Chapter 16

String thermodynamics and black holes

The thermodynamics of strings is governed largely by the exponential growth of the number of quantum states accessible to a string, as a function of its energy. We estimate such growth rates by counting partitions of large integers. As we increase the energy of a string, the behavior of the entropy indicates that the temperature approaches a finite constant: the Hagedorn temperature. We calculate the finite-temperature partition function for bosonic string theory. We explain how the counting of string states can be used to give a statistical mechanics derivation of the entropy of black holes. The calculations give results in qualitative agreement with the expected entropy of Schwarzschild black holes, and in quantitative agreement with the expected entropy of certain charged black holes.

16.1 A review of statistical mechanics

Our study of string thermodynamics will make use of both the microcanonical and canonical ensembles. Recall that the microcanonical ensemble consists of a collection of copies of a particular system $A$, one for each state accessible to $A$ at a particular fixed energy $E$. In the canonical ensemble we consider the system $A$ in thermal contact with a reservoir at a temperature $T$. This ensemble contains copies of system $A$ together with the reservoir, one copy for each allowed state of the combined system. In the canonical ensemble the energy of system $A$ varies among members of the ensemble.
Let’s begin with the microcanonical ensemble. The system $A$ is imagined to be in isolation with a fixed energy. We let $\Omega(E)$ denote the number of possible states of the system $A$ when it has energy $E$. The entropy $S$ of the system is defined in terms of the number of states as

$$S(E) = k \ln \Omega(E),$$

where $k$ is Boltzmann’s constant. The temperature $T$ of the system is defined in terms of derivatives of the entropy with respect to the energy:

$$\frac{1}{T} = \frac{\partial S}{\partial E}.$$

The canonical ensemble is sometimes easier to work with. Imagine a system $A$ which has a fixed volume and which is in thermal contact with a reservoir of temperature $T$. This system could be a box full of strings, or it could be a box containing a single string. It is also not necessary to specify what the reservoir is. Suppose we know the quantum states $\{\alpha\}$ of the system and their associated energies $\{E_\alpha\}$. Then, the partition function $Z$ for system $A$ is defined as

$$Z \equiv \sum_\alpha e^{-\beta E_\alpha}, \quad \beta = \frac{1}{kT}.$$

The partition function is useful because it can be used to calculate interesting quantities. For instance, if system $A$ is known to have temperature $T$, then, using $Z$, we can calculate the probability that $A$ is in a particular quantum state. By definition, the partition function depends both on the temperature $T$ and on the external parameters of the system. These are the parameters that determine the energy levels of the system. The systems we will consider have only one external parameter: the volume $V$ occupied by the system. Thus we will think of $Z$ as $Z(T,V)$, or

$$Z = Z(\beta,V).$$

The probability $P_\alpha$ that the system, in contact with the reservoir of temperature $T$, is in the state $\alpha$ is

$$P_\alpha = \frac{e^{-\beta E_\alpha}}{Z}.$$
Clearly, $\sum_\alpha P_\alpha = 1$, as required by the probabilistic interpretation of $P_\alpha$.

We can calculate the average energy $E$ of the system $A$ in the ensemble by differentiation of the partition function:

$$E = \sum_\alpha P_\alpha E_\alpha = - \frac{\partial \ln Z}{\partial \beta}.$$  \hfill (16.1.6)

The pressure $p$ of the system can also be calculated from the partition function (see Problem 16.1). It is given by

$$p = \frac{1}{\beta} \frac{\partial \ln Z}{\partial V}.$$  \hfill (16.1.7)

Another useful quantity is the Helmholtz free energy $F$. Its basic properties can be obtained in a few steps starting from the first law of thermodynamics. The change $dE$ in the energy of a system whose only external parameter is the volume $V$ is given by

$$dE = TdS - pdV.$$  \hfill (16.1.8)

Here $T$ is the temperature of the system and $p$ is the pressure. Moreover, $T dS$ is the heat transferred into the system, and $(-pdV)$ is the mechanical work done on the system. Equation (16.1.8) implies that $E$ should be viewed as a function $E(S,V)$ of $S$ and $V$, and that

$$T = \left( \frac{\partial E}{\partial S} \right)_V, \quad p = -\left( \frac{\partial E}{\partial V} \right)_S.$$  \hfill (16.1.9)

We can also write the change in energy in (16.1.8) as

$$dE = d(TS) - S dT - pdV,$$  \hfill (16.1.10)

which means that

$$d(E - TS) = -S dT - pdV.$$  \hfill (16.1.11)

The free energy $F$ is defined as

$$F \equiv E - TS,$$  \hfill (16.1.12)

and therefore we have

$$dF = -S dT - pdV.$$  \hfill (16.1.13)
We see that for processes at constant temperature, the free energy represents
the amount of energy that can go into mechanical work. For a chemical
reaction that releases energy, for example, the entropy of the system typically
decreases. Not all of the energy released can then be used for work, only the
free energy can. Since the total entropy cannot decrease, the rest of the
energy goes into heat that increases the entropy of the world. It follows from
(16.1.13) that \( F \) should be viewed as a function \( F(T, V) \) of \( T \) and \( V \), and,

\[
S = -\left( \frac{\partial F}{\partial T} \right)_V, \quad p = -\left( \frac{\partial F}{\partial V} \right)_T. \tag{16.1.14}
\]

The free energy can be calculated from the partition function, as you may
review in Problem 16.1. It is given by

\[
F = -kT \ln Z. \tag{16.1.15}
\]

Our aim is to use the basic thermodynamic relations reviewed above to
compute interesting properties of the string. One central computation is that
of the partition function for a string. This problem is a bit complex, so we
first consider simpler problems that will help us build the necessary tools.

The first result we need is a formula for the number of partitions of in-
tegers. We will obtain this mathematical result using a physical method:
the analysis of the high-temperature behavior of a quantum non-relativistic
string, call it a “quantum violin string”. With this result we calculate the
entropy/energy relation for an idealized quantum relativistic string; a string
where we ignore the momentum labels of the quantum states. The Hagedorn
temperature already emerges in this context. After a discussion of the parti-
tion function for the relativistic point-particle, we assemble all of our results
to compute the partition function of the relativistic string.

In the latter part of this chapter we discuss a significant success of string
theory: giving a statistical-mechanics interpretation of the entropy of black
holes. This entropy, first arrived at via thermodynamical considerations,
arises from the degeneracy of string states that have the macroscopic proper-
ties of the black holes. The agreement between the string calculations and the
thermodynamical expectation is only qualitative for the case of Schwarzchild
black holes, but is quantitave for certain types of extremal black holes.
16.2 Partitions and the quantum violin string

Consider a quantum mechanical non-relativistic string with fixed endpoints: a quantum violin string. This string, studied classically in Chapter 4, has an infinite set of vibrating frequencies, all multiples of a basic frequency $\omega_0$. Its idealization as a quantum string is a collection of simple harmonic oscillators with frequencies $\omega_0, 2\omega_0, 3\omega_0$, and so on. Each simple harmonic oscillator (SHO) has its own creation and annihilation operators, as well as its own Hamiltonian:

$$
\text{SHO}_{\omega_0} : (a_1, a_1^\dagger), \quad H_{\omega_0} = \hbar \omega_0 a_1^\dagger a_1,
$$

$$
\text{SHO}_{2\omega_0} : (a_2, a_2^\dagger), \quad H_{2\omega_0} = 2 \hbar \omega_0 a_2^\dagger a_2, \quad (16.2.1)
$$

$$
\text{SHO}_{3\omega_0} : (a_3, a_3^\dagger), \quad H_{3\omega_0} = 3 \hbar \omega_0 a_3^\dagger a_3,
$$

\[\vdots\]

Here we have discarded zero-point energies, and all oscillators satisfy the conventional commutation relations

$$
[a_l, a_m^\dagger] = \delta_{mn}. \quad (16.2.2)
$$

Since the quantum string is the union of all these oscillators, the Hamiltonian for the string is

$$
H = \sum_{\ell=1}^{\infty} H_{\ell\omega_0} = \hbar \omega_0 \sum_{\ell=1}^{\infty} \ell a_\ell^\dagger a_\ell. \quad (16.2.3)
$$

We recognize here the number operator $\hat{N}$:

$$
\hat{H} = \hbar \omega_0 \hat{N}, \quad \hat{N} = \sum_{\ell=1}^{\infty} \ell a_\ell^\dagger a_\ell. \quad (16.2.4)
$$

The vacuum state of the string is a state $|\Omega\rangle$ such that

$$
a_\ell |\Omega\rangle = 0, \quad \text{for all } \ell. \quad (16.2.5)
$$

A quantum state $|\Psi\rangle$ of this string is obtained by letting creation operators act on the vacuum:

$$
|\Psi\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2} \cdots (a_l^\dagger)^{n_l} \cdots |\Omega\rangle. \quad (16.2.6)
$$
A natural counting question arises here. For a fixed positive integer $N$, how many states are there with $\hat{N}$ eigenvalue equal to $N$? This number, denoted as $p(N)$, is so important that it has been given a name: the partitions of $N$. Before explaining the reason for this terminology, let us determine $p(N)$ for $N = 1, 2, 3$ and 4. Shown in Table 16.1 are the states with those values of $N$. For brevity, we show only the oscillators, omitting the vacuum state $|\Omega\rangle$ that they act on. The fourth line, for example, shows that there are five states with $\hat{N}$ eigenvalue equal to four. Thus $p(4) = 5$.

