in terms of differentials \( d\xi \),

\[
g_{ij}(\xi)d\xi^id\xi^j = \tilde{g}_{pq}(\tilde{\xi}) \frac{\partial \tilde{\xi}^p}{\partial \xi^i} \frac{\partial \tilde{\xi}^q}{\partial \xi^j} d\xi^i d\xi^j. \tag{6.2.13}
\]

Since this result holds for any choice of differentials \( d\xi \), we find a relation between the metric in \( \xi \) and \( \tilde{\xi} \) coordinates:

\[
g_{ij}(\xi) = \tilde{g}_{pq}(\tilde{\xi}) \frac{\partial \tilde{\xi}^p}{\partial \xi^i} \frac{\partial \tilde{\xi}^q}{\partial \xi^j}. \tag{6.2.14}
\]

Making use of the definition of \( \widetilde{M} \) below (6.2.2), we rewrite the above equation as

\[
g_{ij}(\xi) = \tilde{g}_{pq} \tilde{M}_{pi} \tilde{M}_{qj} = (\widetilde{M}^T)_{ip} \tilde{g}_{pq} \tilde{M}_{qj}. \tag{6.2.15}
\]

In matrix notation, the right-hand side is the product of three matrices. Taking the determinant and using the notation in (6.2.10) gives

\[
g = (\det \widetilde{M}^T) \tilde{g} (\det \widetilde{M}) = \tilde{g} (\det \widetilde{M})^2. \tag{6.2.16}
\]

Taking a square root

\[
\sqrt{g} = \sqrt{\tilde{g}} |\det \widetilde{M}|, \tag{6.2.17}
\]

we obtain the transformation property for the square root of the determinant of the metric.

We are finally ready to appreciate the reparameterization invariance of (6.2.11). Making use of (6.2.1), (6.2.17), and (6.2.3) we have

\[
\int d\xi^1 d\xi^2 \sqrt{g} = \int d\tilde{\xi}^1 d\tilde{\xi}^2 |\det M| \sqrt{\tilde{g}} |\det \widetilde{M}|
= \int d\tilde{\xi}^1 d\tilde{\xi}^2 \sqrt{\tilde{g}}, \tag{6.2.18}
\]

which proves the reparameterization invariance of the area functional. To the trained eye the area formula in (6.2.11) is manifestly reparameterization invariant. That is, once you know how metrics transform, the invariance is reasonably simple to establish. No cumbersome calculation is necessary.

Quick Calculation 6.2. Consider the equation \( \partial \xi^i / \partial \xi^j = \delta^i_j \) and use the chain rule to show the matrix property

\[
\widetilde{M}{\widetilde{M}} = 1. \tag{6.2.19}
\]

Show that \( \widetilde{M}{\widetilde{M}} = 1 \) holds as well. Finally, note that as a simple consequence, \( \det M \det \widetilde{M} = 1 \), a result stronger than the one we proved in (6.2.3).
6.3 Area functional for spacetime surfaces

Let us now move to our case of interest, the case of surfaces in spacetime. These surfaces are obtained by representing in spacetime the history of strings, in the same way as a spacetime world-line is obtained by representing the history of a particle. For the case of strings, we obtain a two-dimensional surface called the world-sheets of the string. Spacetime surfaces, such as string world-sheets, are not all that different from the spatial surfaces we considered in the previous section. They are two-dimensional, and require two parameters. Instead of calling the parameters $\xi^1$ and $\xi^2$, we give them special names: $\tau$ and $\sigma$.

Given our usual spacetime coordinates $x^\mu = (x^0, x^1, x^2, x^3, \ldots, x^d)$, the surface is described by the mapping functions

$$x^\mu(\tau, \sigma),$$

taking some region of the $(\tau, \sigma)$ parameter space into spacetime. Following a standard convention in string theory, we change the notation slightly. We will denote the above mapping functions with the capitalized symbols

$$X^\mu(\tau, \sigma).$$

We are not changing the meaning of the functions. Given a fixed point $(\tau, \sigma)$ in the parameter space, this point is mapped to a point with spacetime coordinates

$$(X^0(\tau, \sigma), X^1(\tau, \sigma), \ldots, X^d(\tau, \sigma)).$$

Why do we capitalize the $X$'s? Suppose we used the same symbol to denote spacetime coordinates and mapping functions. Then we could still distinguish between them by writing $x^\mu$ or $x^\mu(\tau, \sigma)$, but we would not have the luxury of dropping the $(\tau, \sigma)$ arguments. On the other hand, with $X^\mu$ we can drop the $(\tau, \sigma)$ arguments and you still know that we are talking about the mapping functions of the string. We will call the $X^\mu$ the string coordinates.

As before, the parameters $\tau$ and $\sigma$ can be viewed as coordinates on the world-sheet, at least locally. The map inverse to $X^\mu$ takes the world-sheet to the parameter space, and locally it assigns to each point on the surface two coordinates: the values of the parameters $\tau$ and $\sigma$. Introducing some potential for confusion, physicists also use the term world-sheet to denote the two-dimensional parameter space whose image under $X^\mu$ gives us
Unless explicitly stated, we will reserve the use of the term world-sheet for the spacetime surface. In Figure 6.4 we consider an open string: in the left side you see the parameter space surface, and to the right, the spacetime surface.

Figure 6.4: Left side: the parameter space \((\tau, \sigma)\), with a little square selected. Right side: The surface in target spacetime with the image of the little square: a parallelogram whose sides are the vectors \(dv^\mu_1\) and \(dv^\mu_2\).

