Chapter 14

D-branes and Gauge Fields

The open strings we have studied so far were described by coordinates all of which satisfy Neumann boundary conditions. They represent open strings moving on the world-volume of a space-filling D25-brane. Here we quantize open strings attached to more general D-branes. We begin with the case of a single $D_p$-brane, with $1 \leq p \leq 25$. We then turn to the case of multiple, parallel $D_p$ branes, where we see the appearance of interacting gauge fields, and the possibility of massive gauge fields. We conclude with the case of parallel $D$-branes of different dimensionalities.

14.1 $D_p$-branes and boundary conditions

A $D_p$-brane is an extended object with $p$ spatial dimensions. In bosonic string theory, where the number of spatial dimensions is 25, a D25-brane is a space-filling brane. The label D in $D_p$-brane stands for Dirichlet. In the presence of a D-brane, the endpoints of open strings must lie on the brane. As we will see in more detail below, this requirement imposes a number of Dirichlet boundary conditions on the motion of the open string endpoints.

Not all extended objects in string theory are D-branes. Strings, for example, are 1-branes because they are extended objects with one spatial dimension, but they are not D1-branes. Branes with $p$ spatial dimensions are generically called $p$-branes. A 0-brane is some kind of particle. Just as the world-line of a particle is one dimensional, the world-volume of a $p$-brane is $(p + 1)$-dimensional. Of these $p + 1$ dimensions, one is the time dimension and the other $p$ are spatial dimensions. We first discussed the concept of
D-branes in section 6.5. In addition, Problem 6.6 examined the motion of open strings ending on D-branes of various dimensions. Our main subject in the present chapter is the quantization of open strings in the presence of various kinds of D-branes. This is a rich subject with important applications for the problem of constructing realistic models of strings. Even more, configurations of D-branes in interplay with gravitation have led to surprising new insights in the study of strongly interacting gauge theories.

Our immediate goal is to set up the notation to describe D-branes and to state the appropriate boundary conditions. We let \( d \) denote the total number of spatial dimensions in the theory; in the present case, \( d = 25 \). A \( D_p \)-brane with \( p < 25 \) extends over a \( p \)-dimensional subspace of the 25-dimensional space. We will focus on simple \( D_p \)-branes: those that are \( p \)-dimensional hyperplanes inside the \( d \)-dimensional space. How can we specify such hyperplanes? We need \( (d - p) \) linear conditions. In three spatial dimensions \( (d = 3) \), a 2-brane \((p = 2)\) is a plane, and it is specified by one linear condition \( (d - p = 3 - 2 = 1) \). For example, \( z = 0 \), specifies the \((x, y)\) plane. Similarly, a string along the \( z \) axis \((p = 1)\) is specified by 2 linear conditions \((d - p = 3 - 1 = 2)\): \( x = 0, y = 0 \). We need as many conditions as there are spatial coordinates normal to the brane.

Consider a \( D_p \)-brane in \( D = d + 1 = 26 \) dimensional spacetime. We will define coordinates \( x^\mu \), with \( \mu = 0, 1, \ldots, 25 \), that are split into two groups. The first group includes the coordinates tangential to the brane world-volume. These are the time coordinate and \( p \) spatial coordinates. The second group includes the \((d - p)\) coordinates normal to the brane world-volume. We write

\[
\begin{align*}
\begin{array}{c}
\text{D}_p \text{ tangential coordinates} \\
\text{D}_p \text{ normal coordinates}
\end{array}
\end{align*}
\]

\[
\begin{array}{c}
x^0, x^1, \ldots, x^p \\quad x^{p+1}, x^{p+2}, \ldots, x^d
\end{array}
\]

The location of the \( D_p \)-brane is specified by fixing the values of the coordinates normal to the brane. With this split in mind we write

\[
x^a = \bar{x}^a, \quad a = p + 1, p + 2, \ldots, d.
\]  

(14.1.2)

Here the \( \bar{x}^a \) are a set of \((d - p)\) constants. In a completely analogous fashion, the string coordinates \( X^\nu(\tau, \sigma) \) are split as

\[
\begin{align*}
\begin{array}{c}
\text{D}_p \text{ tangential coordinates} \\
\text{D}_p \text{ normal coordinates}
\end{array}
\end{align*}
\]

\[
\begin{array}{c}
X^0, X^1, \ldots, X^p \\quad X^{p+1}, X^{p+2}, \ldots, X^d
\end{array}
\]
Since the endpoints of the open string must lie on the D-brane, the string coordinates normal to the brane must satisfy Dirichlet boundary conditions

$$X^a(\tau, \sigma)\big|_{\sigma=0} = X^a(\tau, \sigma)\big|_{\sigma=\pi} = \bar{x}^a, \quad a = p + 1, p + 2, \ldots, d. \quad (14.1.4)$$

The string coordinates $X^a$ are called DD coordinates, because both endpoints satisfy a Dirichlet boundary condition. The open string endpoints can move freely along the directions tangential to the D-brane. As a result, the string coordinates tangential to the D-brane satisfy Neumann boundary conditions:

$$X^{m'}(\tau, \sigma)\big|_{\sigma=0} = X^{m'}(\tau, \sigma)\big|_{\sigma=\pi} = 0, \quad m = 0, 1, \ldots, p. \quad (14.1.5)$$

These string coordinates are called NN coordinates because both endpoints satisfy a Neumann boundary condition. We see that the split (14.1.3) into tangential and normal coordinates is also a split into coordinates which satisfy Neumann and Dirichlet boundary conditions, respectively:

$$\underbrace{X^0, X^1, \ldots, X^p}_{\text{NN coordinates}} \quad \underbrace{X^{p+1}, X^{p+2}, \ldots, X^d}_{\text{DD coordinates}}. \quad (14.1.6)$$

In order to use the light-cone gauge we need at least one spatial NN coordinate that can be used together with $X^0$ to define the coordinates $X^\pm$. We therefore need to take $p \geq 1$, and our analysis does not apply to the case of a D0-brane. We will label the light-cone coordinates as

$$\underbrace{X^+, X^-, \{X^i\}}_{\text{NN}} \quad \underbrace{\{X^a\}}_{\text{DD}} \quad i = 2, \ldots, p, \quad \text{and} \quad a = p + 1, \ldots, d. \quad (14.1.7)$$

## 14.2 Quantizing open strings on Dp-branes

Having specified the boundary conditions on the various string coordinates we can proceed to the quantization of open strings in the presence of a Dp-brane. The purpose of the analysis that follows is to determine the spectrum of open string states, and to use this result to understand more deeply what goes on in the world-volume of a D-brane.

Our earlier work in Chapter 13 is quite useful here. The NN coordinates $X^i(\tau, \sigma)$ satisfy exactly the same conditions that the light-cone coordinates
CHAPTER 14. D-BRANES AND GAUGE FIELDS

$X^I$ satisfy in the quantization of open strings attached to a D25-brane. All expansions and commutation relations for the $X^i$ coordinates can be obtained from those of $X^I$ by replacing $I \rightarrow i$ in the relevant equations.