It is appropriate to name the quantity $p(N)$ partitions of $N$. A partition of $N$ is a set of positive integers that add up to $N$. The order of the elements in the set is immaterial. Thus, for example, $\{3, 2\}$ is a partition of 5, and so is $\{2, 1, 1, 1\}$. The partitions of 4 are

$$\{4\}, \quad \{3, 1\}, \quad \{2, 2\}, \quad \{2, 1, 1\}, \quad \{1, 1, 1, 1\}.$$

---

**Table 16.1:** Counting states of fixed total number eigenvalue $N$. $p(N)$ denotes the number of partitions of the integer $N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>list of states</th>
<th>$p(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_1^\dagger$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$a_2^\dagger, (a_1^\dagger)^2$</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$a_3^\dagger, a_2^\dagger a_1^\dagger, (a_1^\dagger)^3$</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>$a_4^\dagger, a_3^\dagger a_1^\dagger, (a_2^\dagger)^2, a_2^\dagger (a_1^\dagger)^2, (a_1^\dagger)^4$</td>
<td>5</td>
</tr>
</tbody>
</table>

The state is therefore specified by the set $\{n_1, n_2, n_3, \ldots\}$ of occupation numbers. The number operator acting on the state $|\Psi\rangle$ gives us

$$\hat{N} |\Psi\rangle = N |\Psi\rangle,$$  \hspace{1cm} (16.2.7)

where

$$N = n_1 + 2n_2 + 3n_3 + \ldots = \sum_{\ell=1}^{\infty} \ell n_\ell.$$  \hspace{1cm} (16.2.8)

It then follows from (16.2.4) that the energy of $|\Psi\rangle$ is given by

$$\hat{H} |\Psi\rangle = E |\Psi\rangle \quad \rightarrow \quad E = h\omega_0 N.$$  \hspace{1cm} (16.2.9)
The number of states with $\hat{N}$ eigenvalue equal to $N$ coincides with the number of partitions of $N$. Indeed, given a partition of $N$ we can build a state by attaching each element of the partition as a subscript to an oscillator $a^\dagger$, and letting the resulting collection of oscillators act on the vacuum. Note that this is exactly how the states in the last line of Table 16.1 are built from the partitions of 4 given in (16.2.10). Conversely, given a state with number $N$, the set of subscripts of all oscillators in the state gives a partition of $N$.

We would like to find a formula for $p(N)$. But our analysis will not give us that much. We will derive an expression that describes $\ln p(N)$ accurately for large $N$. A more refined calculation gives the famous approximation for $p(N)$ found by Hardy and Ramanujan. There exist closed-form expressions for $p(N)$, but they are extremely complicated!

Our strategy will be as follows. We know that the entropy $S$ is given as a function of the energy $E$ by (16.1.1). For a given $E$, $N = E/(\hbar \omega_0)$, and $\Omega(E)$ is simply $p(N)$. Therefore

$$S(E) = k \ln p \left( \frac{E}{\hbar \omega_0} \right) = k \ln p(N). \quad (16.2.11)$$

If we can find $S(E)$, then we will have found the function $p(N)$. To find $S(E)$ we will calculate the partition function $Z$ for the quantum violin string. From $Z$ we will find the free energy $F$. We will be able to evaluate the free energy explicitly only by assuming high temperature. It is then easy to find the high-energy behavior of the entropy $S(E)$. This can be used to find a large-$N$ approximation for $p(N)$.

Let’s now begin with the calculation of the partition function. We have

$$Z = \sum_\alpha \exp \left( -\frac{E_\alpha}{kT} \right) = \sum_{n_1,n_2,n_3,\ldots} \exp \left[ -\frac{\hbar \omega_0}{kT} (n_1 + 2n_2 + 3n_3 + \cdots) \right]. \quad (16.2.12)$$

In writing this equation we have recognized that the set of all states is labelled by the set of occupation numbers. To sum over all states is to sum over all occupation numbers, each of which ranges from zero to infinity. We have also used the value of the energy $E = \hbar \omega_0 N$. Since the exponential of a sum can be written as a product of exponentials, the sums over the occupation numbers are all independent:

$$Z = \sum_{n_1} \exp \left[ -\frac{\hbar \omega_0}{kT} n_1 \right] \cdot \sum_{n_2} \exp \left[ -\frac{\hbar \omega_0}{kT} 2n_2 \right] \cdot \ldots. \quad (16.2.13)$$
Therefore we have
\[ Z = \prod_{l=1}^{\infty} \sum_{n_l=0}^{\infty} \exp \left( - \frac{\hbar \omega_0 l n_l}{kT} \right). \] (16.2.14)

The sum over \( n_l \) is a geometric series, so we find
\[ Z = \prod_{l=1}^{\infty} \left[ 1 - \exp \left( - \frac{\hbar \omega_0 l}{kT} \right) \right]^{-1}. \] (16.2.15)

Finally, the free energy \( F \) is found using (16.1.15):
\[ F = -kT \ln Z = kT \sum_{l=1}^{\infty} \ln \left[ 1 - \exp \left( - \frac{\hbar \omega_0 l}{kT} \right) \right]. \] (16.2.16)

We cannot go any further unless we do some approximations. If the temperature \( T \) is high enough so that
\[ \frac{\hbar \omega_0}{kT} << 1, \] (16.2.17)

then each term in the sum (16.2.16) differs very little from the previous one. This allows us to approximate the sum by an integral:
\[ F \simeq kT \int_{1}^{\infty} dl \, \ln \left[ 1 - \exp \left( - \frac{\hbar \omega_0 l}{kT} \right) \right]. \] (16.2.18)

The choice \( l = 1 \) for the lower limit of integration, as opposed to zero, will play no role. Indeed, changing variables of integration to
\[ x = \frac{\hbar \omega_0}{kT} l, \] (16.2.19)

we find
\[ F \simeq \frac{(kT)^2}{\hbar \omega_0} \int_{0}^{\infty} dx \ln(1 - e^{-x}). \] (16.2.20)

Using the expansion
\[ \ln(1 - y) = - \left( y + \frac{1}{2} y^2 + \frac{1}{3} y^3 + \frac{1}{4} y^4 + \cdots \right), \] (16.2.21)
which is valid for any $0 \leq y < 1$, we have

$$F \simeq -\frac{(kT)^2}{\hbar \omega_0} \int_0^\infty dx \left( e^{-x} + \frac{1}{2} e^{-2x} + \frac{1}{3} e^{-3x} + \cdots \right),$$

$$\simeq -\frac{(kT)^2}{\hbar \omega_0} \left[ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \right]. \quad (16.2.22)$$

The sum in brackets is a familiar one. It is, in fact, the zeta-function (12.4.14) with argument equal to two:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}. \quad (16.2.23)$$

Thus we finally obtain the high temperature limit of the free energy:

$$F \simeq -\frac{(kT)^2 \pi^2}{6 \hbar \omega_0} = -\frac{1}{\hbar \omega_0} \frac{\pi^2}{6} \frac{1}{\beta^2}. \quad (16.2.24)$$

For this string the free energy has no volume dependence.

We can now calculate the entropy as a function of temperature. Using (16.1.14) we find

$$S = -\frac{\partial F}{\partial T} = k \pi^2 \left( \frac{kT}{\hbar \omega_0} \right). \quad (16.2.25)$$

Since we are interested in the entropy as a function of energy, we also compute the energy. Making use of (16.1.6) we have

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{\partial}{\partial \beta} (\beta F) = \frac{\pi^2}{6} \frac{1}{\hbar \omega_0} \frac{1}{\beta} \frac{\partial}{\partial \beta} \left( \frac{1}{\beta} \right), \quad (16.2.26)$$

which gives

$$E = \frac{\pi^2}{6} \frac{1}{\hbar \omega_0} \frac{1}{\beta^2} = \frac{\pi^2}{6} \left( \frac{kT}{\hbar \omega_0} \right)^2 \hbar \omega_0. \quad (16.2.27)$$

Quick Calculation 16.1. Verify that the energy $E$ can also be calculated from $F = E - TS$.

Combining (16.2.25) and (16.2.27) yields

$$S(E) = k\pi \sqrt{\frac{2E}{3\hbar \omega_0}} = k \frac{2\pi}{\sqrt{6}} \sqrt{\frac{N}{6}}. \quad (16.2.28)$$
Comparing with equation (16.2.11) we finally read
\[ \ln p(N) \simeq 2\pi \sqrt{\frac{N}{6}}. \]  
(16.2.29)

This was our goal, an estimate of \( \ln p(N) \) for large \( N \). Indeed, we must require large \( N \) since
\[ N = \frac{E}{\hbar \omega_0} = \frac{\pi^2}{6} \left( \frac{kT}{\hbar \omega_0} \right)^2 \gg 1, \]  
(16.2.30)

because of our high temperature assumption (16.2.17).