To find the area element, we proceed as in the case of the spatial surface, this time using relativistic notation. The situation is illustrated in Figure 6.4. A little rectangle of sides \(d\tau\) and \(d\sigma\) in parameter-space, becomes a quadrilateral area element in spacetime. This quadrilateral is spanned by the vectors \(dv^\mu_1\) and \(dv^\mu_2\). Furthermore,

\[
dv^\mu_1 = \frac{\partial X^\mu}{\partial \tau} d\tau, \quad dv^\mu_2 = \frac{\partial X^\mu}{\partial \sigma} d\sigma, \quad (6.3.4)
\]

which are analogous to our earlier spatial formulae (6.1.2). We can now use the analog of (6.1.4) as a candidate for the area element \(dA\):

\[
dA \doteq \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2}, \quad (6.3.5)
\]

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1Given the spacetime connotation of the term *world*, and the mathematical flavor of the parameter space, I would call the target space surface the world-sheet, and the parameter space surface the *math-sheet*. 
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where the dot is the relativistic dot product. Using this dot product guarantees that the area element is Lorentz invariant: it is a proper area element. We wrote a question mark on top of the equal sign because there is one problem. Even though this is not obvious to us yet, the sign of the object under the square root is negative. To be able to take the square root we must exchange the two terms under the square root. This change of sign has no effect on the Lorentz invariance. Doing this, and using (6.3.4), we find that the proper area is given as

$$A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial \tau}\right)^2 \left(\frac{\partial X^\nu}{\partial \sigma}\right)^2}.$$  \hspace{1cm} (6.3.6)

Using the relativistic dot product notation,

$$A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2}.$$  \hspace{1cm} (6.3.7)

To understand why the above sign is correct we must convince ourselves that the expression under the square root is positive at any point on the world-sheet of a string.

What characterizes locally the spacetime surface traced by a string? The answer is quite interesting. Consider a point on the world-sheet and the set of all tangent vectors to the surface at that point. These vectors form a two-dimensional vector space. We claim that in this vector space there is a basis made by two vectors, one of which is spacelike, and one of which is timelike. This implies that at each point on the world-sheet there are both timelike and spacelike tangent directions.

The existence of a spacelike direction is easy to visualize: if you took a photograph of the string at some time, every tangent vector along the length of the string would point in a space-like direction. Indeed, in your frame, the events defining the string are simultaneous but spatially-separated.

To appreciate the need for a timelike vector at any point on the world-sheet, consider first the world-line of a point particle. The tangent vector to the world-line is timelike. At each point on the world-line this tangent vector can be used to produce an instantaneous Lorentz observer that sees the particle at rest. Suppose that the tangent vector to the world-line becomes spacelike at some point $P$. We could imagine at $P$ an infinite collection of Lorentz observers with their (spatial) origin at $P$, one for each possible
velocity. None of them can see the particle at rest at the origin, because the
world-line of the origin is timelike for all observers. This is an unphysical
situation.

The argument for the string is a little more subtle since there is no way to
tell how individual points on the string move. As we shall make abundantly
clear later on, the string is not made of constituents whose position we can
keep track of (there is just one exception: one can keep track of the motion of
the endpoints of an open string). For a closed string world-sheet, for example,
consider first the possibility that all along a closed string there is no timelike
tangent vector on the world-sheet. This means that we could display all
possible Lorentz observers at all points on the string, and no observer could
make any point on the string appear to be at rest! A similar unphysical result
would occur if any piece of the string failed to have timelike tangent vectors
on the world-sheet. Since the endpoints of the rest of the string cannot close
up the string instantaneously, a piece of the string would have failed to move
physically. We must have a timelike vector tangent to the world-sheet at all
points on the string.

The existence of both timelike directions and spacelike directions at any
point on the world-sheet is our criterion for physical motion. It guarantees
that equation (6.3.6) makes sense:

**Claim:** For a surface where there is at every point \( P \) both a timelike direction
and a spacelike direction, the quantity under the square root in (6.3.6) is
always positive, namely,

\[
\left( \frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma} \right)^2 - \left( \frac{\partial X}{\partial \sigma} \right)^2 \left( \frac{\partial X}{\partial \tau} \right)^2 > 0 .
\] (6.3.8)

**Proof:** We consider every vector tangent to the surface at some point \( P \), and
show that in this set there are both spacelike vectors and timelike vectors.
First we will parameterize all possible vectors and then search the parameter
space. The situation is illustrated in Figure 6.5.

Consider the set of tangent vectors \( v^\mu(\lambda) \) at \( P \) obtained as:

\[
v^\mu(\lambda) = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma},
\] (6.3.9)

where \( \lambda \) is a parameter that ranges from minus infinity to plus infinity. Since
\( \partial X^\mu/\partial \tau \) and \( \partial X^\mu/\partial \sigma \) are linearly independent tangent vectors, when we vary
\( \lambda \) we get, up to constant scalings, all tangent vectors at \( P \) (see Figure 6.5),
with the exception of $\partial X^\mu/\partial \sigma$, which is obtained in the limit $\lambda \to \infty$. Constant scalings does not matter in determining whether the vector is timelike or spacelike. To determine whether $v^\mu(\lambda)$ is timelike or spacelike, we consider its square:

$$v^2(\lambda) = v^\mu(\lambda)v_\mu(\lambda) = \lambda^2 \left( \frac{\partial X}{\partial \sigma} \right)^2 + 2\lambda \left( \frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma} \right) + \left( \frac{\partial X}{\partial \tau} \right)^2. \quad (6.3.10)$$

The derivatives of $X$ appearing here are just numbers, so we have a quadratic polynomial in $\lambda$. To have both timelike and spacelike tangent vectors at $P$, $v^2(\lambda)$ must take both negative and positive values as we vary $\lambda$. In other words, the equation $v^2(\lambda) = 0$ must have two real roots. For this to happen, the discriminant of the quadratic equation $v^2(\lambda) = 0$ must be positive. From (6.3.10) we see that this requires

$$\left( \frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma} \right)^2 - \left( \frac{\partial X}{\partial \sigma} \right)^2 \left( \frac{\partial X}{\partial \tau} \right)^2 > 0, \quad (6.3.11)$$

which is precisely the condition (6.3.8) we set out to prove!

**Quick Calculation 6.3.** Consider a point on the world-sheet where all tangent vectors are spacelike with the exception of one vector (or any number times the vector) that is null. Why is the quantity under the square root in (6.3.6) zero at this point?
6.4 The Nambu-Goto string action

Now that we are sure that the proper area functional in (6.3.7) is correctly defined, we can introduce the action of the relativistic string. This action is proportional to the proper area of the world-sheet. To have the units of action we must multiply the area functional by some suitable constants.