Additionally, we recall that the $X^-$ coordinate was determined in terms of the transverse light-cone coordinates as in equation (9.5.5):

$$\dot{X}^- \pm X^- = \frac{1}{2\alpha'} \frac{1}{2p^+} (\dot{X}^I \pm X'^I)^2. \quad (14.2.1)$$

Moreover, the mode expansion of $\dot{X}^I \pm X'^I$ was given in (9.5.12):

$$\dot{X}^I \pm X'^I = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^I_n e^{-in(\tau \pm \sigma)}. \quad (14.2.2)$$

A completely analogous expression holds for the mode expansion of the coordinate $X^-$. These equations led eventually to equations (12.4.10) and (12.4.11), summarized here as

$$2p^+ p^- = \frac{1}{\alpha'} (\frac{1}{2} \alpha^I_0 \alpha^I_0 + \sum_{n=1}^{\infty} \alpha^I_{-n} \alpha^I_n - a). \quad (14.2.3)$$

The subtraction constant $a$ was determined to be equal to one for the quantization of strings on a D25-brane. The light-cone index $I = 2, \ldots, 25$, takes values that presently run over NN coordinates labelled by $i$ and DD coordinates labeled by $a$. As a result, (14.2.1) becomes

$$\dot{X}^- \pm X'^- = \frac{1}{2\alpha'} \frac{1}{2p^+} \left\{ (\dot{X}^i \pm X'^i)^2 + (\dot{X}^a \pm X'^a)^2 \right\}. \quad (14.2.4)$$

As explained before, the $X^i$ coordinates are expanded as

$$\dot{X}^i \pm X'^i = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^i_n e^{-in(\tau \pm \sigma)}. \quad (14.2.5)$$

The $X^a$ coordinates are the ones we must investigate. If an expansion analogous to (14.2.5) holds for $X^a$, we will be able to find $p^-$ by letting $I \rightarrow (i, a)$ in (14.2.3), just as we did to obtain equation (14.2.4).

We are finally in a position to address the novel part of the quantization of open strings attached to a $Dp$-brane. The coordinates $X^a$ transverse to
the brane satisfy the wave equation. The general solution is a superposition of two waves:

\[ X^a(\tau, \sigma) = \frac{1}{2} \left( f^a(\tau + \sigma) + g^a(\tau - \sigma) \right), \quad (14.2.6) \]

Let’s examine the boundary conditions (14.1.4). At \( \sigma = 0 \) we obtain

\[ X^a(\tau, 0) = \frac{1}{2} (f^a(\tau) + g^a(\tau)) = \bar{x}^a, \quad (14.2.7) \]

so that \( g^a(\tau) = -f^a(\tau) + 2\bar{x}^a \), and as a result

\[ X^a(\tau, \sigma) = \bar{x}^a + \frac{1}{2} \left( f^a(\tau + \sigma) - f^a(\tau - \sigma) \right). \quad (14.2.8) \]

The boundary condition at \( \sigma = \pi \) then gives us

\[ f^a(\tau + \pi) = f^a(\tau - \pi). \quad (14.2.9) \]

This simply means that \( f^a(u) \) is a periodic function with period \( 2\pi \). This information is incorporated into the following expansion for \( f(u) \):

\[ f^a(u) = \tilde{f}_0^a + \sum_{n=1}^{\infty} \left( \tilde{f}_n^a \cos nu + \tilde{g}_n^a \sin nu \right). \quad (14.2.10) \]

It is interesting to note that there is no term linear in \( u \). Such a term was present when the coordinate satisfied a Neumann boundary condition because in that case the derivative \( f'(u) \) was periodic. Returning to (14.2.8), with (14.2.10) we have, after some trigonometric simplification,

\[ X^a(\tau, \sigma) = \bar{x}^a + \sum_{n=1}^{\infty} \left( -\tilde{f}_n^a \sin n\tau \sin n\sigma + \tilde{g}_n^a \cos n\tau \sin n\sigma \right). \quad (14.2.11) \]

Redefining the expansion coefficients that are arbitrary anyway, we can write

\[ X^a(\tau, \sigma) = \bar{x}^a + \sum_{n=1}^{\infty} \left( f_n^a \cos n\tau + \bar{f}_n^a \sin n\tau \right) \sin n\sigma. \quad (14.2.12) \]

There is no term linear in \( \tau \) here. This means that the string has no momentum in the direction \( x^a \). This is reasonable: strings must remain attached to the brane. With a \( p^a \tau \) term, the endpoint \( \sigma = 0 \) would not remain at
\( x^a = \bar{x}^a \) for \( \tau \neq 0 \).

In order to define the quantum theory associated to \( X^a \), we focus on the classical parameters describing the motion of the open string in equation (14.2.12). Since we are trying to quantize strings attached to a fixed Dp-brane, the values \( \bar{x}^a \) are not parameters that can be adjusted to describe various open string motions. The \((f^a,\tilde{f}^a)\) are on a different footing; they are parameters of the classical motion of the open string. Therefore, in quantizing the open string, the \( \bar{x}^a \) remain numbers and do not become operators, while the \( f^a \) and \( \tilde{f}^a \) turn into operators.

We now rewrite (14.2.12) in terms of oscillators, defined conveniently to obtain familiar commutation relations:

\[
X^a(\tau,\sigma) = \bar{x}^a + \sqrt{2\alpha'} \sum_{n \neq 0}^{\infty} \frac{1}{n} \alpha_n^a e^{-in\tau} \sin n\sigma.
\]

The string coordinate \( X^a \) is Hermitian if \((\alpha_n^a)^\dagger = \alpha_{-n}^a\), the usual Hermiticity property of oscillators. Note that the zero mode \( \alpha_0^a \) does not exist. Additionally,

\[
\dot{X}^a = -i\sqrt{2\alpha'} \sum_{n \neq 0} \alpha_n^a e^{-in\tau} \sin n\sigma, \quad X^{a\prime} = \sqrt{2\alpha'} \sum_{n \neq 0} \alpha_n^a e^{-in\tau} \cos n\sigma,
\]

and therefore

\[
X^{a\prime} \pm \dot{X}^a = \sqrt{2\alpha'} \sum_{n \neq 0} \alpha_n^a e^{-in(\tau \pm \sigma)}.
\]

The analogy with (14.2.5) is quite close, but there are two differences. First, when the lower sign applies, the combinations of derivatives differ by an overall minus sign. Second, the zero mode is absent for (14.2.15).

The quantization is now straightforward. With \( P_a^{(\tau,\sigma)} = \dot{X}^a / 2\pi\alpha' \), the non-vanishing commutators are postulated to be

\[
\left[ X^a(\tau,\sigma), \dot{X}^b(\tau,\sigma') \right] = 2\pi\alpha' i \delta^{ab} \delta(\sigma - \sigma').
\]
sign difference alluded above is of no import since such terms appear twice is the relevant formulae. We thus find

\[ [\alpha^a_m, \alpha^b_n] = m \delta^{ab} \delta_{m+n,0}, \quad m, n \neq 0. \tag{14.2.17} \]

There is no mismatch for zero modes: \( \bar{x}^a \) is a constant, and there is no conjugate momentum since \( \alpha^a_0 \equiv 0 \). The sign difference is also immaterial for the evaluation of (14.2.4). As a result, equation (14.2.3) can be split as

\[ 2p^+ p^- \equiv \frac{1}{\alpha'} \left( \alpha' p^i p^i + \sum_{n=1}^{\infty} \left[ \alpha^i_{-n} \alpha^i_n + \alpha^a_{-n} \alpha^a_n \right] - 1 \right). \tag{14.2.18} \]

A few comments are needed here. Since the momentum \( p^a \sim \alpha^a_0 \equiv 0 \), the term \( \frac{1}{2} \alpha^a_0 \alpha^i_0 \) simply became \( \alpha' p^i p^i \) (recall that \( \alpha^a_0 = \sqrt{2} \alpha' p^a \)). The ordering constant has been set to minus one, as for the D25-brane. Nor is the critical dimension changed. This is reasonable since only the zero mode structure differs between the \( X^a \) and the \( X^i \) coordinates. In particular, note that the naive contributions needed to normal order \( L_0^+ \) are the same for \( X^a \) and for \( X^i \). If follows from (14.2.18) that

\[ M^2 = -p^2 = 2p^+ p^- - p^i p^i = \frac{1}{\alpha'} \left( \sum_{n=1}^{\infty} \left[ \alpha^i_{-n} \alpha^i_n + \alpha^a_{-n} \alpha^a_n \right] - 1 \right). \tag{14.2.19} \]