The result (16.2.29) is only the leading term of the celebrated Hardy-Ramanujan asymptotic expansion of \( p(N) \):
\[ p(N) \simeq \frac{1}{4N\sqrt{3}} \exp\left(2\pi \sqrt{\frac{N}{6}} \right). \]  
(16.2.31)

This is not an exact formula either, but is an accurate estimate of \( p(N) \), as opposed to our accurate estimate of the logarithm of \( p(N) \). We will not give here a derivation of the Hardy-Ramanujan result. It is fun, however, to test the accuracy of the Hardy-Ramanujan expansion. In Table 16.2 we compare the values of \( p(N) \), as calculated exactly, with the estimate \( p_{est}(N) \) provided by (16.2.31). The estimate gives an error of about one-half of a percent for \( N = 10000 \).

We now need a minor generalization of (16.2.31). Assume the string can vibrate in \( d \) transverse directions. Then, for each frequency \( \ell \omega_0 \), we must have \( d \) harmonic oscillators representing the possible polarizations of the motion. Furthermore, the associated occupation numbers need a superscript labelling the \( d \) polarizations:
\[
\begin{array}{cccc}
  n^{(1)}_1 & n^{(2)}_1 & & n^{(d)}_1 \\
  n^{(1)}_2 & n^{(2)}_2 & & n^{(d)}_2 \\
  & & \vdots & \vdots \\
  n^{(1)}_l & n^{(2)}_l & & n^{(d)}_l \\
  & & \vdots & \vdots \\
\end{array}
\]  
(16.2.32)

In order to sum over all possible states in the new partition function \( Z_d \), we must sum over all possible values of the occupation numbers \( n^{(q)}_k \), where
16.2. PARTITIONS AND THE QUANTUM VIOLIN STRING

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$N$ & $p(N)$ & $p(N)_{\text{est}}$ & $p(N)/p_{\text{est}}(N)$ \\
\hline
5 & 7 & 8.94 & 0.7829 \\
10 & 42 & 48.10 & 0.8731 \\
100 & 190569292 & 199281893.25 & 0.9563 \\
1000 & $2.406 \times 10^{31}$ & $2.440 \times 10^{31}$ & 0.9860 \\
10000 & $3.617 \times 10^{106}$ & $3.633 \times 10^{106}$ & 0.9956 \\
\hline
\end{tabular}
\caption{Comparing the exact values of $p(N)$ with the estimate $p(N)_{\text{est}}$ provided by the Hardy-Ramanujan formula.}
\end{table}

$k = 1, 2, \ldots, \infty$, and $q = 1, 2, \ldots, d$. This gives

$$Z_d = \sum_{n_k^{(1)}, \ldots, n_k^{(d)}} \exp\left[ -\frac{\hbar \omega_0}{kT} \sum_{\ell=0}^{\infty} \sum_{q=1}^{d} \ell n^{(q)}_{\ell} \right] . \quad (16.2.33)$$

The sums over the various $n^{(q)}$ factorize, so,

$$Z_d = \prod_{n_k^{(1)}} \exp\left[ -\frac{\hbar \omega_0}{kT} \sum_{\ell=0}^{\infty} \ell n^{(1)}_{\ell} \right] \cdots \prod_{n_k^{(d)}} \exp\left[ -\frac{\hbar \omega_0}{kT} \sum_{\ell=0}^{\infty} \ell n^{(d)}_{\ell} \right] . \quad (16.2.34)$$

Each factor here is equal to the previously calculated partition function $Z_d$. We therefore have

$$Z_d = (Z)^d . \quad (16.2.35)$$

The new free energy $F_d$ is also easy to calculate:

$$F_d = -kT \ln Z_d = -kT d \ln Z = F d . \quad (16.2.36)$$

The entropy, obtained by differentiation of the free energy, also acquires a multiplicative factor of $d$:

$$S_d = S d . \quad (16.2.37)$$

For the energy $E_d$, the same multiplicative factor exists on account of (16.1.6). We also note that $E_d$ is equal to $\hbar \omega_0 N$, where $N$ is now the total occupation
number

\[ E_d = E d = \hbar \omega_0 N , \quad N = \sum_{\ell,q} \ell \tau^{(q)}_\ell . \]  

(16.2.38)

Using our earlier result for \( S \) in (16.2.28) we now have

\[ S_d = d (k 2\pi) \sqrt{\frac{1}{6} \frac{E}{\hbar \omega_0}} = k 2\pi \sqrt{\frac{d E d}{6 \hbar \omega_0}} = k 2\pi \sqrt{\frac{Nd}{6}} , \]  

(16.2.39)

where we made use of (16.2.38).

Let us call \( p_d(N) \) the number of partitions of \( N \) when we have a \( d \)-fold degeneracy. This means, for example, that the partition \( \{3, 2, 1\} \) of 6 now gives rise to many partitions written like \( \{3_{p_1}, 2_{p_2}, 1_{p_3}\} \), where we include subscripts \( p_i \) that can take all possible values from one to \( d \). A partition with different subscripts is considered a different partition. We now see that, for a given energy, with associated number \( N \), the number of states is \( p_d(N) \). Therefore \( S_d = k \ln p_d(N) \), and comparing with (16.2.39) we conclude that for large \( N \)

\[ \ln p_d(N) \simeq 2\pi \sqrt{\frac{Nd}{6}} . \]  

(16.2.40)

The more accurate version of this result can be shown to be

\[ p_d(N) \simeq \frac{1}{\sqrt{2}} \left( \frac{d}{24} \right)^{(d+1)/4} N^{-(d+3)/4} \exp \left( 2\pi \sqrt{\frac{Nd}{6}} \right) . \]  

(16.2.41)

You can see that for \( d = 1 \) this reduces to \( p(N) \), as given in (16.2.31). For \( d = 24 \), the number of transverse light-cone directions in the bosonic string, the expression simplifies a little:

\[ p_{24}(N) \simeq \frac{1}{\sqrt{2}} N^{-27/4} \exp \left( 4\pi \sqrt{N} \right) . \]  

(16.2.42)

Quick Calculation 16.2. Show that for large \( N \)

\[ \frac{p_{24}(N + 1)}{p_{24}(N)} \simeq \exp \left( \frac{2\pi}{\sqrt{N}} \right) . \]  

(16.2.43)

This means that the fractional change in the number of partitions when the argument is increased by one unit goes down to zero as \( N \to \infty \).
Quick Calculation 16.3. Use direct counting to confirm that \( p_{24}(1) = 24 \), \( p_{24}(2) = 324 \), \( p_{24}(3) = 3200 \), and \( p_{24}(4) = 25650 \).

Counting other types of partitions is also interesting. Consider, for example, partitions of integers into unequal parts. The possible partitions of 6 into unequal integers are

\[
\{6\}, \{5, 1\}, \{4, 2\}, \{3, 2, 1\}.
\] (16.2.44)

We denote by \( q(N) \) the number of partitions of \( N \) into unequal parts, so \( q(6) \), for example, is equal to four. We can use a fermionic version of the violin string problem to determine the large \( N \) behavior of \( q(N) \). The frequencies of the oscillators are not changed, but this time we demand that each occupation number can only be equal to zero or to one. Since no creation operator can be used more than once, the total number \( N \) of any state is effectively split into contributions all of whose parts are unequal. Creation operators that cannot be used more than once create fermionic excitations. We therefore say that we have fermionic oscillators. With a little abuse of language, the numbers entering an unequal partition are called fermionic numbers. You will show in Problem 16.2 that for large \( N \),

\[
\ln q(N) \sim 2\pi \sqrt{\frac{N}{12}}.
\] (16.2.45)

We extended the earlier counting of \( p(N) \) to the case where the elements of a partition can carry \( d \) labels. If the elements of an unequal partition can carry \( d_f \) labels, the number \( q_{d_f}(N) \) of partitions is obtained from (16.2.45) by replacing \( N \rightarrow N d_f \). In such partitions a fermionic number can appear more than once if it uses a different label each time. This counting corresponds to a system with \( d_f \) species of fermionic oscillators.

A final generalization is useful. We consider partitions of \( N \) with \( d \) labels for the ordinary numbers, and with \( d_f \) labels for the fermionic numbers. In this case (Problem 16.4) we find that for large \( N \)

\[
\ln P(N; d, d_f) \sim 2\pi \sqrt{\frac{N}{6} \left( d + \frac{d_f}{2} \right)}.
\] (16.2.46)

As an example, let’s calculate \( P(2; 1, 2) \), that is, the partitions of 2 into ordinary and fermionic numbers, with the latter having two possible labels.
The list of partitions is
\[ \{2\}, \{2_1\}, \{2_2\}, \{1, 1\}, \{1_1, 1\}, \{1_2, 1\}, \{1_1, 1_2\}. \] (16.2.47)

The labels on the fermionic numbers are shown as subscripts. We see that \( P(2; 1, 2) = 7 \).

The general partition in (16.2.46) is useful for calculations in superstring theories. The states in these theories are built with both bosonic and fermionic creation operators. An application to a supersymmetric black hole will be considered in section 16.7.