The area functional in (6.3.7) has units of length squared, as it must be. This is because $X^\mu$ has units of length, and each term under the square root has four $X$’s. The units of $\tau$ and $\sigma$ cancel out. Each term in the square root has two $\sigma$-derivatives and two $\tau$-derivatives. Their units cancel against the units of the differentials. Nevertheless, we will take $\sigma$ to have units of length and $\tau$ to have units of time. To summarize:

$$\begin{align*}
[\sigma] &= L, \quad [\tau] = T, \quad [X^\mu] = L, \quad [A] = L^2.
\end{align*}$$

(6.4.1)

Since $S$ must have units of $ML^2/T$ and $A$ has units of $L^2$, we must multiply the proper area by a quantity with units of $M/T$. The string tension $T_0$ has units of force, and force divided by velocity has the desired units of $M/T$. We can therefore multiply the proper area by $T_0/c$ to get a quantity with the units of action. Making use of (6.3.7) we set the string action equal to

$$S = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2(X')^2}.$$  

(6.4.2)

In writing this action we have introduced some notation for derivatives:

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma}.$$  

(6.4.3)

Of course, we have not yet confirmed that the symbol $T_0$ in the string action has the precise interpretation of tension, but we will do so below. We will also confirm that the overall negative sign multiplying the action is correct. The action $S$ is the Nambu-Goto action for the relativistic string.

It is crucial that this action be reparameterization-invariant. We can proceed just as we did with spatial surfaces to write the Nambu-Goto action in a manifestly reparameterization-invariant way. In this case we have

$$-ds^2 = dX^\mu dX_\mu = \eta_{\mu\nu} dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} d\xi^\alpha d\xi^\beta.$$  

(6.4.4)
Here \( \eta_{\mu\nu} \) is the target-space metric. Just as in our study of two-dimensional surfaces, we are motivated to define a metric \( \gamma = [\gamma_{\alpha\beta}] \) on the world-sheet:

\[
\gamma_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} = \frac{\partial X}{\partial \xi^\alpha} \cdot \frac{\partial X}{\partial \xi^\beta}.
\]

With \( \xi^1 = \tau \) and \( \xi^2 = \sigma \), the matrix \( \gamma_{\alpha\beta} \) is

\[
\gamma_{\alpha\beta} = \begin{bmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{bmatrix}.
\]

With the help of this metric we can write the Nambu-Goto action in the manifestly reparameterization-invariant form

\[
S = -\frac{T_0}{c} \int d\tau d\sigma \sqrt{-\gamma}, \quad \gamma = \det(\gamma_{\alpha\beta}).
\]

The analysis in section 6.2 of reparameterization invariance for spatial surfaces holds, without change, in the present case. Not only is the action (6.4.7) manifestly reparameterization-invariant, it is also more compact. In this form, one can readily generalize the Nambu-Goto action to describe the dynamics of objects that have more dimensions than strings. An action of this kind is useful as a first approximation to the dynamics of D-branes.

### 6.5 Equations of motion, boundary conditions and D-branes

In this section we will obtain the equations of motion that follow by variation of the string action. In doing so we will also have an opportunity to discuss the various boundary conditions that can be imposed on the ends of open strings. Dirichlet boundary conditions will be interpreted to arise due to the existence of D-branes.

Let us begin by writing the Nambu-Goto action (6.4.2) as the double integral of a Lagrangian density \( \mathcal{L} \):

\[
S = \int_{\tau_i}^{\tau_f} d\tau L = \int_{\tau_i}^{\tau_f} d\tau \int_{\sigma_1}^{\sigma_f} d\sigma \, \mathcal{L}(\dot{X}^\mu, X^\mu),
\]

\[ (6.5.1) \]
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where \( \mathcal{L} \) is given by

\[
\mathcal{L}(\dot{X}^\mu, X'^\mu) = -\frac{T_0}{c} \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}. \tag{6.5.2}
\]

We can obtain the equations of motion for the relativistic string by setting the variation of the action (6.5.1) equal to zero. The variation is simply

\[
\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[ \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} \frac{\partial (\delta X^\mu)}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial X'^\mu} \frac{\partial (\delta X^\mu)}{\partial \sigma} \right], \tag{6.5.3}
\]

where we have used

\[
\delta \dot{X}^\mu = \delta \left( \frac{\partial X^\mu}{\partial \tau} \right) = \frac{\partial (\delta X^\mu)}{\partial \tau}, \tag{6.5.4}
\]

and an analogous equation for \( \delta X'^\mu \).

The quantities \( \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} \) and \( \frac{\partial \mathcal{L}}{\partial X'^\mu} \) will appear frequently throughout the remainder of our discussion, so it is useful to introduce new symbols for them. This is just what we did when we studied the nonrelativistic string in Section 4.6. This time we find

\[
P^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}, \tag{6.5.5}
\]

\[
P^\sigma_\mu \equiv \frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}. \tag{6.5.6}
\]

Quick Calculation 6.4. Verify equations (6.5.5) and (6.5.6).

Using this notation, the variation \( \delta S \) in (6.5.3) becomes

\[
\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[ \frac{\partial}{\partial \tau} (\delta X^\mu P^\tau_\mu) + \frac{\partial}{\partial \sigma} (\delta X^\mu P^\sigma_\mu) - \delta X^\mu \left( \frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} \right) \right]. \tag{6.5.7}
\]

The first term on the right-hand side, being a full derivative in \( \tau \), will contribute terms proportional to \( \delta X^\mu(\tau_f, \sigma) \) and \( \delta X^\mu(\tau_i, \sigma) \). If the initial and final states of the string are specified, we can restrict ourselves to variations for which \( \delta X^\mu(\tau_f, \sigma) = \delta X^\mu(\tau_i, \sigma) = 0 \). We will always assume such variations, so we can forget about these terms. The variation then becomes

\[
\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \delta X^\mu \left( \frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} \right). \tag{6.5.8}
\]
The first term on the right-hand side has to do with the string endpoints. As before, there are two natural types of boundary conditions which one can impose on the endpoints. The first is Dirichlet boundary conditions, which require that the endpoints of the string remain fixed throughout the motion:

\[
\text{Dirichlet Boundary Condition: } \frac{\partial X^\mu}{\partial \tau}(0, \tau) = \frac{\partial X^\mu}{\partial \tau}(\sigma_1, \tau) = 0. \tag{6.5.9}
\]

Alternatively, rather than requiring that the \(\tau\)-derivatives vanish, we could simply specify constant values for \(X^\mu(0, \tau)\) and \(X^\mu(\sigma_1, \tau)\). If the string endpoints are fixed, the variations are set to vanish at the endpoints: \(\delta X^\mu(0, \tau) = 0\), and \(\delta X^\mu(\sigma_1, \tau) = 0\). This will guarantee that the first term in \(\delta S\) vanishes. Alternatively, setting

\[
\text{Free Boundary Condition: } P_\sigma^\mu(0, \tau) = P_\sigma^\mu(\sigma_1, \tau) = 0, \tag{6.5.10}
\]

would also result in the vanishing of the boundary term. This is the “free-endpoints” boundary condition for the relativistic string. For the non-relativistic string, the free-endpoints boundary condition implies the vanishing of \(P^x\), which imposes a Neumann boundary condition on the string coordinate (see (4.6.16)). While it will take us some work to get there, we will eventually understand (6.5.10) in terms of a Neumann boundary condition. Similarly, the Dirichlet boundary (6.5.9) will be shown to imply the vanishing of \(P_\mu^x\) at the string endpoints.