Using creation and annihilation operators, we get

\[ M^2 = \frac{1}{\alpha'} \left( -1 + \sum_{n=1}^{\infty} \sum_{i=2}^{p} n a^i_n a^i_n + \sum_{m=1}^{d} \sum_{a=p+1}^{d} m a^a_m a^a_m \right). \tag{14.2.20} \]

Let us now consider the state space for the quantum string. The ground states of the quantum string in the D25-brane background were \( |p^+, \vec{p}_T\rangle \), where \( \vec{p}_T = (p^2, \ldots, p^{25}) \) is the vector with components \( p^I \). The \( I \) index now runs over \( i \) and \( a \) values, but there are no \( p^a \) momenta since no \( p^a \) operators exist. Therefore the ground states of the theory are labelled by \( p^+ \) and \( p^i \)

\[ |p^+, \vec{p}\rangle \quad \text{with} \quad \vec{p} = (p^2, \ldots, p^p). \tag{14.2.21} \]

We build states by acting with oscillators on the ground states. We have oscillators along the brane:

\[ a^i_n, \quad n \geq 1, \quad i = 2, \ldots, p, \tag{14.2.22} \]
and oscillators transverse to the brane:

\[ a_n^\dagger, \quad n \geq 1, \quad a = p + 1, \ldots, d. \]  (14.2.23)

So the states take the form

\[
\prod_{n=1}^{\infty} \prod_{i=2}^{p} (a_n^\dagger)^{\lambda_{n,i}} \prod_{m=1}^{\infty} \prod_{a=p+1}^{d} (a_m^\dagger)^{\lambda_{m,a}} |p^+, \vec{p}\rangle.
\]  (14.2.24)

Schrödinger wavefunctions take the schematic form

\[
\psi_{i_1 \ldots i_p a_1 \ldots a_q} (\tau, p^+, \vec{p}).
\]  (14.2.25)

Just as the indices on the oscillators, the indices on the wavefunctions are of two types: indices along the directions tangent to the brane (\(i\)-type), and indices along the directions normal to the brane (\(a\)-type).

In considering the field theories that arise from string quantization, we have seen that the Schrödinger wavefunctions become the fields. We can therefore ask: where do these fields live? Do these fields live over all of spacetime, or only in some subspace thereof? Since a field is a function, we are really asking about the arguments of this function. Is it a function over all of spacetime, or only over some subspace thereof?

Since the wavefunctions depend on all the \(p^i\), we have dependence on all the \(x^i\) coordinates. Because of the \(\tau\) and \(p^+\) arguments, we have \(x^+\) and \(x^-\) dependence. All together, we have fields that depend on \(x^+, x^-\), and \(x^i\), with \(i = 2, \ldots, p\). These are precisely the \((p+1)\) coordinates that span the world-volume of the Dp-brane. It is reasonable to conclude that the fields actually live on the Dp-brane. Indeed, this world-volume is the only natural candidate for a \((p+1)\)-dimensional subspace of spacetime.

Our analysis suggests, but does not prove that the fields live on the Dp-brane; the positions \(\vec{x}^a\) of the Dp-brane did not appear in the states nor in the wavefunctions. How could we prove that the fields in question live on the Dp-brane? We would have to study interactions. Since closed strings have no endpoints, they are not fixed by D-branes and can exist over all of spacetime. By scattering closed strings off the Dp-brane we can investigate if the interactions between fields from the closed string sector and fields from the open string sector take place on the D-brane world-volume. The answer appears to be yes, at least in some computational schemes. It is likely, however, that statements about where open string fields live are ambiguous or even gauge dependent. Different answers could be completely consistent.
14.2. QUANTIZING OPEN STRINGS ON DP-BRANES

We conclude our analysis of the Dp-brane by giving a list and detailed description of the fields that have $M^2 \leq 0$. Since all these fields live on the Dp brane, we must decide whether they are scalars or vectors with respect to the Lorentz symmetry of the Dp-brane. Let us begin with the simplest states, the ground states

$$|p^+, \vec{p}\rangle, \quad M^2 = -\frac{1}{\alpha}. \quad (14.2.26)$$

These states are tachyon states on the brane, and have exactly the same mass as the tachyon states we found on the D25-brane. The corresponding tachyon field, of course, is just a Lorentz scalar on the brane.

The next states have one oscillator acting on them. Consider first the case when the oscillator arises from the coordinates tangent to the brane:

$$a^i_1 |p^+, \vec{p}\rangle, \quad i = 2, \ldots, p, \quad M^2 = 0, \quad (14.2.27)$$

For any momenta, these give $(p + 1) - 2$ massless states. Moreover, they carry one index, which lives on the brane. They are therefore states that transform as a Lorentz vector. Since the number of states equals the space-time dimensionality of the brane minus two, these are clearly photon states. The associated field is a Maxwell gauge field living on the brane. This is a fundamental result:

A Dp-brane has a Maxwell field living on its world-volume. \hfill (14.2.28)

Finally, let us consider the case when the oscillator acting on the vacuum states arises from coordinates normal to the brane:

$$a^a_1 |p^+, \vec{p}\rangle, \quad a = p + 1, \ldots, d, \quad M^2 = 0. \quad (14.2.29)$$

For any momenta, these are $(d - p)$ states living on the brane. Since the index $a$ is not a Lorentz index for the brane, as far as the brane is concerned this index is just a counting label. The brane considers these states to be states that transform as Lorentz scalars. Therefore, we get a massless scalar field for each direction normal to the Dp-brane:

A Dp-brane has a massless scalar for each normal direction. \hfill (14.2.30)
These massless scalars have an interpretation. In section 12.8 we indicated that open string states represent D-brane excitations. Our $D_p$-brane and a slightly displaced parallel $D_p$-brane are actually states of the same energy. Such a displacement, constant over all of the $D_p$-brane, corresponds to a zero-momentum excitation with zero energy. These excitations arise from the massless scalars: massless excitations obey $E = p$, and in the limit of zero-momentum they have zero energy. Supporting this interpretation, we have as many massless fields as there are directions normal to the $D_p$-brane. Those are the independent directions in which the $D_p$-brane can be moved. Finally, note that the space-filling $D_{25}$-brane has no massless scalars on its world-volume, consistent with the fact that this D-brane cannot be displaced.

All in all, the massless states on the $D_p$-brane are $(p - 1)$ photon states and $(d - p)$ scalar field states. Apart from the momentum labels which are different, we have the same number of massless states as on the $D_{25}$-brane. The $(d - 1)$ states on the $D_{25}$-brane are accounted on the $D_p$-brane by $(p - 1)$ photon states and $(d - p)$ scalar states.

### 14.3 Open strings between parallel $D_p$-branes

We will now consider the quantization of open strings that can exist between two parallel $D_p$-branes. In describing such branes we will continue to use the notation of the previous sections. Two parallel branes of the same dimensionality have the same set of longitudinal coordinates and the same set of normal coordinates. Recall that the values $\bar{x}^a$ of the normal coordinates specify the position of a $D_p$-brane. This time the first $D_p$-brane is located at $x^a = \bar{x}_1^a$ and the second at $x^a = \bar{x}_2^a$. If we happen to have that $\bar{x}_1^a = \bar{x}_2^a$ for all $a$, the two $D_p$-branes coincide – they are on top of each other. Otherwise, they are separated. In Figure 14.1 we illustrate the situation with an example of two parallel, separated $D_2$-branes.