### 16.3 Hagedorn temperature

Let’s now return to the subject of relativistic strings. We will consider open string theory in the case where the open string states carry no spatial momentum. This will happen, for example, if the open string endpoints end on a D0-brane. With zero spatial momentum, the energy levels of the string are simply given by the rest masses of its quantum states. The mass-squared of a given state can be expressed in terms of the number operator \( N^\perp \) (12.6.6):

\[
M^2 = \frac{1}{\alpha'} (N^\perp - 1) \simeq \frac{N^\perp}{\alpha'}, \quad (16.3.1)
\]

in the approximation of large \( N^\perp \). It follows that the energy \( E = M \) is related to the number operator by the simple equality

\[
\sqrt{N^\perp} = \sqrt{\alpha'} E. \quad (16.3.2)
\]

In the micro-canonical ensemble, the number of states \( \Omega(E) \) equals \( p_{24}(N^\perp) \), because we have 24 transverse light-cone directions, and consequently 24 oscillator labels for each mode number. Therefore, equation (16.2.39) gives

\[
S(E) = k \log p_{24}(N^\perp) = k \cdot 2\pi \sqrt{\frac{N^\perp \cdot 24}{6}} = k \cdot 4\pi \sqrt{N^\perp}. \quad (16.3.3)
\]

Making use of the number/energy relation in (16.3.2) we find

\[
S = k \cdot 4\pi \sqrt{\alpha'} E. \quad (16.3.4)
\]
This is the entropy/energy relation at high energy. An entropy proportional to the energy is unusual because it leads to a constant temperature:

\[
\frac{1}{kT} = \frac{1}{k} \frac{\partial S}{\partial E} = 4\pi \sqrt{\alpha'}. \tag{16.3.5}
\]

This temperature is called the Hagedorn temperature \( T_H \):

\[
\frac{1}{\beta_H} = kT_H = \frac{1}{4\pi \sqrt{\alpha'}.} \tag{16.3.6}
\]

Here \( kT_H \) is the thermal energy associated with the Hagedorn temperature. In this high-energy approximation we are working with, we can increase arbitrarily the energy of strings and their temperature will remain fixed at the Hagedorn temperature. It is interesting to compare the energy \( kT_H \) to the rest mass of the particles found in the first massive level of the string. This corresponds to \( N_\perp = 2 \) in (16.3.1) and gives \( E = M = 1/\sqrt{\alpha'} \). The ratio of the Hagedorn thermal energy to this rest energy is

\[
\frac{kT_H}{\sqrt{\alpha'}} = \frac{1}{4\pi} \approx \frac{1}{12.6}. \tag{16.3.7}
\]

This shows that the Hagedorn thermal energy is quite small compared with the rest energy of almost any particle state of the string. This is an important result that will play a role in our later work in this chapter.

The entropy/energy relation in (16.3.4) holds also for closed strings having no spatial momentum. Recalling (13.2.14), we find

\[
M^2 = \frac{2}{\alpha'} (N_\perp + \overline{N}_\perp - 2) \approx \frac{4}{\alpha'} N_\perp, \tag{16.3.8}
\]

since closed string states satisfy \( N_\perp = \overline{N}_\perp \). It follows that the energy \( E = M \) is related to the number operator as

\[
2\sqrt{N_\perp} = \sqrt{\alpha'} E. \tag{16.3.9}
\]

This time, the number of states \( \Omega(E) \) is equal to the product of available states in the left-moving and in the right-moving sectors:

\[
\Omega(E) = p_{24}(N_\perp) p_{24}(\overline{N}_\perp) = (p_{24}(N_\perp))^2. \tag{16.3.10}
\]
As a result, the entropy $S$ is precisely twice that indicated in (16.3.3):

$$S(E) = k \, 4\pi \sqrt{N} \, = k \, 4\pi \sqrt{\alpha'} E , \quad (16.3.11)$$

making use of (16.3.9) in the last step. We see that the Hagedorn temperature $T_H$ is also the approximate temperature of very energetic closed strings.

16.4 Relativistic particle partition function

As a warmup to our computation of the partition function for a string, we compute here the partition function for a particle. We will work with a relativistic particle of mass $m$ that lives in a $D$-dimensional spacetime, or equivalently in $d = D - 1$ space dimensions. Moreover, we assume that this particle is confined to a box of volume

$$V = L_1 L_2 \cdots L_d . \quad (16.4.1)$$

This box is in thermal contact with a reservoir at temperature $T$. The particle has an energy/momentum relation

$$E(\vec{p}) = \sqrt{\vec{p}^2 + m^2} . \quad (16.4.2)$$

The quantum states of the particle in the box are labelled by the momenta $\vec{p}$, and therefore the partition function $Z(m^2)$ is given by

$$Z(m^2) = \sum_{\vec{p}} \exp(-\beta E(\vec{p})) . \quad (16.4.3)$$

In order to evaluate this partition function one must turn the sum over quantized momenta into an integral; this is where the volume dependence of $Z$ comes in. The quantum wavefunctions with momentum $\vec{p} = \hbar \vec{k}$ have spatial dependence $\exp(ik \cdot \vec{x})$. The periodicity of these wavefunctions in the box requires that for each spatial direction $i$

$$k_i L_i = 2\pi n_i , \quad i = 1, 2, \cdots, d . \quad (16.4.4)$$

Here the $n_i$'s are integers. Equivalently, in terms of momenta,

$$n_i = p_i \frac{L_i}{(2\pi \hbar)} . \quad (16.4.5)$$
It follows that summing over the various momenta is the same as summing over the various $n_i$. For an arbitrary smooth function $f[E]$ of the energy, we can thus write

$$
\sum_{\vec{\nu}} f[E(\vec{\nu})] = \sum_{\vec{n}} f[E(\vec{\nu}(\vec{n}))] \approx \int dn_1 dn_2 \ldots dn_d f[E(\vec{\nu}(\vec{n}))],
$$

where the approximation by an integral is allowed because, for large boxes, the momenta change very little when a counter $n_i$ changes by one unit. Using (16.4.5) and (16.4.1) we obtain

$$
\sum_{\vec{\nu}} f[E(\vec{\nu})] \approx V \int \frac{d^d \vec{p}}{(2\pi \hbar)^d} f[E(\vec{\nu})].
$$

This is the general prescription for dealing with sums over momenta. Applied to our case of interest (16.4.3) it gives

$$
Z(m^2) = V \int \frac{d^d \vec{\nu}}{(2\pi \hbar)^d} \exp\left(-\beta \sqrt{\vec{\nu}^2 + m^2}\right).
$$

This is the integral representation of the partition function for a relativistic point particle of rest mass $m$. The temperature and volume arguments of $Z$ are left implicit. Working with $\hbar = 1$, and letting $\vec{p} = m \vec{u}$, we find

$$
Z(m^2) = V m^d \int \frac{d^d \vec{u}}{(2\pi)^d} \exp\left(-\beta m \sqrt{1 + \vec{u}^2}\right).
$$

This integral is not elementary, but it can be written in terms of derivatives of the modified Bessel functions with argument $\beta m$ (Problem 16.6). Rather than doing so, we will examine the integral in the domain of interest. For our string theory applications, this is the case when the thermal energy is much smaller than the rest energy of the particle. Indeed, as we saw earlier, for temperatures below the Hagedorn temperature, all but a few string states satisfy this condition. Thus we consider the situation where

$$
\beta m >> 1, \; \text{low temperature}.
$$

We now claim that the leading approximation to the integral can be found by expanding the square root in (16.4.9) for $\vec{u}^2$ small. This is explained as follows. Using spherical coordinates and letting $\vec{u}^2 = u^2$, we note that $d^d \vec{u} \sim$
$u^{d-1}du$ (recall the familiar cases of $d = 2, 3$). As a plain one-dimensional integral, the integrand in (16.4.9) is thus of the form

$$\text{integrand} \sim u^{d-1} e^{-\beta m \sqrt{1+u^2}}. \quad (16.4.11)$$

This integrand vanishes at $u = 0$ and $u = \infty$, and it peaks somewhere in between, giving the largest contribution to the integral. The maximum of the integrand can be found by setting the $u$-derivative of (16.4.11) equal to zero. This equation gives

$$\frac{d-1}{\beta m} = \frac{u^2}{\sqrt{1+u^2}}. \quad (16.4.12)$$

Since $\beta m$ is large, the left hand side is very small, and $u^2$ must also be small. We can therefore neglect the $u^2$ in the square root and we find that the integrand is largest for

$$u^2 \simeq \frac{d-1}{\beta m} \ll 1. \quad (16.4.13)$$

We are therefore allowed to expand the square root in (16.4.9) to write

$$Z(m^2) \simeq V m^d e^{-\beta m} \int \frac{d^d \vec{u}}{(2\pi)^d} \exp\left(-\frac{1}{2} \beta m \vec{u}^2\right). \quad (16.4.14)$$

The integral is now gaussian, and is readily evaluated

$$Z(m^2) \simeq V e^{-\beta m} \left(\frac{m}{2\pi \beta}\right)^{\frac{d}{2}}. \quad (16.4.15)$$

This is our final form for the partition function of a relativistic particle in the low-temperature limit. One can verify that this partition function is dimensionless, as it should be. Except for the additional factor $e^{-\beta m}$, this partition function coincides with the exact partition function for a non-relativistic particle. The exponential factor accounts for the contribution of the relativistic rest energy to the energy of the particle.