The above boundary conditions can be imposed in many possible ways. We need not use the same boundary condition for all values of the index \(\mu\). Some string coordinates may have a Dirichlet-type condition, and some others may have a free-type condition. Even more, for any given \(\mu\), the two endpoints of the open string need not satisfy the same boundary conditions: one end could be fixed and the other free. For closed strings there are no boundary conditions.

Let us digress for a while on the case of Dirichlet boundary conditions. It is clear from the study of non-relativistic strings that Dirichlet boundary conditions arise if string endpoints are attached to some physical objects. Consider, for example, Figure 4.2. On the left side of the figure, the string is attached to two points. On the right side of the figure the string is free to slide up and down at the endpoints, but the string endpoints are forced to stay on one-dimensional lines – horizontal motion of the endpoints is forbidden.
The objects where open string endpoint must lie on, are characterized by their dimensionality, more precisely, by the number of spatial dimensions they have. They are called D-branes, where the D stands for Dirichlet. The objects fixing the string on the left-side of Figure 4.2, are zero-dimensional. They are called D0-branes. The lines fixing the string on the right-side of the figure, are one-dimensional. They are called D1-branes.

Figure 6.6: A D2-brane stretched over the \((x^1, x^2)\) plane. The endpoints of the open string can move freely on the plane, but must remain attached to it. The coordinate \(x^3\) of the endpoints must vanish at all times. This is a Dirichlet boundary condition for the string coordinate \(X^3\).

A Dp-brane is an object with \(p\) spatial dimensions. Since the string endpoints must lie on the Dp-brane, a set of Dirichlet boundary conditions are specified. A flat D2-brane in a three-dimensional space, for example, is specified by one condition, say \(x^3 = 0\) (see Figure 6.6). This means that the D2-brane extends over the \((x^1, x^2)\) plane. The Dirichlet boundary condition applies to the string coordinate \(X^3\), which must vanish for the string endpoints. Since the motion of the string endpoint is free along the directions of the brane, the string coordinates \(X^1\) and \(X^2\) satisfy free boundary conditions. When the open string endpoints have free boundary conditions along all spatial directions, we still have a D-brane, but this time it is a space-filling D-brane. The D-brane extends all over space, and since open
string endpoints can be anywhere on the D-brane, open string endpoints are completely free.

For (quantum) relativistic strings the consistency of Dirichlet boundary conditions allows one to discover the properties of D-branes. D-branes are physical objects that exist in a theory of strings, and they are not introduced by hand. They have calculable energy densities, and a host of remarkable properties. We will study them in more detail beginning in Chapter 12.

Returning after this long aside to the variation of the action, since the second term in (6.5.8) must vanish for all variations of the motion, we set

\[ \frac{\partial P_\mu}{\partial \tau} + \frac{\partial P_\mu}{\partial \sigma} = 0. \]  

(6.5.11)

This is the equation of motion for the relativistic string, open or closed. A quick glance at definitions (6.5.5) and (6.5.6) shows that this equation is incredibly complicated. The key to its solution will lie in the reparameterization invariance of the Nambu-Goto action. Choosing a clever parameterization will simplify our work enormously.

6.6 The static gauge

To make progress in understanding the action for the relativistic string, we must parameterize the string surface in a useful way. We are allowed to freely choose the parameterization because of the reparameterization invariance of the string action. Reparameterization invariance in string theory is analogous to gauge invariance in electrodynamics. Maxwell’s equations possess a symmetry under gauge transformations that allows us to use different potentials \( A_\mu \) to represent the same electromagnetic fields \( \vec{E} \) and \( \vec{B} \). A suitable choice of gauge helps to uncover the physics. Similarly, we may use many different grids on the world-sheet to describe the same physical motion of the string. A suitable choice of grid can make this task much easier. A good choice of parameterization was useful even for the relativistic point particle – its equation of motion is simplest when the trajectory is parameterized by proper time.

In this section, we will discuss only a partial parameterization on the world-sheet. We will fix the lines of constant \( \tau \) by relating \( \tau \) to the time coordinate \( X^0 = ct \), the time in some chosen Lorentz frame.
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Figure 6.7: Left side: the parameter space strip for an open string. The vertical segment \( AB \) is the line \( \tau = t_0 \). Right side: the open string world-sheet in target space. The string at time \( t = t_0 \) is the intersection of the world-sheet with the hyperplane \( t = t_0 \). In the static gauge, the string at time \( t = t_0 \) is the image of the \( \tau = t_0 \) segment \( AB \).

We proceed as in Figure 6.7. Suppose we draw a hyperplane of constant \( t \) in the target space, say the \( t = t_0 \) plane. This plane will intersect the world-sheet along a curve – the string at time \( t_0 \) according to observers in our chosen Lorentz frame. We declare this curve to be a curve of constant \( \tau \); in fact, we declare it to be the curve \( \tau = t_0 \). Extending this definition to all times \( t \), we declare that for any point \( Q \) on the world-sheet

\[
\tau(Q) = t(Q). \tag{6.6.1}
\]

This choice of \( \tau \) parameterization is called the static gauge because lines of constant \( \tau \) are “static strings” in the chosen Lorentz frame.