What kinds of open strings does this configuration of parallel $D_p$-branes support? There are actually four different classes of strings, each of which must be analyzed separately. The first two classes are made up of open strings that begin and end on the same D-brane, either brane one or brane two. These strings we already studied and quantized in the previous section. The other two classes are made up of strings that start on one of the branes and end on the other brane. These are *stretched strings*. The strings that begin on brane one and end on brane two are different from the strings that
begin on brane two and end on brane one. This is clear from the coordinate expansions that we will write. These strings are oppositely oriented, and the orientation of the string (the direction of increasing $\sigma$) matters. As we will show in Chapter 15, the string charge of a string changes sign when we reverse its orientation. The classes of open strings that exist in a configuration of D-branes are called sectors. The quantum theory of open strings in a configuration of two parallel D$p$-branes has four sectors. In Figure 14.1 we show a string for each of the four sectors.

Let us consider the sector where open strings begin on brane one and end on brane two. The NN string coordinates $X^+$, $X^-$, and $X^i$ are quantized just as before, since the boundary conditions are not changed from (14.1.5). On the other hand, for the DD string coordinates equation (14.1.4) is changed into

$$X^a(\tau, \sigma) \bigg|_{\sigma=0} = \bar{x}^a_1, \quad X^a(\tau, \sigma) \bigg|_{\sigma=\pi} = \bar{x}^a_2.$$  \hfill (14.3.1)

The solution of the wave equation subject to these boundary conditions can be studied starting from (14.2.8), which already incorporates the boundary conditions.
condition at $\sigma = 0$. In the present case we just change $\bar{x}^a$ to $\bar{x}_1^a$:

$$X^a(\tau, \sigma) = \bar{x}_1^a + \frac{1}{2} \left( f^a(\tau + \sigma) - f^a(\tau - \sigma) \right). \quad (14.3.2)$$

The boundary condition at $\sigma = \pi$ then gives us

$$f^a(\tau + \pi) - f^a(\tau - \pi) = 2 (\bar{x}_2^a - \bar{x}_1^a), \quad (14.3.3)$$

or, equivalently,

$$f^a(u + 2\pi) - f^a(u) = 2 (\bar{x}_2^a - \bar{x}_1^a). \quad (14.3.4)$$

This means that the derivative $f^a'(u)$ is a periodic function with period $2\pi$, and has an expansion of the type indicated in (14.2.10). Integrating, the function $f^a(u)$ must have an expansion of the form

$$f^a(u) = f_0^a u + \sum_{n=1}^{\infty} (h_n^a \cos nu + g_n^a \sin nu). \quad (14.3.5)$$

We have not included a constant term because it would drop out of $X^a$, as can be seen in (14.3.2). The constant $f_0^a$ is fixed by the boundary condition (14.3.4):

$$f_0^a = \frac{1}{\pi} (\bar{x}_2^a - \bar{x}_1^a). \quad (14.3.6)$$

It is now possible to substitute $f^a(u)$ into (14.3.2). Except for the zero modes, the computations are identical to those that led to (14.2.12). This time,

$$X^a(\tau, \sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\frac{\sigma}{\pi} + \sum_{n=1}^{\infty} \left( f_n^a \cos n\tau + \tilde{f}_n^a \sin n\tau \right) \sin n\sigma. \quad (14.3.7)$$

Note that the boundary conditions are manifestly satisfied. For strings that go from brane two to brane one we would have to exchange $\bar{x}_1^a$ and $\bar{x}_2^a$ in the equation above. Using (14.2.13) as a model, we can rewrite (14.3.7) in terms of oscillators:

$$X^a(\tau, \sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\frac{\sigma}{\pi} + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^a e^{-im\tau} \sin m\sigma. \quad (14.3.8)$$

As before, the constants $x_1^a$ and $x_2^a$ are not parameters of the open string fluctuations for fixed D-branes, and therefore they do not become quantum
operators. Note the absence of terms linear in $\tau$: the open strings do not have momentum in the $x^a$ directions. Even though we are not giving the oscillators above different names, they are not the same operators we had for the quantization of the strings beginning and ending on the same Dp-brane. The oscillators in different sectors must not be confused. This time the derivatives give

$$
\dot{X}^a = -i\sqrt{2}\alpha' \sum_{n \in \mathbb{Z}} \alpha^a_n e^{-i n\tau} \sin n\sigma, \quad X^{a'} = \sqrt{2}\alpha' \sum_{n \in \mathbb{Z}} \alpha^a_n e^{-i n\tau} \cos n\sigma,
$$

(14.3.9)

where

$$\sqrt{2}\alpha' \alpha^a_0 = \frac{1}{\pi} (\bar{x}_2^a - \bar{x}_1^a).$$

(14.3.10)

Although the strings do not carry momentum in the $x^a$ direction, there is an $\alpha^a_0$. The interpretation of $\alpha_0$ as momentum requires that $\alpha_0$ appear in $\dot{X}$. As you can see, $\alpha_0^a$ appears in $X^{a'}$, but does not appear in $\dot{X}^a$. A non-vanishing $\alpha^a_0$ implies stretched strings: $\alpha^a_0$ vanishes if the two D-branes coincide. Similar facts emerge for closed strings that wrap around compact dimensions (see Chapter 17).

The two derivatives in (14.3.9) can be combined into

$$
X^{a'} \pm \dot{X}^a = \sqrt{2}\alpha' \sum_{n \in \mathbb{Z}} \alpha^a_n e^{-in(\tau \pm \sigma)}.
$$

(14.3.11)

It follows from this result, and our comments in the previous section, that the oscillators satisfy the expected commutation relations. To calculate the mass squared operator, we reconsider equation (14.2.3). As before, we let $I \rightarrow (i, a)$, and set the subtraction constant $a$ equal to one, finding

$$
2p^+ p^- = \frac{1}{\alpha'} \left( \alpha' p^i p^j + \frac{1}{2} \alpha^a_0 \alpha_0^a + \sum_{n=1}^{\infty} \left[ \alpha^i_{-n} \alpha^i_n + \alpha^{a}_{-n} \alpha^{a}_n \right] - 1 \right).
$$

(14.3.12)

We therefore have

$$
M^2 = 2p^+ p^- - p^i p^i = \frac{1}{2\alpha'} \alpha^a_0 \alpha_0^a + \frac{1}{\alpha'} \left( \sum_{n=1}^{\infty} \left[ \alpha^i_{-n} \alpha^i_n + \alpha^{a}_{-n} \alpha^{a}_n \right] - 1 \right).
$$

(14.3.13)

Using the explicit value of $\alpha^a_0$ we finally get:

$$
M^2 = \left( \frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'} \right)^2 + \frac{1}{\alpha'} (N - 1),
$$

(14.3.14)
where
\[
N^\perp = \sum_{n=1}^{\infty} \sum_{i=2}^{p} n a_n^i a_n^i + \sum_{m=1}^{d} \sum_{a=p+1} \; ma^a_m a^a_m.
\] (14.3.15)

The first term in the right hand side of (14.3.14) is a new contribution to the mass-squared of the states. Since the string tension is \( T_0 = 1/(2\pi\alpha') \), the term is simply the square of the energy of a classical static string stretched between the two D-branes. It is eminently reasonable to find that the mass-squared operator is changed by the addition of this constant. If the branes coincide the constant vanishes.