### 16.5 Single string partition function

We are now finally ready to evaluate the partition function for a single open string placed in a box of volume $V$. In order to calculate this, we must
enumerate the quantum states of the string. The states are obtained by acting with the light-cone creation operators on the momentum eigenstates. A generic state is written as in (12.6.4):

\[ |\lambda, p\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{25} (a_n^{I\dagger})^{\lambda_n,I} |p^+, \vec{p}_T\rangle, \]  

(16.5.1)

where the notation \( |\lambda, p\rangle \) emphasizes that the momentum components as well as the occupation numbers \( \lambda_n,I \) are labels of the string states. The \( d \) components \((p^+, \vec{p}_T)\) listed in the momentum eigenstate specify the light-cone energy \( p^- \) via the on-shell condition:

\[ M^2(\{\lambda_n,I\}) = -p^2 = 2p^+p^- - p^I p^I, \]  

(16.5.2)

where

\[ M^2(\{\lambda_n,I\}) = \frac{1}{\alpha'}(N^\perp - 1), \quad N^\perp = \sum_{n,I} n \lambda_n,I. \]  

(16.5.3)

Since both the spatial momentum and the energy are determined for the above states, we can label the string states with the set \( \{\lambda_{n,I}\} \) of occupation numbers and the spatial momentum \( \vec{p} \). We then write

\[ E(\{\lambda_{n,I}\}, \vec{p}) = \sqrt{M^2(\{\lambda_{n,I}\}) + \vec{p}^2}. \]  

(16.5.4)

To find the partition function \( Z_{\text{str}} \) of a single string, we must sum over all states \( |\lambda, p\rangle \), or equivalently over all spatial momenta \( \vec{p} \) and all values of the occupation numbers \( \lambda_{n,I} \):

\[ Z_{\text{str}} = \sum_\alpha \exp(-\beta E_\alpha) = \sum_{\lambda_{n,I}} \sum_{\vec{p}} \exp\left[ -\beta \sqrt{M^2(\{\lambda_{n,I}\}) + \vec{p}^2} \right]. \]  

(16.5.5)

We recognize, however, that the momentum sum simply gives the partition function for a relativistic particle of mass-squared \( M^2(\{\lambda_{n,I}\}) \). We thus write

\[ Z_{\text{str}} = \sum_{\lambda_{n,I}} Z(M^2(\{\lambda_{n,I}\})). \]  

(16.5.6)

Since the mass \( M^2 \) depends only on \( N^\perp \), and there are \( p_{24}(N^\perp) \) states with number eigenvalue \( N^\perp \), the sum over occupation numbers \( \{\lambda_{n,I}\} \) can be
traded for a sum over $N^\perp \equiv N$:

$$Z_{\text{str}} = \sum_{N=0}^{\infty} p_{24}(N) Z(M^2(N)).$$  \hfill (16.5.7)

So far, no approximations have been made, and the above result is exact.

In order to proceed further, we approximate this sum by turning it into an integral; this is accurate for large $N$. We get

$$Z_{\text{str}} \simeq \int_0^\infty dN \, p_{24}(N) Z(M^2(N)).$$  \hfill (16.5.8)

It is customary to define a density of states $\rho(M)$ as a function of the mass $M$, and to use the mass as the variable of integration. This is done using the relation

$$p_{24}(N)dN = \rho(M)dM.$$  \hfill (16.5.9)

We express the left-hand side in terms of mass by using $\alpha' M^2 \simeq N$:

$$dN = 2\alpha' M dM = 2(\sqrt{\alpha'} M) d(\sqrt{\alpha'} M).$$  \hfill (16.5.10)

Moreover, using (16.2.42) and (16.3.6) we find

$$p_{24}(N) \simeq \frac{1}{\sqrt{2}} (\sqrt{\alpha'} M)^{-27/2} \exp(\beta_H M).$$  \hfill (16.5.11)

Substituting these two equations back into (16.5.9) gives us

$$\rho(M)dM = \sqrt{2} (\sqrt{\alpha'} M)^{-25/2} \exp(\beta_H M) d(\sqrt{\alpha'} M).$$  \hfill (16.5.12)

Note incidentally that

$$\rho(M) \sim M^{-25/2} \exp(\beta_H M),$$  \hfill (16.5.13)

showing that the exponential growth in the density of states is controlled by the Hagedorn temperature. As we will see shortly, the partition function does not converge for temperatures higher than the Hagedorn temperature. With (16.5.12) and (16.5.9), the partition function in (16.5.8) becomes

$$Z_{\text{str}} \simeq \sqrt{2} \int_0^\infty (\sqrt{\alpha'} M)^{-25/2} \exp(\beta_H M) Z(M^2) d(\sqrt{\alpha'} M).$$  \hfill (16.5.14)
16.5. SINGLE STRING PARTITION FUNCTION

It only remains to write the relativistic particle partition function (16.4.15) in terms of \( M \) and \( kT_H \). With the help of

\[
\frac{M}{2\pi\beta} = 2(\sqrt{\alpha' M}) kT kT_H, \quad \beta M = 4\pi(\sqrt{\alpha' M}) \frac{T_H}{T}, \quad (16.5.15)
\]

we find

\[
Z(M^2) \simeq V^{25/2} (kT kT_H)^{25/2} (\sqrt{\alpha' M})^{25/2} \exp\left(-4\pi \sqrt{\alpha' M} \frac{T_H}{T}\right). \quad (16.5.16)
\]

Substituting this result into (16.5.14) the string partition function becomes

\[
Z_{\text{str}} \simeq 2^{13} V (kT kT_H)^{25/2} \int_0^\infty d(\sqrt{\alpha' M}) \exp\left(-4\pi \sqrt{\alpha' M} \left[\frac{T_H}{T} - 1\right]\right). \quad (16.5.17)
\]

Notice that the powers of \( M \) in the integrand cancelled out. Setting \( x = \sqrt{\alpha' M} \), the above expression turns into

\[
Z_{\text{str}} \simeq 2^{13} V (kT kT_H)^{25/2} \int_0^\infty dx \exp\left(-4\pi x \left[\frac{T_H}{T} - 1\right]\right). \quad (16.5.18)
\]

The integral only converges for \( T < T_H \), where we have

\[
Z_{\text{str}} \simeq \frac{2^{11}}{\pi} V (kT kT_H)^{25/2} \left(\frac{T}{T_H - T}\right). \quad (16.5.19)
\]

This is our final expression for the approximate partition function of a single open string in a box of volume \( V \), in thermal contact with a reservoir at temperature \( T \). We made several approximations. In particular, we assumed that the largest contributions arise from large \( N^\perp \) and turned the sum over \( N^\perp \) into an integral. This approximation is not delicate enough to represent the contributions from massless states and from tachyon states. The partition function of a tachyon is indeed problematic: equation (16.4.8) tells us that \( Z \) is complex number if \( m^2 < 0 \). Since the tachyon represents an instability, and (16.5.19) ignores this complication, we can expect this result to capture roughly the physics of an open string in a theory without a tachyon. We have also assumed no string interactions. In this approximation, the string in the box cannot not break to become a set of short strings.
Despite all of these limitations, equation (16.5.19) gives us some interesting information. Using (16.1.7), and since the volume dependence of \( Z_{\text{str}} \) is only multiplicative, we find

\[
p = \frac{1}{\beta} \frac{\partial \ln V}{\partial V} = \frac{1}{\beta V} \quad \longrightarrow \quad pV = kT.
\]  

(16.5.20)

This is not different from the equation of state for a non-relativistic particle in a box of volume \( V \). Indeed, the volume dependence of the partition function is the same for a particle and for a string.

The calculation of the average energy (16.1.6) is a little more nontrivial. For this computation we need only the \( \beta \)-dependence of \( \ln Z \). Using (16.5.19) we find

\[
\ln Z_{\text{str}} = -\frac{25}{2} \ln \beta - \ln(\beta - \beta_H) + \ldots,
\]  

(16.5.21)

where the dots represent terms without \( \beta \) dependence. It now follows that

\[
E = -\frac{\partial \ln Z}{\partial \beta} = \frac{25}{2} kT + kT_H \left( \frac{T}{T_H - T} \right).
\]  

(16.5.22)

For low temperatures the energy is approximately given by

\[
E \simeq \frac{25}{2} kT + kT = \left( \frac{25}{2} + 1 \right) kT, \quad T << T_H.
\]  

(16.5.23)

The first term, proportional to \( d/2 \), is the standard average energy of a particle in \( d \) spatial dimensions at a temperature \( T \). The additional \( kT \) appearing in the energy is due to string effects. For temperatures smaller, but very close to the Hagedorn temperature, the energy is roughly

\[
E = \frac{kT_H}{1 - \frac{T}{T_H}}, \quad T \approx T_H.
\]  

(16.5.24)

It follows that the energy grows without bound as the temperature approaches the Hagedorn temperature. This is, of course, a string effect.

16.6 Black holes and entropy

A black hole is formed when the mass of an object is increased while keeping the object of the same size, or when the size of an object is reduced while
keeping its mass constant. Black holes exist in our universe. The existence of a supermassive black hole at the center of our galaxy has been established beyond reasonable doubt. Most likely, there are millions of black holes in every galaxy. They are the remnants of ordinary stars that are a few times more massive than the sun.

Black holes pose very significant theoretical challenges. In Einstein’s theory of general relativity they appear as classical solutions representing matter that has collapsed down to a point with infinite density: a singularity. Although dealing with singularities is quite delicate, the real puzzles of black holes arise at the quantum level. Black holes have temperature and can radiate. They have entropy as well. String theory has had definite successes in understanding some of these properties. In this section we review basic features of black holes and use string theory to discuss the entropy of four-dimensional Schwarzschild black holes. In the following section we will examine a particular five-dimensional black hole whose entropy can be calculated exactly in string theory.