We will not try to make a sophisticated choice of \( \sigma \) at this time. For an open string, we will choose one edge of the worldsheet to be the curve \( \sigma = 0 \), and the other edge to be the curve \( \sigma = \sigma_1 \):

\[
\sigma \in [0, \sigma_1], \quad \text{for an open string.} \tag{6.6.2}
\]

We draw lines of constant \( \sigma \) on the surface quite arbitrarily, provided, of course, that constant \( \sigma \) lines vary smoothly, do not intersect, and are consis-
tent with the two curves which are the boundary of the world-sheet. Drawing constant $\sigma$ lines is equivalent to giving an explicit $\sigma$-parameterization to all the strings.

For closed strings the same ideas apply, but there is a significant proviso: there must be an identification in the $(\tau, \sigma)$ parameter space. The $\sigma$ direction must be made into a circle, making the $(\tau, \sigma)$ parameter space into a cylinder. This is needed because the closed string world-sheet is topologically a cylinder. Letting $\sigma_c$ denote the circumference of the $\sigma$ circle, the identification is

$$ (\sigma, \tau) \sim (\sigma + \sigma_c, \tau). \quad (6.6.3) $$

Points that are identified by this relation on the parameter space map to the same point on the closed string world-sheet. The closed strings can be parameterized using any $\sigma$ interval of length $\sigma_c$, for example

$$ \sigma \in [0, \sigma_c], \quad \text{for a closed string.} \quad (6.6.4) $$

Let us now explore the implications of our choice of $\tau$. We can write (6.6.1) as

$$ X^0(\sigma, \tau) \equiv c t(\sigma, \tau) = c \tau, \quad (6.6.5) $$

or simply

$$ \tau = t. \quad (6.6.6) $$

We can thus describe the collection of string coordinates $X^\mu$ as

$$ X^\mu(\tau, \sigma) = X^\mu(t, \sigma) = \{c t, \vec{X}(t, \sigma)\}, \quad (6.6.7) $$

letting the vector $\vec{X}$ represent the spatial string coordinates. We then find

$$ \frac{\partial X^\mu}{\partial \sigma} = \left( \frac{\partial X^0}{\partial \sigma}, \frac{\partial \vec{X}}{\partial \sigma} \right) = \left( 0, \frac{\partial \vec{X}}{\partial \sigma} \right), $$

$$ \frac{\partial X^\mu}{\partial \tau} = \left( \frac{\partial X^0}{\partial t}, \frac{\partial \vec{X}}{\partial t} \right) = \left( c, \frac{\partial \vec{X}}{\partial t} \right). \quad (6.6.8) $$

As you can see, this parameterization separates the time and space components quite neatly.

Now that we have made a choice of $\tau$ coordinates, we can do a simple test to confirm that we got the right sign under the radical in the Nambu-Goto
action (6.4.2). Imagine a little piece of string with no velocity. Because it is not moving, $\partial \vec{X}/\partial t = 0$, and using (6.6.8), the square root in (6.4.2) becomes

$$\sqrt{0 - \left(\frac{\partial \vec{X}}{\partial \sigma}\right)^2 (-c^2)}. \quad (6.6.9)$$

The quantity under the square root is positive, just as we expected. If some day you forget the sign under the radical in the string action, this is a good way to check it quickly.

### 6.7 Tension and energy of a stretched string

Let us now do our first calculation with the Nambu-Goto action – our first calculation in string theory! We are going to analyze a stretched relativistic string. The endpoints of the string are fixed at $x^1 = 0$, and at $x^1 = a > 0$, with vanishing values for the coordinates of the additional spatial dimensions. We therefore write the string endpoints as the (space) points $(0, \vec{0})$ and $(a, \vec{0})$. The inclusion of the common $(d-1)$-dimensional vector $\vec{0}$ tells us that the string is only stretched along the first spatial coordinate.

We now evaluate the string action for this stretched string using the static gauge $X^0 = c \tau$. Because this is a static string stretched from $x^1 = 0$ to $x^1 = a$, we can write

$$X^1(t, \sigma) = f(\sigma), \quad X^2 = X^3 = \cdots = X^d = 0, \quad (6.7.1)$$

where

$$f(0) = 0, \quad f(\sigma_1) = a, \quad (6.7.2)$$

and the function $f(\sigma)$ is strictly increasing and continuous on the interval $\sigma \in [0, \sigma_1]$. The setup is illustrated in Figure 6.8. The function $f$ must be strictly increasing to ensure that each point along the string is assigned a unique $\sigma$ coordinate.

It now follows that

$$\dot{X}^\mu = (c, 0, \vec{0}), \quad X'^\mu = (0, f', \vec{0}), \quad (6.7.3)$$

with $f' = df/d\sigma$. Therefore

$$(\dot{X})^2 = -c^2, \quad (X')^2 = (f')^2, \quad \dot{X} \cdot X' = 0. \quad (6.7.4)$$
6.7. TENSION AND ENERGY OF A STRETCHED STRING

Figure 6.8: A string of length $a$ stretched along the $x^1$-axis. The string is parameterized as $X^1(t, \sigma) = f(\sigma)$.

We can now evaluate the action (6.4.2):

$$S = -\frac{T_0}{c} \int_{t_i}^{t_f} dt \int_0^{\sigma_1} d\sigma \sqrt{c^2 - (f'(\sigma))^2} = -T_0 \int_{t_i}^{t_f} dt \int_0^{\sigma_1} d\sigma \frac{df}{d\sigma}.$$

The $\sigma$ integrand is a total derivative, so

$$S = -T_0 \int_{t_i}^{t_f} dt \left( f(\sigma_1) - f(0) \right) = \int_{t_i}^{t_f} dt (-T_0 a),$$

where we used (6.7.2). It is interesting to see that the value of the action does not depend on the function $f$ used to parameterize the string. This is an explicit confirmation of the reparameterization invariance of the string action.

We would like to interpret our result. For this, recall that the action is the time integral of the Lagrangian $L$. When the kinetic energy vanishes, $L = -V$, where $V$ is the potential energy. Since our string is static, there is no kinetic energy, so

$$S = \int_{t_i}^{t_f} dt (-V).$$

Comparing this with (6.7.6) we conclude that

$$V = T_0 a.$$
The potential energy of our stretched string is just $T_0 a$. What does this mean? If the tension of a static string is $T_0$, regardless of its length, then $T_0 a$ is the amount of energy you must spend to create a string of length $a$. Imagine that you start with an infinitesimal string and you start pulling it. As you do work you are giving energy to the string, in fact, you are creating rest energy, or rest mass. The rest mass $\mu_0$ per unit length is

$$\mu_0 c^2 = \frac{V}{a} = T_0 \quad \rightarrow \quad \mu_0 = \frac{T_0}{c^2}. \quad (6.7.9)$$

The mass (or rest energy) arises only because the string has a tension. Because of this, the relativistic string is sometimes referred to as a massless string. The above calculation also confirms that the minus sign in front of the action (6.4.2) is necessary – otherwise the potential energy of the stretched string would have come out negative. Finally, the constant $T_0$ was identified as the string tension.