Let’s now consider the ground states. In fact, let’s consider the ground states for the four sectors describing open strings on this D-brane configuration. The momenta labels of these states are the same: \( p^+ \) and \( \vec{p} \). To distinguish the various sectors, we include as additional ground-state labels two integers \([ij]\) with values one or two. For any open string sector, the first integer denotes the brane where the \( \sigma = 0 \) endpoint lies, and the second integer denotes the brane where the \( \sigma = \pi \) endpoint lies. Alternatively, in the \([ij]\) sector, the open strings go from brane \( i \) to brane \( j \). The ground states are thus written as \( |p^+,\vec{p};[ij]\rangle \) and the four types of vacuum states are

\[
|p^+,\vec{p};[11]\rangle, \quad |p^+,\vec{p};[22]\rangle, \quad |p^+\vec{p};[12]\rangle, \quad |p^+,\vec{p};[21]\rangle.
\] (14.3.16)

The states of open strings that are fully attached to the first brane are built with oscillators acting on the first of the above states. The states of open strings that are fully attached to the second brane are built on the second of the above states. The states of open strings stretched from brane one to brane two are built on the third of the above ground states. Finally, the states of open strings stretched from brane two to brane one are built on the last of the above ground states. These four sets of states all take the form indicated in (14.2.24), with the exception that the vacuum state is replaced by the appropriate \( |p^+,\vec{p};[ij]\rangle \). The oscillators in the four sectors are the same in number and in type, but are really different operators. We could label them with the \([ij]\) labels, but this is seldom necessary.

Where do the fields corresponding to the states built on \( |p^+,\vec{p};[12]\rangle \) live? This question is difficult to answer. They are clearly \((p+1)\)-dimensional fields, since the momentum structure of the states is the same as the one we had for string states fully-attached to a single brane. Since the two D-branes are on a similar footing as far as the stretched strings is concerned, we cannot
say that the fields live in any single one of the two D-branes. In some sense, the fields must live on both of them. Operationally, the fields are declared to live on some fixed \((p + 1)\)-dimensional space (not necessarily identified with any of the two D-branes), and are seen to have non-local interactions reflecting the fact that the D-branes are separated. More conceptually, the issue appears to require a new way of thinking, the basis of which may be provided by a branch of mathematics called non-commutative geometry.

We continue our discussion of the state space by giving a list and a detailed description of the fields making up the two lowest levels of the stretched strings. Just as in the case of the single brane, we will determine whether they are scalars or vectors with respect to \((p + 1)\)-dimensional Lorentz symmetry.

The simplest states are the ground states

\[
|p^+, \vec{p}; [12]\rangle, \quad M^2 = -\frac{1}{\alpha} + \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2. \tag{14.3.17}
\]

These states are tachyon states of the usual mass-squared if the separation between the branes is zero. If the branes are separated the mass-squared gets a positive contribution. In fact, for the critical separation

\[
|\bar{x}_2^a - \bar{x}_1^a| = 2\pi\sqrt{\alpha'}, \tag{14.3.18}
\]

the ground states represent a massless scalar field. For larger separations, the ground states represent a massive scalar.

The next states have one oscillator acting on them. Assume, until stated otherwise, that the separation between the branes is non-zero. If the oscillator acting on the ground states arises from the coordinates normal to the brane we have

\[
a_1^a |p^+, \vec{p}; [12]\rangle, \quad a = p + 1, \ldots, d, \quad M^2 = \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2. \tag{14.3.19}
\]

For any momenta, these are \((d - p)\) massive states. Since the index \(a\) is not a Lorentz index for the \((p + 1)\)-dimensional spacetime, these states are Lorentz scalars. Therefore, we get \((d - p)\) massive scalar fields. If the oscillator arises from the coordinates tangent to the brane we have

\[
a_i^a |p^+, \vec{p}; [12]\rangle, \quad i = 2, \ldots, p, \quad M^2 = \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2. \tag{14.3.20}
\]

For any momenta, these are \((p + 1) - 2 = p - 1\) massive states. Moreover, they carry an index corresponding to the \((p + 1)\)-dimensional spacetime. We
might think that these states make up a massive Maxwell gauge field, but this is not correct.

A massive gauge field has more degrees of freedom than a massless gauge field. The results of Problem 10.7 indicated that a massive gauge field has, for each value of the momentum, one more state than a massless gauge field. In a D-dimensional spacetime, a massless gauge field has, for each momentum, \( D - 2 \) states, while a massive gauge field has \( D - 1 \) states. Therefore, in the case at hand, one of the states in (14.3.19) must join the \((p - 1)\) states in (14.3.20) to form the massive vector. At the end we have one massive vector and \((d - p - 1)\) massive scalars.

Can we make an educated guess as to which scalar state in (14.3.19) becomes part of the massive gauge field? If \( p = d - 1 \) the answer is simple. The D-branes are only separated along one coordinate, and there is just one scalar in (14.3.19). The scalar uses the oscillator labelled with the direction along which the branes are separated. For \( p < d - 1 \) there are several states in (14.3.19). The scalar state that becomes part of the vector must be that which arises as the linear combination

\[
\sum_a \left( \delta x_2^a - \delta x_1^a \right) a_1^{a\dagger} \left| p^+, \vec{p}; [12] \right>.
\]

From all directions normal to the D-branes, the direction defined by the spatial vector with non-vanishing components \( \delta x_2^a - \delta x_1^a \) is unique: it takes us from one brane to the other one. To visualize this concretely you can think of two parallel D1-branes in three-dimensional space. There are clearly many normal directions that do not take us from one brane to the other, and just one direction that does. More generally, since the vectors with nonvanishing components \( \delta x_1^a \) and \( \delta x_2^a \) are points on the first and second branes, respectively, the vector difference manifestly joins one brane to the other. This direction can be said to be the direction along which the branes are separated. The guess (14.3.21) can be proven to be the correct one.

In the limit as the separation between the branes goes to zero, we obtain a very interesting situation. Even though the D-branes are coincident, they are still distinguishable and we have the four open-string sectors. The massless open-string states representing strings from brane one to brane two include a massless gauge field and \((d - p)\) massless scalars. This is the same field content as that for the sector where strings begin and end on the same D-brane. When the two D-branes coincide we therefore get a total of four massless
gauge fields. These gauge fields actually interact with one another – in the string picture they do so by the process of joining endpoints. Theories of interacting gauge fields are called Yang-Mills theories. They were discovered in the 1950’s and later used successfully to build the theories of electroweak and of strong interactions. In the world-volume of two coincident D-branes we indeed get a particular Yang-Mills theory: it is a $U(2)$ Yang-Mills theory. The two in $U(2)$ is there precisely because we have two coincident D-branes.

![Figure 14.2](image)

Figure 14.2: The interaction where a string in the sector $[ij]$ combines with a string in the sector $[jk]$ to form a string in the sector $[ik]$. The interaction occurs in (b), when the end of one string coincides with the beginning of the next.

Suppose we have $N$ D$^p$-branes. This time the sectors will be labeled by pairs $[ij]$ where $i$ and $j$ are integers that run from one to $N$. The $[ij]$ sector represents open strings starting on the $i$-th brane and ending on the $j$-th brane. It is clear that there are as many sectors as there are entries in an $N \times N$ matrix, thus $N^2$ sectors. String interactions can be visualized neatly. In a typical process, a first open string joins a second open string to form a third open string. To do so, the end of the first string ($\sigma = \pi$) joins with the beginning of the second string ($\sigma = 0$) to form a third open string. If the open strings are stretched between D-branes, a first string from the $[ij]$ sector can be joined by a second open string from the $[jk]$ sector to give a
product open string in the $[ik]$ sector. We write this possible interaction as

$$[ij] \ast [jk] = [ik], \quad j \text{ not summed}. \quad (14.3.22)$$

This interaction is possible since both the end of the first string and the beginning of the second string lie on the same D-brane, the $j$ D-brane. The physical process can be imagined to take place as in Figure 14.2. In part (a) we see the three D-branes, labelled $i, j, k$, and two strings in the $[ij]$ and $[jk]$ sectors, respectively. The end of the $[ij]$ string meets the beginning of the $[jk]$ string in (b). At this stage the interaction takes place and the strings join to form a single string. The resulting string does not remain attached to the $j$ D-brane since the joining point is not anymore the endpoint of any string. As the string moves away from the $j$ D-brane in (c), the string is clearly recognized to belong to the $[ik]$ sector.