The simplest black holes are Schwarzschild black holes. These black holes are spherically symmetric, static solutions that represent the gravitational field of a point mass $M$. In this black hole, the point singularity is separated from the outside world by an event horizon. This is a two-sphere centered at the singularity, whose radius $R$ is called the Schwarzschild radius, or the radius of the hole. If any object ventures inside the event horizon it will irrevocably fall into the singularity. Classically, nothing can escape from the region enclosed by the event horizon. The value $R$ of the Schwarzschild radius can be estimated by assuming that the total energy of any particle at the horizon is equal to zero. For a particle of mass $m$, this energy includes the rest energy $mc^2$ and the gravitational potential energy $-GMm/R$. Setting the sum of these to equal to zero, we find

$$mc^2 - \frac{GMm}{R} = 0 \quad \longrightarrow \quad R \simeq \frac{GM}{c^2}. \quad (16.6.1)$$

In fact, the exact answer is

$$R = \frac{2GM}{c^2}. \quad (16.6.2)$$

The Schwarzschild radius of the sun is about 3 km. The Schwarzschild radius of the earth is about 1cm. The Schwarzschild radius of a billion-ton asteroid is of the order of $10^{-15}$ m. It is possible to use Newtonian gravitation to
estimate the gravitational field at the horizon:
\[
|\vec{g}| = \frac{GM}{R^2} = \frac{c^4}{4GM}.
\] (16.6.3)

This gravitational field becomes small for very massive black holes.

**Quick Calculation 16.4.** Show that an object of uniform mass density \(\rho\) forms a black hole if its radius is larger than \(c/\sqrt{8\pi G \rho / 3}\).

If we believe that the second law of thermodynamics holds generally, the existence of black holes leads to some surprising conclusions. Assume that a certain amount of hot gas falls into a black hole forming a new black hole with a slightly higher mass. Since the total entropy of the system made by the gas and the black hole cannot decrease, the new black hole must have increased its entropy by at least the amount of entropy carried by the gas. We are thus led to believe that black holes must have entropy. You know that a system has entropy when there are many microscopic states of the system that are consistent with its macroscopic properties. On the other hand, if the black hole represents a point mass singularity at the origin, it is hard to see what are the microstates that give rise to the entropy.

Black holes are also assigned a temperature \(T\). This temperature, in fact, behaves as the gravitational field at the horizon: it is inversely proportional to the mass of the hole. In natural units, \(kT = 1/(8\pi M)\), so inserting back the factors of \(\hbar, c,\) and \(G\) (Problem 3.7), we find
\[
kT = \frac{\hbar c^3}{8\pi GM}.
\] (16.6.4)

This equation allows us to calculate the entropy of the black hole using \(E = Mc^2\) for the energy of the black hole and the first law of thermodynamics \(dE = TdS\):
\[
dE = c^2 dM = TdS = \frac{\hbar c^3}{8\pi GM} \frac{1}{k} dS.
\] (16.6.5)

A little rearrangement yields:
\[
\frac{1}{k} dS = \frac{4\pi G}{\hbar c} dM^2.
\] (16.6.6)

Integrating this equation, and assuming that the entropy of a zero-mass black hole is zero, we find
\[
\frac{S}{k} = \frac{4\pi G}{\hbar c} M^2.
\] (16.6.7)
The entropy of the black hole is proportional to the square of its mass. A useful alternative expression for the entropy uses the area $A$ of the event horizon. With $A = 4\pi R^2$ and $R$ given in (16.6.2), one readily obtains

$$\frac{S}{k} = \frac{1}{4\hbar G^2} A = \frac{A}{4\ell_P^2},$$

(16.6.8)

where $\ell_P$ is the Planck length. The last right-hand side in this equation has a simple interpretation: the entropy is one-fourth of the area of the horizon expressed in units of Planck-length squared. Since $\ell_P^2$ is a remarkably small area, the entropy of any astrophysical-size black hole is extremely large. The entropy of a black hole is roughly reproduced if one imagines having a degree of freedom with a finite number of states for each horizon element of area $\ell_P^2$. String theory provides candidate degrees of freedom for black holes, but they do not relate directly to the horizon area.

In string theory, we attempt to relate a stationary Schwarzschild black hole to a string with a high degree of excitation but zero momentum. In the microcanonical ensemble, a string state with energy $E$ has an entropy (16.3.4). This is true both for open and for closed strings (see (16.3.11)). Identifying $E = M$ and working henceforth with $\hbar = c = 1$, we have

$$\frac{S_{\text{str}}}{k} = 4\pi \sqrt{\alpha'} M,$$

(16.6.9)

where we have added the subscript ‘str’ to refer to the entropy of the string. This result should be compared to the black-hole entropy (16.6.7):

$$\frac{S_{\text{bh}}}{k} = 4\pi G M^2.$$

(16.6.10)

The disagreement appears to be clear: the entropy of a black hole goes like the mass squared, while the entropy of a string goes like the mass. We will soon show, however, that the apparent disagreement was to be expected. Properly understood, there is a surprising agreement between these equations. The linear dependence of the string entropy on the mass $M$ of the string is not surprising. Entropy is an extensive quantity, and for a string, the mass $M$ is roughly proportional to its length $L$. The black hole entropy, on the other hand, exhibits a surprising feature. It is not proportional to the volume enclosed by the event horizon, but rather, to the area of the horizon. This failure of extensivity is a feature of gravitational physics.
Before considering the relation between equations (16.6.9) and (16.6.10), let’s give a heuristic derivation of the string entropy. For this, we consider a string of mass \( M \) and estimate its length \( L \) to be roughly given by

\[
M \sim T_0 L \sim \frac{1}{\alpha'} L ,
\]

where \( T_0 \sim 1/\alpha' \) is the string tension. We now imagine the string built by joining together string bits, each of which is of length \( \ell_s = \sqrt{\alpha'} \). Assume that each time we add a bit, it can point in any of \( n \) possible directions. The number \( n \) may be equal to the number of spatial dimensions, but since our arguments are rough, we will not attempt to be specific. Since the number of string bits is \( L/\sqrt{\alpha'} \), the number of ways \( \Omega \) that we can build this string is roughly

\[
\Omega \sim n^{L/\sqrt{\alpha'}} \sim n^{M\sqrt{\alpha'}} \sim e^{M\sqrt{\alpha'} \ln n}.
\]

(16.6.12)

The entropy of the string is obtained by taking the logarithm of \( \Omega \):

\[
\frac{S_{str}}{k} \sim M\sqrt{\alpha'} \sim M\ell_s ,
\]

(16.6.13)

where we discarded the \( \ln n \) factor, in keeping with the accuracy of the estimate. This result is consistent with the expression given in (16.6.10).

The reason equations (16.6.9) and (16.6.10) disagree is that the black hole entropy \( S_{bh} \) was calculated in a regime where interactions are necessary, while the string entropy \( S_{str} \) was calculated for free strings. We did not have the right to expect agreement, unless for some reason, interactions did not affect the calculation of the entropy of strings. There is no such reason in the theory of bosonic strings.

Interactions are necessary in the black hole entropy calculation because Newton’s constant \( G \) vanishes if the string coupling constant \( g \) is set to zero. Indeed, we recall (13.4.6), which states that

\[
G \sim g^2 \alpha' = g^2 \ell_s^2 .
\]

(16.6.14)

The black hole entropy and its radius are then given as

\[
\frac{S_{bh}}{k} \sim GM^2 \sim g^2 \ell_s^2 M^2 ,
\]

\[
R \sim GM \sim g^2 \ell_s^2 M .
\]

(16.6.15)
While they incorporate the string coupling dependence via Newton’s constant, the above results use classical general relativity, where, for example, the concept of a horizon makes sense. We are allowed to neglect string theory corrections to general relativity as long as black holes are larger than the string length.

Consider now a large black hole with entropy $S_0$, mass $M_0$ and radius $R_0 \gg \ell_s$. Fix also the string coupling at some finite value $g_0$. Equations (16.6.15) then give us

$$\frac{S_0}{k} \sim g_0^2 \ell_s^2 M_0^2,$$

$$R_0 \sim g_0^2 \ell_s^2 M_0.$$  \hspace{1cm} (16.6.16)

Since the calculation of the string entropy is valid for zero, or possibly small string coupling, imagine now the process of dialing down the value of the string coupling. This is done by changing the expectation value of the dilaton, as explained in section 13.4. It is reasonable to assume that this process can be carried out reversibly, so we can expect the black hole entropy to remain unchanged. On the other hand, as we dial down the coupling $g$ the mass of the black hole increases like $1/g$ to keep the entropy constant in (16.6.15).

The mass is not increasing, however, if measured in units of Planck mass, since $G \sim 1/m_P^2$. The radius $R$ of the black hole decreases, as it follows from the second relation in (16.6.15) bearing in mind that $M \sim 1/g$.