There is one point we have glossed over. We assumed in our analysis that the configuration in (6.7.1) satisfies the string equations of motion. If it does not, then the configuration cannot be physically realized. Let us check that the equations of motion are satisfied.

First note that on account of (6.7.3) neither $\dot{X}^\mu$ nor $X'^{\mu'}$ has $\tau$ dependence. Therefore neither $P^\tau$ nor $P^\sigma$ has $\tau$ dependence (see (6.5.5) and (6.5.6)). This being the case, the equation of motion (6.5.11) reduces to

$$\frac{\partial P^\sigma}{\partial \sigma} = 0. \quad (6.7.10)$$

This requires that $P^\sigma_\mu$ be $\sigma$-independent. We look again at (6.5.6) and use (6.7.4) to find

$$P^\sigma_\mu = -\frac{T_0}{c} \frac{c^2 X'_\mu}{\sqrt{c^2(f')^2}} = -T_0 \frac{X'_\mu}{f'}. \quad (6.7.11)$$

This is non-vanishing only for $\mu = 1$, in which case $X'_1 = f'$, so $P^\sigma_\mu$ is indeed $\sigma$-independent. Thus the equation of motion is satisfied. Even the boundary conditions are satisfied. As we discussed in section 6.5, there is no condition to be checked for string coordinates when the endpoints satisfy Dirichlet boundary conditions. In our problem this means that there are no extra conditions to be checked for any of the spatial coordinates. On the other hand, for the zeroth coordinate our choice of gauge required $X^0 = c\tau$. This
is not a Dirichlet boundary condition since $X^0$ is not a constant anywhere on the string. It follows that $X^0$ must be treated as a coordinate with free endpoints, and the condition in (6.5.10) must be checked. This just requires $\mathcal{P}_0^0 = 0$, a fact that holds on account of (6.7.11).

### 6.8 Action in terms of transverse velocity

We have begun to choose a specific parameterization of the string surface by fixing $\tau$ via the condition $X^0 = ct = c\tau$. With this choice, a line of constant $\tau$ on the string spacetime surface corresponds to the string as seen by our chosen Lorentz observer at the particular time $t = \tau$.

Can we define some sort of string velocity? Since $\vec{X}(t, \sigma)$ are the string spatial coordinates, the derivative $\partial \vec{X}/\partial t$ seems to be the closest thing we have to a velocity. This velocity, however, depends upon the choice of $\sigma$. Its direction, for example, goes along the lines of constant $\sigma$. Since $\sigma$ can be chosen quite arbitrarily, keeping $\sigma$ constant in taking the derivative is clearly not very physically significant!

Fixing physically the $\sigma$ parameterization of a string is subtle because the string is an object with no substructure. When comparing a string at two nearby times, it is not possible to say that a point moved from one location to the next. To speak of points on the string we need a $\sigma$ parameterization, and reparameterization invariance makes it clear to us that this parameterization is not unique. This suggests that longitudinal motion on the string is not physically meaningful.

There is an invariant velocity that can be defined on the string. This is, however, a transverse velocity. We consider the string motion in space, and imagine that each point on the string moves transversely to the string. Consider a string at some fixed time and pick a point $p$ on it. Draw the hyperplane orthogonal to the string at $p$. An infinitesimal instant later the string has moved, but it will still intersect the plane, this time at a point $p'$. The transverse velocity is what we get if we presume that the point $p$ moved to $p'$. No string parameterization is needed to define this velocity.

When speaking of evolving strings there are two surfaces we can discuss. One is the world-sheet, the surface in spacetime which represents the history of the string. The other is a surface in space. This spatial surface is put together by combining the strings that we observe at all times. This is the surface that would be generated if the string were to leave a wake as it moved.
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The transverse velocity at any point on the string is a vector orthogonal to the string and tangent to the string spatial surface.

We are therefore motivated to define, for each point on the string, a velocity $\vec{v}_\perp$ perpendicular to the string itself. Our discussion above indicates that this is a reparameterization-invariant notion of velocity, and therefore we expect it to enter naturally into the evaluation of the string action for a string moving arbitrarily.

In order to define the perpendicular velocity, it is useful to introduce a unit vector tangent to the string. To this end, we now introduce a parameter $s$ which is more physical than our nearly-arbitrary $\sigma$. Let us work with a fixed string, and define $s(\sigma)$ to be the length of the string in the interval $[0, \sigma]$. Thus, for example, $s(0) = 0$, and $s(\sigma_1)$ is the length of an entire open string. Since $ds$ is the length of the infinitesimal vector $d\vec{X}$ arising from a world-sheet segment $d\sigma$ along the string, we have:

$$ds = |d\vec{X}| = \left| \frac{\partial \vec{X}}{\partial \sigma} \right| |d\sigma|.$$  \hspace{1cm} (6.8.1)

Now consider the quantity $\partial \vec{X}/\partial s$, which is the variation of $\vec{X}$ with the length of the string. First note that it is a unit vector:

$$\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial s} = \frac{\partial \vec{X}}{\partial \sigma} \cdot \frac{\partial \vec{X}}{\partial \sigma} \left( \frac{d\sigma}{ds} \right)^2 = \left| \frac{\partial \vec{X}}{\partial \sigma} \right|^2 \left( \frac{d\sigma}{ds} \right)^2 = 1.$$ \hspace{1cm} (6.8.2)

The derivative $\partial \vec{X}/\partial \sigma$, as the notation indicates, is taken with $t$ held fixed, and therefore it lies along a line of constant $t$. Since the lines of constant $t$ are precisely the strings, it is tangent to the string. In addition

$$\frac{\partial \vec{X}}{\partial s} = \frac{\partial \vec{X}}{\partial \sigma} \frac{d\sigma}{ds},$$ \hspace{1cm} (6.8.3)

and thus $\partial \vec{X}/\partial s$ is also tangent to the string. Because it has unit length,

$$\frac{\partial \vec{X}}{\partial s} \text{ is a unit vector tangent to the string.}$$ \hspace{1cm} (6.8.4)