If the $N$ D$p$-branes are coincident the $N^2$ sectors result in $N^2$ interacting massless gauge fields. This defines a $U(N)$ Yang-Mills theory on the world-volume of the $N$ coincident D-branes:

$$N \text{ coincident D-branes carry } U(N) \text{ massless gauge fields}. \quad (14.3.23)$$

The full spectrum of the open string theory consists of $N^2$ copies of the spectrum of a single D$p$-brane. Each copy is a sector of the theory that carries the appropriate $[ij]$ labels.

If we have a single brane, $N$ equals one, and we get a $U(1)$ Yang-Mills theory. The $U(1)$ Yang-Mills theory, having just one massless gauge field, coincides with the Maxwell theory. This is consistent with (14.2.28). Here $U(1)$ denotes a group; the group whose elements are complex numbers of unit length, and where group multiplication is just multiplication. The relevance of the $U(1)$ group arises because the gauge parameters in Maxwell theory are actually elements of the group. So far, our study of Maxwell gauge transformations has made no use of the $U(1)$ group structure. We will see in Chapter 18 that this group structure is needed to understand the gauge symmetry of Maxwell theory in the presence of compact spatial dimensions. For the case of $U(N)$ Yang-Mills theory, $U(N)$ is also a group of symmetries: the group whose elements are $N \times N$ unitary matrices (matrices whose Hermitian conjugates and inverses coincide) and where group multiplication is matrix multiplication. The gauge symmetries of $U(N)$ Yang-Mills theory are described by the group $U(N)$, each group element giving rise to a gauge transformation.
Quick Calculation 14.1. Consult a math book for the definition of a group, and verify that $U(1)$ and $U(N)$, as described above, are groups.

The discrete labels $i, j$ used to label the branes and the various open string sectors are sometimes called *Chan-Paton* indices. They were introduced much before we knew about D-branes precisely in order to obtain Yang-Mills theories with open strings. With the discovery of D-branes it became clear that the Chan-Paton indices were simply labels for the various D-branes in a multi-D-brane configuration.

The appearance of Yang-Mills theories on the world-volume of D-brane configurations is of great relevance because Yang-Mills theories are used to describe the Standard Model of particle physics. Electromagnetism is a $U(1)$ gauge theory, and the photon $\gamma$ is the quantum of the electromagnetic field. The electroweak theory is described by a $U(2)$ Yang-Mills theory. The four gauge bosons of this theory include the photon $\gamma$, the $W^+$, the $W^-$, and the $Z^0$. The latter three are massive gauge fields. The mechanism by which massless gauge fields become massive, known as the Higgs mechanism of field theory, has a D-brane counterpart. It corresponds to separating the D-branes that, when coincident, gives the corresponding massless gauge particles. If we have two coincident D3-branes, we obtain a $U(2)$ Yang-Mills theory, with four massless gauge fields living on the four-dimensional world-volume of the brane. Is this a good model for the electroweak gauge theory? Not quite. If we separate the D3-branes to give mass to some gauge bosons, two of them acquire a mass – the two arising from the stretched strings – and two remain massless. In the electroweak gauge theory only one gauge field remains massless. A more sophisticated D-brane configuration is needed to produce a model of the electroweak theory.

14.4 Strings between $Dp$-, and $Dq$-branes

In this section we consider a fairly general case: the case of two parallel D-branes of different dimensionality. Let $p$ and $q$ be two integers both of which satisfy $1 \leq p, q \leq 25$. We are interested in a configuration where we have two D-branes: a $Dp$-brane and a $Dq$-brane. We take $p > q$, since the case of branes of equal dimensionality was considered before. The branes can be coincident, if the $Dq$-brane world-volume is fully contained in the $Dp$-brane world-volume, or they can be separate. The branes are taken to be parallel.
Table 14.1: In the second and third row we note by \( \times \) coordinates along which the D-brane stretches, and by \( \bullet \) coordinates normal to the brane. In the last row we indicate the boundary conditions (BC) for open strings belonging to the sector \([D_2, D_1]\) of strings that stretch from the D2-brane to the D1-brane.

This means the same as what we mean when we say that a line is parallel to a plane: a second plane parallel to the given plane contains the line. Thus, if separate, there is a \( p \)-dimensional hyperplane, parallel to the \( p \)-dimensional \( D_p \)-brane, that contains the \( D_q \)-brane.

Figure 14.3: A D2-brane stretched on the \((x, y)\) plane and a parallel D1-brane stretched along the \( x \)-axis, and located at \( y = 0, z = z_0 \). Also shown is an open string going from the D2-brane to the D1-brane. For such a string, the string coordinate \( Y \) is of ND type.

A simple case that can be easily visualized is that of a D2-brane and
14.4. STRINGS BETWEEN DP-, AND DQ-BRANES

There is a parallel D1-brane, as illustrated in Figure 14.3. The D2-brane stretches along the x- and y-directions and is located at \( z = 0 \). The D1 stretches along the x-direction and is located at \( y = 0 \) and at \( z = z_0 \). This D1-brane is parallel to the D2-brane, and they are not coincident when \( z_0 \neq 0 \). In Table 14.1 the relevant information about the D-branes is summarized. With a (\( \times \)) we indicate the directions along the world-volume of the D-brane. With a (\( \bullet \)) we indicate the directions normal to the D-brane. As a result, the coordinates \( t \) and \( x \) are common tangential directions. The coordinate \( y \) is a mixed direction: one brane extends on this direction, the other brane does not. Finally the \( z \)-coordinate is a common normal direction. More generally, for the Dp-, Dq-brane configuration, we have (with \( p > q \))

\[
\begin{align*}
\{x_0, x_1, \ldots, x_q\} & \quad \text{common tangential coordinates} \\
\{x_{q+1}, x_{q+2}, \ldots, x_p\} & \quad \text{mixed coordinates} \\
\{x_{p+1}, x_{p+2}, \ldots, x_d\} & \quad \text{common normal coordinates}
\end{align*}
\]

We have \((q + 1)\) common tangential coordinates – all the world-volume coordinates of the Dq-brane. We have \((p - q)\) directions that are tangential to the Dp-brane and normal to the Dq-brane. Finally, we have \((d - p)\) common normal directions.

We have already studied in some detail strings that begin and end on the same D-brane, so we focus here on the strings that go from one D-brane to the other. For definiteness, consider the strings that stretch from the Dp-brane to the Dq-brane. We already have partial knowledge about these strings. For example the common tangential coordinates are of NN type since they have Neumann boundary conditions on both endpoints. The common normal coordinates are of DD type, since they have Dirichlet boundary conditions on both endpoints. We had the opportunity to study these coordinates explicitly in the previous sections. There is, however, a new type of string coordinate arising for the directions that are tangential to one of the branes and normal to the other brane. In the case of our D2-, D1-brane example, this is the \( y \)-direction. For a string going from the D2-brane to the D1-brane, the string coordinate \( Y \) associated to \( y \) would be \( N \) at its beginning endpoint, since \( y \) is tangential to the D2-brane, and \( D \) at its final endpoint, since \( y \) is normal to the D1-brane. In short, for a string going from the D2-brane to the D1-brane, \( Y \) is an ND coordinate. For a string going from the D1-brane to the D2-brane, \( Y \) is a DN coordinate.