Let $g_*, R_*$, and $M_*$ denote the final values of the string coupling, black hole radius, and black hole mass, respectively. The constancy of the entropy, and the formula for the radius give

$$\frac{S_0}{k} \sim g_0^2 \ell_s^2 M_0^2 = g_*^2 \ell_s^2 M_*^2,$$

$$R_* \sim g_*^2 \ell_s^2 M_*.$$  \hspace{1cm} (16.6.17)

We do not expect these results to hold when the black hole becomes smaller that the string length, so let’s fix $R_* = \ell_s$ as the minimum radius for which equations (16.6.17) can be trusted. The condition $R_* = \ell_s$ tells us that

$$g_*^2 \ell_s^2 M_* \sim \ell_s \quad \longrightarrow \quad M_* \sim \frac{1}{g_*^2 \ell_s}.$$  \hspace{1cm} (16.6.18)

Back into the expression for the entropy $S_0$, we find

$$\frac{S_0}{k} \sim \frac{1}{g_*^2},$$  \hspace{1cm} (16.6.19)
The coupling $g_*$ is clearly very small since $S_0$ was assumed to be very large. At such weak coupling we can reasonably trust the free string theory expression (16.6.13) for the entropy. Since the black hole we are comparing with has mass $M_*$, we consider a string of mass $M_*$. The entropy is then given by

$$\frac{S_{\text{str}}}{k} \sim M_* \ell_s \sim \left( \frac{1}{g_*^2 \ell_s} \right) \ell_s \sim \frac{1}{g_*^2},$$

(16.6.20)

where we made use of (16.6.18). Comparing with $S_0$ in (16.6.19), we see that $S_{\text{str}} \sim S_0$. This agreement is evidence for the hypothesis that a Schwarzschild black hole is the strong coupling version of a string with a very high degree of excitation. It is far from a proof, however. As you have seen, we have only written approximate relations, and we have made a series of assumptions about the ranges of validity of certain results. A proof remains to be found at this time. Nevertheless, there is additional circumstantial evidence that this picture is at least roughly correct. It is possible to estimate the “size” of a string using the picture of string bits and assuming that the string is a random walk. One can then show that for any fixed coupling $g$ there is a mass beyond which any excited string state is smaller than its Schwarzschild radius (Problem 16.8). This suggests that very heavy string states will form black holes.

### 16.7 Counting states of a black hole

Our computation of the entropy of strings can be done in the limit when we neglect the effects of interactions. Since a black hole can only exist once interactions are turned on, an exact computation of the entropy of a black hole in string theory requires that the counting of states done with string coupling $g = 0$ remain valid when $g \neq 0$.

For the Schwarzschild black holes considered in the previous section this does not happen. As a result, we could only confirm qualitative agreement over a narrow range of couplings where the gravity computation and the free string theory computation could both hold. In this section we wish to consider a particular five-dimensional black hole that appears in superstring theory. In this black hole, as we will explain below, the counting of states at zero string coupling will remain valid when the coupling becomes non-zero. It is the simplest known black hole with this property. Four-dimensional black
holes with the same property are known, but are slightly more complicated. This is why we focus here on the five-dimensional black hole.

Such remarkable property is due to supersymmetry, a symmetry that relates bosons to fermions. As long as this symmetry is present, certain quantities can be calculated at zero coupling, and the results are valid for all values of the coupling. Superstring theories living in ten-dimensional Minkowski spacetime have supersymmetry. It is a challenge to compactify spacetime and preserve supersymmetry, but this happens if we curl up dimensions into circles. If we now include a black hole solution in the compactified spacetime, supersymmetry can be lost. The black hole we are interested in is special: some supersymmetry survives.

The starting point is type IIB superstring theory, a ten-dimensional theory of closed strings. One can search for black hole solutions in the regime where the string theory is well-approximated by a field theory of gravity, Kalb-Ramond fields, and other fields, including fermions. Such a theory is called type IIB supergravity. The black hole in question is obtained after curling up five of the spatial dimensions into circles. These curled up dimensions are $x^5, x^6, x^7, x^8,$ and $x^9$. The black hole is a spherically symmetric configuration in the un-compactified effective spacetime $M^5$ defined by the coordinates $x^0, x^1, x^2, x^3$, and $x^4$. We cannot discuss here the full construction of the black hole, so we will simply summarize the results that are obtained:

1. The black hole carries three different electric charges with respect to three Maxwell-like gauge fields that live on $M^5$. These charges are denoted by the integers

$$Q_1, Q_5, \text{ and } N.$$  \hspace{1cm} (16.7.1)

A specific black-hole is obtained by choosing these three integers.

2. The black-holes is \textit{extremal}: it has the minimal mass that is compatible with its charges. It does not radiate, since radiation would reduce its mass without the necessary change of charge. The black hole has zero temperature. In addition, its presence preserves a large part of the original supersymmetry of the IIB theory in ten-dimensional Minkowski space.

3. The black-hole horizon is a three-sphere with finite volume $A_H$. The thermodynamically expected black hole entropy $S_{bh}$ is calculated using
the five-dimensional analog of (16.6.8):

\[
\frac{S_{bh}}{k} = \frac{A_H}{4G^{(5)}} = 2\pi \sqrt{NQ_1 Q_5}.
\]  

(16.7.2)

Here \(G^{(5)}\) is the five-dimensional Newton constant and we have set \(\hbar = c = 1\). Interestingly, the entropy only depends on the charges carried by the black hole, and not on other parameters, like the string coupling, or the size of the circles used for the compactification.

The goal of a string theory computation is to reproduce the entropy (16.7.2) by a counting of states. String theory must explain why this black hole can be constructed in many possible ways. As before, we know how to count states in non-interacting string theory. This time, however, the black hole respects supersymmetry and this guarantees that the zero-coupling counting holds for non-zero coupling.

At zero coupling the black hole is constructed by considering type IIB superstring theory, with the five coordinates \(x^5, \ldots, x^9\) curled up into circles. The charges \(Q_1\) and \(Q_5\) are generated by wrapping a number \(Q_1\) of D1-branes along the circle \(x^5\), and a number \(Q_5\) of D5-branes around the five circles. Since a D5-brane has five spatial dimensions, the D5-branes wrap completely around the compact extra dimensions. How does this look to the five-dimensional observer in \(M^5\)? Since all directions along \(M^5\) are Dirichlet for the D5-branes, the D5-branes have fixed positions on \(M^5\). They appear as a collection of points. The same is true for the D1-branes. In the configuration we are trying to build, we require that all these points coincide. Thus all D-branes are coincident, and are seen by the observer as a single point in \(M^5\). This point is the center of the would-be black hole that forms when the coupling is turned on. So far, this configuration of D-branes cannot be built in different ways preserving supersymmetry. A few discrete choices are possible, we can choose, for example, another coordinate to wrap all of the D1-branes. But any constant number, independent of the charges will not help us get the correct entropy. So, where does the entropy come from?

We recall that the macroscopic black hole had an additional charge \(N\). What does it correspond to in the brane construction? It is a momentum quantum number. The momentum around the circle \(x^5\) must equal

\[
p^5 = \frac{N}{R},
\]  

(16.7.3)
where $R$ is the radius of the circle. This momentum cannot be carried by
the D-branes since they are translationally invariant along the $x^5$-direction.
The momentum is carried by open strings attached to the D-branes! We
can now see how it is possible to get many states: there are many kinds
of strings stretching between the $Q_1$ D1-branes and the $Q_5$ D5-branes. We
have $(1,1)$ strings going from D1-branes to D1-branes. We have $(5,5)$ strings
going from D5-branes to D5-branes. Finally, there are $(1,5)$ and $(5,1)$ strings,
going from D1-branes to D5-branes and vice versa, respectively. Moreover,
the total momentum quantum number $N$ can be split between many open
strings. Supersymmetry, however, makes one extra demand: all of the open
strings must carry momentum in the same direction along $x^5$.

To proceed further, we need some known facts about the combined system
of coincident D1- and D5-branes:

(1) The D1/D5 brane system is a bound-state system. Open strings of
type $(1,1)$ and $(5,5)$ become massive and do not become excited in the
configuration we are interested in. These strings can be dropped from
the counting.

(2) The total number of ground states of a $(1,5)$ string and the oppositely-
oriented $(5,1)$ string is eight: four bosonic ground states and four
fermionic ground states.

(3) The $Q_1$ D1-branes may join to form a single D1-brane wrapped $Q_1$
times around the circle. Similarly, the $Q_5$ D5-branes can join to form a
single D5-brane wrapped $Q_5$ times around the circle. If this happens,
the charges are not changed.

Bearing this information in mind, we see that the momentum number
$N$ must be split among open strings that go in between D1-branes and D5-
branes. We need a partition of $N$, but of which kind? Let’s assume, for the
time being that $N \gg Q_1 Q_5$ and do a preliminary counting that will work
but is not generally valid.

We have to partition $N$ and for each element of a partition we have
to tell what kind of state is carrying such momentum quantum number.
There are $Q_1 Q_5$ ways of picking a D1-brane and a D5-brane. But then, four
additional ways to pick a bosonic excitation or, alternatively, four ways to
pick a fermionic excitation (see (2)). As a result, we have $d = 4 Q_1 Q_5$ bosonic
labels and $d_f = 4Q_1Q_5$ fermionic labels. Making use of (16.2.46), the entropy is then

$$\frac{S_{str}}{k} = \ln P\left(N; 4Q_1Q_5, 4Q_1Q_5\right) \sim 2\pi \sqrt{\frac{N}{6} (4Q_1Q_5)^3} = 2\pi \sqrt{NQ_1Q_5},$$

in perfect agreement with (16.7.2). This is very nice, but not general enough. The restriction $N \gg Q_1Q_5$ is needed because in (16.2.46) $N$ must be much larger than both $d$ and $d_f$. It can be shown that if $N, Q_1$ and $Q_5$ all grow large simultaneously, $\ln P$ actually fails to give the expected entropy. This means that we have not quite yet identified the general counting that gives the entropy.