We define $\vec{v}_\perp$ to be the component of the velocity $\partial \vec{X}/\partial t$, in the direction perpendicular to the string (see Figure 6.9). For any vector $\vec{u}$, its component
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perpendicular to a unit vector \( \vec{n} \) is \( \vec{u} - (\vec{u} \cdot \vec{n})\vec{n} \). Therefore, using our unit vector \( \partial \vec{X}/\partial s \) along the string, we have

\[
\vec{v}_\perp = \frac{\partial \vec{X}}{\partial t} - \left( \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \right) \frac{\partial \vec{X}}{\partial s}. \tag{6.8.5}
\]

Figure 6.9: A small piece of the world-sheet showing the vector \( \partial \vec{X}/\partial t \), the transverse velocity \( \vec{v}_\perp \) and the unit vector \( \partial \vec{X}/\partial s \).

For future use, let’s calculate \( v_\perp^2 \):

\[
v_\perp^2 = \left( \frac{\partial \vec{X}}{\partial t} \right)^2 - 2 \left( \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \right) \left( \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \right) + \left( \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \right)^2,
\]

\[
= \left( \frac{\partial \vec{X}}{\partial t} \right)^2 - \left( \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \right)^2. \tag{6.8.6}
\]

The string action depends only upon products of the vectors \( \vec{X}^\mu \) and \( X'^\mu \). Our goal now is to write it in terms of \( \vec{v}_\perp \) and other quantities, if necessary. Using the static gauge \( \tau = t \), and equations (6.6.8), we find

\[
(\dot{\vec{X}})^2 = -c^2 + \left( \frac{\partial \vec{X}}{\partial t} \right)^2, \quad (X')^2 = \left( \frac{\partial \vec{X}}{\partial \sigma} \right)^2, \quad \dot{\vec{X}} \cdot \vec{X}' = \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma}. \tag{6.8.7}
\]
With these relations we simplify the square root in the string action:

\[
(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2 = \left( \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} \right)^2 + \left[ c^2 - \left( \frac{\partial \vec{X}}{\partial t} \right)^2 \right] \left( \frac{\partial \vec{X}}{\partial \sigma} \right)^2
\]

\[
= \left( \frac{ds}{d\sigma} \right)^2 \left[ \left( \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \right)^2 + c^2 - \left( \frac{\partial \vec{X}}{\partial t} \right)^2 \right].
\]

(6.8.8)

The terms on the right-hand side above can be neatly expressed in terms of \( v_\perp^2 \). Making use of (6.8.6),

\[
(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2 = \left( \frac{ds}{d\sigma} \right)^2 \left( c^2 - v_\perp^2 \right),
\]

(6.8.9)

or, alternatively,

\[
\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2} = c \frac{ds}{d\sigma} \sqrt{1 - \frac{v_\perp^2}{c^2}}.
\]

(6.8.10)

This simple expression for the string Lagrangian density shows that \( \vec{v}_\perp \) is a natural dynamical variable. Moreover, the longitudinal component of the velocity is completely irrelevant. Now we can write the string action as

\[
S = -T_0 \int dt \int_{\sigma_1}^{\sigma_2} d\sigma \left( \frac{ds}{d\sigma} \right) \sqrt{1 - \frac{v_\perp^2}{c^2}}.
\]

(6.8.11)

Here \( ds/d\sigma = |\partial \vec{X} / \partial \sigma| \). Moreover, we did not cancel the \( d\sigma \)'s because it is typically useful to have an integral over a fixed parameter range. While the range of \( \sigma \) is constant, the length \( s \) of a string is time-dependent.

The associated Lagrangian is given by

\[
L = -T_0 \int ds \sqrt{1 - \frac{v_\perp^2}{c^2}}.
\]

(6.8.12)

This formula was written as an integral over the length parameter in order to give an interpretation. For each piece of string, \( T_0 ds \) is its rest energy. As a result, the Lagrangian is an integral over the string of (minus) the rest energy times a local relativistic factor. In this form, we recognize (6.8.12) as the natural generalization of the relativistic particle Lagrangian (5.1.8).

The action (6.8.11) is valid both for open strings and for closed strings. Although relatively simple, it still leads to rather complicated equations of motion in all but the most symmetrical situations. In order to obtain simple equations of motion, we will have to be clever in our choice of \( \sigma \). For open strings, in addition, we must understand how the endpoints move. We turn now to this question.
6.9 Motion of open string endpoints

We will now analyze the motion of the endpoints of an open relativistic string. We consider endpoints that are free to move in all directions. Given our discussion in section 6.5, this means that we have a space-filling D-brane. Free endpoints are specified by the boundary conditions (6.5.10), which require the vanishing of $P_{\mu}^{\sigma}$ at the endpoints. We will discover two important properties of the free motion of open string endpoints:

- The endpoints move with the speed of light.
- The endpoints move transversely to the string.

On the interior of the string the notion of a velocity was ambiguous. For the string endpoints, however, the velocity is well-defined – there is no ambiguity defining the velocity of points! Therefore, our statements about endpoint motion have content. In the second statement, motion transverse to the string means that the velocity of an endpoint is orthogonal to the tangent to the string at the endpoint.