For the Dp-, Dq-brane configuration, the mixed coordinates were listed in (14.4.1). For open strings stretching from the Dp-brane to the Dq-brane, the
analogously labeled string coordinates satisfy a Neumann boundary condition on the starting $D_p$-brane, and a Dirichlet boundary condition on the ending $D_q$-brane. They are ND coordinates. In summary, the string coordinates split into

$$X^0, X^1, \ldots, X^q \quad X^{q+1}, X^{q+2}, \ldots, X^p \quad X^{p+1}, X^{p+2}, \ldots, X^d.$$  \hspace{1cm} (14.4.2)$$

In light-cone, we need three types of indices to label the string coordinates:

$$X^+, X^-, \{X^i\} \quad \{X^r\} \quad \{X^a\}.$$  \hspace{1cm} (14.4.3)$$

where

$$i = 2, \ldots, q, \quad r = q + 1, \ldots, p, \quad \text{and} \quad a = p + 1, \ldots, d.$$  \hspace{1cm} (14.4.4)$$

We think of the $D_p$-brane as the first brane, and the $D_q$-brane as the second brane. The position of the $D_p$-brane is specified by the coordinates $\bar{x}^a_1$. The position of the $D_q$-brane is specified by the coordinates $\bar{x}^r_2$ and $\bar{x}^a_2$. In our $D2$, $D1$-brane example of Figure 14.3, the role of $\bar{x}^r_2$ is played by the $y$-coordinate of the $D1$-brane. This coordinate can be set to zero by a suitable choice of axes.

Let us begin our analysis of the ND coordinates $X^r$. The boundary conditions are

$$\frac{\partial X^r}{\partial \sigma}(\tau, \sigma)\bigg|_{\sigma=0} = 0, \quad X^r(\tau, \sigma)\bigg|_{\sigma=\pi} = \bar{x}^r_2.$$  \hspace{1cm} (14.4.5)$$

The $\bar{x}^r_2$ coordinates, as mentioned above, could have been chosen to be equal to zero by a suitable choice of axes. As opposed to the coordinate differences $\bar{x}^a_1 - \bar{x}^a_2$ that define the separation between the D-branes, the $\bar{x}^r_2$ coordinates will not play any significant role. Consider now the usual expansion

$$X^r(\tau, \sigma) = \frac{1}{2} \left( f^r(\tau + \sigma) + g^r(\tau - \sigma) \right).$$  \hspace{1cm} (14.4.6)$$

The boundary condition at $\sigma = 0$ gives us

$$f^r(u) = g^r(u), \quad \Rightarrow \quad g^r(u) = f^r(u) + c_0^r.$$  \hspace{1cm} (14.4.7)$$
14.4. STRINGS BETWEEN DP-, AND DQ-BRANES

Bearing in mind that the second boundary condition will set $X^r$ equal to $\bar{x}^r_2$ at $\sigma = \pi$, we choose $c_0^r = 2\bar{x}^r_2$ so that

$$X^r(\tau, \sigma) = \bar{x}^r_2 + \frac{1}{2}f^r(\tau + \sigma) + f^r(\tau - \sigma). \quad (14.4.8)$$

The condition at $\sigma = \pi$ gives us

$$f^r(u + 2\pi) = -f^r(u). \quad (14.4.9)$$

The function $f^r(u)$ goes into minus itself when its argument increases by $2\pi$. To find an appropriate mode expansion, we first note that $f^r(u)$ is periodic with period $4\pi$. This is a necessary but not sufficient condition for (14.4.9) to hold. Any function with period $T$ can be expanded in terms of the basis functions

$$\left\{ \cos\left(\frac{2\pi nu}{T}\right), \sin\left(\frac{2\pi nu}{T}\right) \right\} \quad n = 0, 1, 2, \ldots, \infty. \quad (14.4.10)$$

For the case of interest $T = 4\pi$ and therefore we can write

$$f^r(u) = \sum_{n=0}^{\infty} \left[ f^r_n \cos\left(\frac{nu}{2}\right) + h^r_n \sin\left(\frac{nu}{2}\right) \right]. \quad (14.4.11)$$

We now impose the condition (14.4.9) of anti-periodicity. Indeed, computing $f^r(u + 2\pi)$:

$$f^r(u + 2\pi) = \sum_{n=0}^{\infty} \left[ f^r_n \cos\left(\frac{nu}{2} + n\pi\right) + h^r_n \sin\left(\frac{nu}{2} + n\pi\right) \right],$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[ f^r_n \cos\left(\frac{nu}{2}\right) + h^r_n \sin\left(\frac{nu}{2}\right) \right]. \quad (14.4.12)$$

For this right hand side to be precisely equal to the negative of the right hand side in (14.4.11) we need to restrict the sums to odd $n$. Therefore, we have

$$f^r(u) = \sum_{n \text{ odd}} \left[ f^r_n \cos\left(\frac{nu}{2}\right) + h^r_n \sin\left(\frac{nu}{2}\right) \right]. \quad (14.4.13)$$

Finally, substituting back into (14.4.8) and relabeling the expansion coefficients, we get

$$X^r(\tau, \sigma) = \bar{x}^r_2 + \sum_{n \text{ odd}} \left[ A^r_n \cos\left(\frac{n\tau}{2}\right) + B^r_n \sin\left(\frac{n\tau}{2}\right) \right] \cos\left(\frac{n\sigma}{2}\right). \quad (14.4.14)$$
This is our expansion of the ND coordinates.

To proceed with the quantization we define useful oscillators. Note that in all the examples considered thus far the $\alpha_n$ oscillator is matched with an exponential $e^{-in\tau}$. In the present case, this suggests that the oscillators must carry fractional mode labels! Another useful guidance is the desired simplicity of $\dot{X}^r \pm \dot{X}^r'$. With these pointers, we are led to write

$$X^r(\tau, \sigma) = \bar{x}_2^r + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}_{\text{odd}}} \frac{2}{n} \alpha_n^r e^{-i\frac{n}{2} \tau} \cos\left(\frac{n\sigma}{2}\right),$$  \hspace{1cm} (14.4.15)

where the sum runs over both positive and negative odd integers. The factor of $i$ in front of the sum is necessary so that the Hermiticity of $X^r$ imposes the standard Hermiticity property:

$$\left(\alpha_{\frac{r}{2}}^+\right) = \alpha_{-\frac{n}{2}}^-.$$  \hspace{1cm} (14.4.16)

Note that the $\bar{x}_2^r$ are constants and do not become operators. There are no zero modes in the expansion of $X^r$, and therefore ND coordinates carry no momentum. We also record the derivatives

$$\dot{X}^r \pm \dot{X}^r' = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}_{\text{odd}}} \alpha_n^r e^{-i\frac{n}{2} (\tau \pm \sigma)},$$  \hspace{1cm} (14.4.17)

which are indeed of the expected form. The (non-trivial) commutation relations take the form

$$[X^r(\tau, \sigma), \dot{X}^s(\tau, \sigma')] = i(2\pi\alpha')\delta(\sigma - \sigma')\delta^{rs}. \hspace{1cm} (14.4.18)$$

Since both the expansions (14.4.17) and the commutators (14.4.18) take standard form, we can use (12.2.11), which for the present case reads

$$\left[ (\dot{X}^r \pm \dot{X}'^r)(\tau, \sigma), (\dot{X}^s \pm \dot{X}'^s)(\tau, \sigma') \right] = 4\pi\alpha' i\eta^{rs} \frac{d}{d\sigma} \delta(\sigma - \sigma'). \hspace{1cm} (14.4.19)$$

Additionally, equation (12.2.14), with minor modifications also holds for $\sigma, \sigma' \in [0, 2\pi]$:

$$\sum_{m',n' \in \mathbb{Z}_{\text{odd}}} e^{-i\frac{m'}{2} (\tau + \sigma)} e^{-i\frac{n'}{2} (\tau + \sigma')} [\alpha_{m'}^r, \alpha_{n'}^s] = 2\pi i\eta^{rs} \frac{d}{d\sigma} \delta(\sigma - \sigma'). \hspace{1cm} (14.4.20)$$
14.4. STRINGS BETWEEN DP-, AND DQ-BRANES

To extract the commutators we apply the following integral operators to the left and to the right of the equation. The operations are

\[
\frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{i\frac{m}{2}\sigma} \cdot \frac{1}{2\pi} \int_0^{2\pi} d\sigma' e^{i\frac{n}{2}\sigma'},
\]

(14.4.21)

where \( m \) and \( n \) are odd integers. The set of functions \( e^{i\frac{k}{2}\sigma} \) with \( k \in \mathbb{Z}_{\text{odd}} \) are all orthogonal over the interval \([0, 2\pi]\). This happens because the sum or difference of two odd integers is an even integer. The integrations in (14.4.21) therefore select a single commutator, and one can show that

\[
\left[ \alpha^r_{\frac{m}{2}}, \alpha^s_{\frac{n}{2}} \right] = \frac{m}{2} \delta^{rs} \delta_{m+n,0}.
\]

(14.4.22)

This is the expected form of the commutation relations.