The clue is given in item (3) of the list above. Imagine the D1-brane wrapped $Q_1$ times around the circle $x^5$. Consider then, a (1,1) string moving along the D1-brane. How is the momentum of the string quantized? For such string, the circle has effectively become $Q_1$ times longer: $(2\pi R)Q_1$ is the distance the string must travel to return to its original starting point on the D1-brane. Accordingly, the string momentum is quantized in units of $1/(Q_1 R)$. This is true with one proviso. The individual open strings can have their momentum quantized with this finer unit, but the total momentum of all the open strings must still be quantized in units of $1/(Q_1 R)$. This is because the system comprised by the D1-brane and the attached open strings must be invariant under a $2\pi R$-long translation along the circle. As a result, the total momentum of the system must be quantized in units of $1/R$. Since the D1-brane has no momentum, the claim follows.

We must focus, however, on the strings stretching between D1- and D5-branes. Imagine now that the D5-branes are also wrapped. For simplicity, assume that $Q_1$ and $Q_5$ are relatively prime (we will relax this assumption shortly). Consider now a (1,5) string. How many times must it go around the circle so that both of its endpoints return to their original positions? After $Q_1$ turns the first endpoint does, but not the second. After $Q_5$ turns the second endpoint returns to its starting point, but the first does not. Being relatively prime, it takes $Q_1Q_5$ turns to have both endpoints return to their original positions on the respective branes. As a result, the momentum of (1,5) and (5,1) strings is quantized with the even finer unit of $1/(Q_1Q_5 R)$. This can be arranged to be approximately true even if $Q_1$ and $Q_5$ are not relatively prime. Take, for example $Q_1 = Q_5 = 100$. We can take the D1-brane and split off one turn, to get a system with $Q_1' = 99$, plus one extra D1-brane.
Since $Q'_1$ and $Q_5$ are relatively prime, the momentum of most open strings is then quantized in units of $1/(Q'_1Q_5 R)$, which is approximately equal to $1/(Q_1Q_5 R)$. In general, for large $Q_1$ and $Q_5$ we can find relatively prime numbers $Q'_1 < Q_1$ and $Q'_5 < Q_5$, such that $Q'_1 \sim Q_1$ and $Q'_2 \sim Q_2$.

With this finer unit of quantization, the total momentum in (16.7.3) is suggestively written as

$$p^5 = \frac{NQ_1 Q_5}{Q_1 Q_5 R}, \quad (16.7.5)$$

This time we must partition the quantum number $NQ_1 Q_5$. Since we just have one long D1-brane and one D5-brane, there is just one kind of string stretching across the branes. Therefore, the labels on the elements of a partition are either four bosonic ones, or four fermionic ones: $d = d_f = 4$. As a result, the entropy is given by

$$S_{\text{str}} = \ln P(NQ_1 Q_5 ; 4, 4) \sim 2\pi \sqrt{\frac{NQ_1 Q_5}{6}} \frac{3}{2} = 2\pi \sqrt{NQ_1 Q_5}. \quad (16.7.6)$$

The agreement with the black hole entropy is now complete and holds generally.

Giving a statistical mechanics derivation of the black hole entropy is a significant accomplishment of string theory. After all, black holes do exist, and they have entropy. Much work remains to be done in string theory to understand fully black holes. As we have seen, Schwarzschild black holes are not under such precise control. Moreover, there are puzzles associated to the fate of the information that falls into a black hole.

String theory gives a clear picture of the zero-coupling degrees of freedom of the would-be black hole. Moreover, we know that the counting continues to hold for non-zero coupling. Nevertheless we do not know how these degrees of freedom look by the time the black hole is formed. Thus mysteries remain.
CHAPTER 16. STRING THERMODYNAMICS AND BLACK HOLES

Problems


(a) Prove equation (16.1.7). Hint: The energy levels $E_\alpha(V)$ of the system depend on the volume. As the volume changes quasi-statically, the change of mean energy is calculated using the equilibrium distribution of states. The change of mean energy can be interpreted as due to work against the pressure.

(b) Prove equation (16.1.15). Hint: Consider the differential $d \ln Z(T,V)$.

Problem 16.2. Fermionic violin string and counting unequal partitions.

Consider a system of simple harmonic oscillators of frequencies $\omega_0, 2\omega_0, \ldots$ identical to those of the bosonic violin string oscillators of section 16.2. This time, however, each occupation number $n_\ell$ can only take the values 0 or 1. Oscillators with such property are said to be fermionic oscillators.

(a) Calculate the free energy of such string in the high temperature limit. The answer involves the sum

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots$$

You can calculate this sum using the result in (16.2.23).

(b) Let $q(N)$ denote the number of partitions of $N$ into unequal pieces. Use your result in (a) above to show that the large $N$ expansion for $\ln q(N)$ is given by (16.2.45).

(c) Now assume this string is relativistic, with the energy related to the mode number as in (16.3.2): $\sqrt{N} = \sqrt{\alpha'} E$. What would be the “Hagedorn” temperature for a string with very high energy?

Problem 16.3. Generating functions for partitions.

A particularly simple infinite product provides a generating function for the partitions $p(n)$:

$$\prod_{n=1}^{\infty} \frac{1}{(1-x^n)} = \sum_{n=0}^{\infty} p(n)x^n.$$  

Here $p(0) \equiv 1$. To evaluate the left-hand side, each factor is expanded as an infinite Taylor series around $x = 0$. Test this formula for $n \leq 4$ and explain (in words) why it works in general. Find a generating function for unequal partitions $q(n)$ and test it for low values of $n$. 

Problems for Chapter 16

Problem 16.4. Generalized counting of partitions.

Prove the formula (16.2.46) for the counting of partitions $P(N; d, d_f)$ of $N$ into ordinary integers with $d$ labels and fermionic integers with $d_f$ labels. Calling $Z$ the partition function of ordinary oscillators, and $Z_f$ the partition function of Problem 16.2, begin your derivation by explaining why the partition function $Z_T$ for the composite system of bosonic and fermionic labelled oscillators is given by

$$Z_T = (Z)^d(Z_f)^{d_f}.$$ 

Problem 16.5. Open superstring Hagedorn temperature

Consider the supersymmetric open superstring theory described in section 13.5.

(a) Show that the total number of states (NS and R sectors) with number $N^\perp$ is $16P(N^\perp; 8, 8)$. Hints: One of the two sectors is easier to count, then use supersymmetry.

(b) Following the method of section 16.3 calculate the Hagedorn temperature for an open superstring. Show that it is a factor of $\sqrt{2}$ larger than the Hagedorn temperature of the bosonic string.

Problem 16.6. Partition function of the relativistic particle

Evaluate exactly the partition function (16.4.9) for the relativistic point particle in terms of (derivatives of) modified Bessel functions making use of the integral definition

$$K_\nu(z) = \frac{\sqrt{\pi} \left(\frac{1}{2} z\right)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-z \cosh t} \sinh^{2\nu} t \, dt.$$ 

Use the following asymptotic expansion, valid for large $z$

$$K'_\nu(z) \sim -\sqrt{\frac{\pi}{2z}} \left[1 + \frac{4\nu^2 + 3}{8z} + \cdots\right],$$

to confirm our low temperature result in (16.4.15). Calculate the first nontrivial correction to this result.

Problem 16.7. Corrections to temperature/energy relation in the idealized string.

We found the Hagedorn temperature in the idealized string model by computing the entropy/energy relation in the high energy approximation where $\ln p_{24}(N) \sim 4\pi \sqrt{N}$. Use the more accurate expression for the partitions $p_{24}(N)$ as given in (16.2.42) to find the corrections to the temperature/energy relation. You will find the surprising result that as the energy goes to infinity the temperature goes to $T_H$ from above! Plot $T(E)$ and calculate the specific heat $C$ in the large energy regime.

We used the heuristic picture of a string made out of string bits to estimate correctly the entropy (16.6.13) of a string. We now want to use this picture to estimate the size of a string state. Assume that each string bit can point randomly in any of $d$ orthogonal directions. The string can then be viewed as a random walk with a number of steps equal to the number of bits.

(a) Use the random-walk formula for the average value of the square of the displacement to show that the “size” $R_{str}$ of a string of mass $M$ is

$$R_{str}(M) \sim M^{1/2} \ell_s^{3/2} \sim N^{1/4} \ell_s,$$

where $N$ is the number eigenvalue associated to the mass $M$. Note that the size grows like the square root of the mass, while the length of the string grows like the mass.

(b) Show that the size of a string becomes smaller than its Schwarzschild radius if its mass exceeds $M$, where

$$\overline{M} \sim \frac{1}{g^4 \ell_s} \sim \frac{m_P}{g^3}.$$

Give a rough estimate of $\overline{M}$ in kg. when $g \sim 0.01$. What is the corresponding value of $N$?

(c) Consider a black hole of a million solar masses. Assume $g = 0.01$ and calculate the value of $N$ and the size of the string that models this black hole. Compare the string size to the Schwarzschild radius.

The random walk model of string states applies to strings with little or no angular momentum (recall that the size of a rigidly rotating open string is proportional to the mass). In this model the string is much smaller than its length. The effect of string interactions appears to reduce further the size of the strings.