To prove the above properties, we investigate our expression (6.5.6) for $P^{\sigma\mu}$, which we know must vanish at the endpoints. The denominator of $P^{\sigma\mu}$ is given in (6.8.10) and the numerator is simplified using relations (6.8.7). We find

$$P^{\sigma\mu} = -\frac{T_0}{c} \left( \frac{\partial \vec{X}}{\partial \sigma} \cdot \frac{\partial \vec{X}}{\partial t} \right) \dot{X}^\mu - \frac{1}{c^2} \frac{\partial^2}{\partial \sigma^2} \left( \partial \vec{X}/\partial t \right)^2 \partial X^\mu/\partial s \sqrt{1 - v^2/c^2}. \quad (6.9.1)$$

Bringing the $ds/d\sigma$ from the denominator up to the numerator, we can turn derivatives with respect to $\sigma$ into derivatives with respect to $s$:

$$P^{\sigma\mu} = -\frac{T_0}{c^2} \left( \frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial t} \right) \dot{X}^\mu + \frac{c^2}{\sqrt{1 - v^2/c^2}} \frac{\partial X^\mu/\partial s}{\partial s}. \quad (6.9.2)$$

Now consider the $\mu = 0$ component of this quantity. In this case we can make some simplifications: $\dot{X}^0 = c$ and $\partial X^0/\partial s = c \partial t/\partial s = 0$. We find that

$$P^{\sigma,0} = -\frac{T_0}{c} \frac{\partial \vec{X}/\partial s \cdot \partial \vec{X}/\partial t}{\sqrt{1 - v^2/c^2}}. \quad (6.9.3)$$
Since $P^{\sigma,0}$ vanishes at the endpoints, and the square root in the denominator is manifestly finite, we deduce that

$$\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial t} = 0 \text{ at the endpoints.} \tag{6.9.4}$$

Since $\partial \vec{X}/\partial s$ is a unit vector tangent to the string, and $\partial \vec{X}/\partial t$ is the endpoint velocity, this equation proves that the endpoints move transversely to the string – one of our two claims. In agreement with this interpretation, using (6.9.4) in (6.8.5) we see that at the endpoints $\vec{v} = \vec{v}_\perp$. Equation (6.9.4) actually allows for vanishing endpoint velocity, in which case the transversality property would be trivially satisfied. But this cannot happen; the endpoints move with the speed of light, as we now show.

Using (6.9.4), we simplify the expression (6.9.2) for $P^{\sigma,\mu}$ at the endpoints:

$$P^{\sigma,\mu} = -\frac{T_0 c^2 (1 - \frac{v^2}{c^2})(\frac{\partial X^\mu}{\partial s})}{\sqrt{1 - \frac{v^2}{c^2}}} = -T_0 \sqrt{1 - \frac{v^2}{c^2}} \frac{\partial X^\mu}{\partial s}, \text{ at the endpoints.} \tag{6.9.5}$$

For the space coordinates, $\mu = 1, 2, 3, \ldots, d$, equation (6.9.5) gives

$$\vec{P}^\sigma = -T_0 \sqrt{1 - \frac{v^2}{c^2}} \frac{\partial \vec{X}}{\partial s} = 0, \text{ at the endpoints.} \tag{6.9.6}$$

Since $\partial \vec{X}/\partial s$ is a unit vector, we conclude that

$$v^2 = c^2. \tag{6.9.7}$$

This proves that free endpoints move with the speed of light.
Problems


A Nambu-Goto string with endpoints attached at \((0, \vec{0})\) and \((a, \vec{0})\) (as in section 6.7) is vibrating non-relativistically. Show that the action (6.8.11) reduces to that of a non-relativistic string with transverse oscillations. What is the tension and linear mass density of that non-relativistic string?


At \(t = 0\), a closed string forms a circle of radius \(R\) on the \((x, y)\) plane and has zero velocity. The time development of this string can be studied using the action (6.8.11). The string will remain circular, but its radius will be time-dependent. Calculate the radius and velocity as functions of time. Sketch the spacetime surface traced by the string in a 3-dimensional plot with \(x, y,\) and \(ct\) axes.

Problem 6.3. *Covariant analysis of open string endpoint motion.*

Use the explicit form of \(P^\sigma_\mu\) to calculate \(P^\sigma_\mu P^{\sigma\mu}\) explicitly. Show that the result of this calculation can be used to prove that free open string endpoints move with the speed of light.

Problem 6.4. *Hamiltonian density for relativistic strings.*

Consider the string Lagrangian density \(L\) in the static gauge, and written in terms of \(\partial_\sigma \vec{X}\) and \(\partial_t \vec{X}\). Calculate the canonical momentum \(\vec{P}(\sigma, t)\):

\[
\vec{P}(\sigma, t) = \frac{\partial L}{\partial (\partial_t \vec{X})}.
\]

Recall that \(L\) can be written in terms of \(\vec{v}_\perp\) and \(\frac{d\vec{X}}{d\sigma}\). Show that this is also possible for \(\vec{P}\). Calculate the Hamiltonian density \(\mathcal{H}\). Write the total Hamiltonian as \(H = \int d\sigma \mathcal{H} = \int ds(\cdots)\) and show that your answer is consistent with the interpretation that the energy of the string arises as energy of transverse motion of a string whose rest mass arises solely from the tension.
Problem 6.5. *Open strings ending on D-branes of various dimensions.*

Consider a world with $d$ spatial dimensions. A $D_p$-brane is an extended object with $p$ spatial dimensions: a $p$-dimensional hyperplane inside the $d$-dimensional space. We will examine properties of strings ending on a $D_p$-brane, where $0 \leq p < d$. The case $p = d$, where the D-brane is space-filling was discussed in section 6.9.

For a $D_p$-brane, let $x^i$, with $i = 1, 2, \ldots, p$, correspond to directions on the $D_p$-brane, and $x^a$ with $a = p + 1, p + 2, \ldots, d$, correspond to directions orthogonal to the $D_p$-brane. The $D_p$-brane position would be specified, for example by $x^a = 0$, with $a = p + 1, \ldots, d$. Open string endpoints must lie on the $D_p$-brane, and, focusing on the $\sigma = 0$ endpoint we have

$$X^a(\sigma = 0, t) = 0, \quad a = p + 1, p + 2, \ldots, d.$$ 

There are no constraints on $X^i(\sigma = 0, t)$.

(a) State the boundary conditions that the various components of $P^\sigma_\mu$ must satisfy. Distinguish three cases: the case of $P^\sigma_0$, the case of the components $P^\sigma_i$, and the case of the components $P^\sigma_a$.

(b) Show that all constraints are automatically satisfied when the string ends on a $D0$-brane.

(c) Show that for a string ending on a $D1$-brane, the tangent to the string at the endpoint must be orthogonal to the $D1$-brane, and the endpoint velocity is unconstrained.

(d) For $p \geq 2$ show that there are two possibilities:

(i) the string is orthogonal to the $D_p$-brane at the endpoint and the endpoint velocity is unconstrained, or,

(ii) the tangent to the string at the endpoint is not orthogonal to the $D_p$-brane and the endpoint moves with the speed of light.