Quick Calculation 14.2. Prove equation (14.4.22).

Let’s now calculate the mass-squared operator. This operator receives contributions from all the coordinates in this sector: the NN-, the ND-, and the DD-coordinates. This is clear from (14.2.1) since the original light-cone index \( I \) now runs over \( i, r, \) and \( a \) labels. Given that the linear combination of derivatives (14.4.17) take the standard form, the contribution from the ND coordinates takes a familiar form. At the classical level can be read as a minor modification of equation (14.3.12), finding

\[
2p^+ p^- = \frac{1}{\alpha'} \left( \alpha' p^i p^i + \frac{1}{2} \alpha_0^a \alpha_0^a + \sum_{n=1}^{\infty} \left[ \alpha^i_{-n} \alpha^i_n + \alpha^a_{-n} \alpha^a_n \right] \\
+ \sum_{m \in \mathbb{Z}_{\text{odd}}^+} \alpha^r_{-\frac{m}{2}} \alpha^r_{\frac{m}{2}} - a \right).
\]

(14.4.23)

Here \( \mathbb{Z}_{\text{odd}}^+ \) denotes the odd positive integers. In writing this equation we have restored the subtraction constant \( a \) because an issue arises with the ordering of the oscillators. This issue can be settled using the heuristic arguments described in the discussion of (12.4.7) and of (12.4.15). Since all the oscillators for the NN and DD directions are integrally-moded, their normal-ordering constant is the same, and each of these coordinates contributes to \( a \) an amount

\[
\frac{1}{2} \left( 1 + 2 + 3 + 4 + \cdots \right) = \frac{1}{2} \left( -\frac{1}{12} \right) = -\frac{1}{24}.
\]

(14.4.24)
With a total of 24 transverse light-cone coordinates, if we only have NN and DD coordinates we get a normal ordering constant of \((-1\)). The ND coordinates, however, give a different subtraction. The sum that must be rearranged in this case is

\[
\frac{1}{2} \sum_{m \in \mathbb{Z}_{\text{odd}}} \alpha_{m}^{r} \alpha_{m}^{r} = \sum_{m \in \mathbb{Z}_{\text{odd}}}^{+} \alpha_{m}^{r} \alpha_{m}^{r} + \frac{1}{2} \sum_{m \in \mathbb{Z}_{\text{odd}}}^{+} \left[ \alpha_{m}^{r}, \alpha_{-m}^{r} \right] \tag{14.4.25}
\]

The first term in the right hand side is the one that appears in (14.4.23), and the second term is the ordering contribution. Since we have \((p - q)\) ND-coordinates, the ordering constant is

\[
\frac{1}{2} \sum_{m \in \mathbb{Z}_{\text{odd}}}^{+} \left[ \alpha_{m}^{r}, \alpha_{-m}^{r} \right] = \frac{1}{4} (p - q) \sum_{m \in \mathbb{Z}_{\text{odd}}}^{+} m, \tag{14.4.26}
\]

where we used (14.4.22). We need to calculate the sum of all odd integers. This is an infinite sum that can be given a concrete finite value using (14.4.24). This is done as follows:

\[
\sum_{n=1}^{\infty} n = \sum_{n \in \mathbb{Z}_{\text{odd}}} n + \sum_{n \in \mathbb{Z}_{\text{even}}} n = \sum_{n \in \mathbb{Z}_{\text{odd}}} n + 2 \sum_{n=1}^{\infty} n, \tag{14.4.27}
\]

and, as a result,

\[
\sum_{n \in \mathbb{Z}_{\text{odd}}} n = - \sum_{n=1}^{\infty} n = \frac{1}{12}. \tag{14.4.28}
\]

It follows from this result that the contribution (14.4.26) to the normal ordering constant from the ND coordinates is

\[
\frac{1}{4} (p - q) \sum_{m \in \mathbb{Z}_{\text{odd}}}^{+} m = \frac{1}{48} (p - q). \tag{14.4.29}
\]

This calculation has shown that each ND coordinate contributes \(+1/48\) to the normal ordering constant in \(\alpha'M^2\). A DN string coordinate will contribute exactly the same constant. In summary, the normal ordering contributions to \(\alpha'M^2\) for the various string coordinates are

\[
a_{NN} = a_{DD} = -\frac{1}{24}, \quad a_{ND} = a_{DN} = \frac{1}{48}. \tag{14.4.30}
\]
Returning to the problem at hand, the total ordering constant $a$ is given by (14.4.29) plus the contribution of the $(24 - (p - q))$ coordinates that are either NN or DD:

$$a = -\frac{1}{24}(24 - (p - q)) + \frac{1}{48}(p - q) = -1 + \frac{1}{16}(p - q).$$

With this information we can now write the expression for $M^2$. Following the same steps as in (14.3.13) we now find

$$M^2 = \left(\frac{x_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}\left(N_{\perp} - 1 + \frac{1}{16}(p - q)\right),$$

where

$$N_{\perp} = \sum_{n=1}^{\infty} \sum_{i=2}^{q} n a_n^{\dagger} a_i + \sum_{k \in \mathbb{Z}_{\text{odd}}}^{p} \frac{k}{2} a_k^{\dagger} a_k + \sum_{m=1}^{d} m a_m^{\dagger} a_m.$$ (14.4.33)

This formula incorporates all the effects we have discussed – a normal ordering constant that is shifted upwards, a contribution due to the stretched strings if the branes do not coincide, and a number operator that now includes contributions from NN, DD, and ND coordinates.

What is the state space, and what are the fields associated with the two lowest mass levels of the quantum open strings stretching between the branes? The ground states are labeled as

$$|p^+, \vec{p}; [12]\rangle, \quad \vec{p} = (p^2, \ldots, p^q).$$ (14.4.34)

The momentum labels of the states indicate that the quantum states give rise to fields that live in a $(q+1)$-dimensional spacetime. Roughly, they live on the world-volume of the D$q$-brane, the brane of lower dimensionality. The general rule is clear – the spacetime dimensionality for the fields arising in any given sector of the state space equals the number of NN string coordinates in the sector. The state space is built by letting the three types of oscillators $a_p^{\dagger}, a_{k/2}^{\dagger}$, and $a_m^{\dagger}$ – act on the vacuum states.

The ground states have $N_{\perp} = 0$ and correspond to a single scalar field on the D$q$-brane. This scalar is in general massive, but can be tachyonic or massless depending on the separation of the branes and the value of $p - q$. Assume, for simplicity, that the branes coincide. If additionally $p - q = 16$, the scalar would be massless. The next-level states are of the form

$$a_{\frac{k}{2}}^{\dagger} |p^+, \vec{p}; [12]\rangle, \quad N_{\perp} = 1/2.$$ (14.4.35)
These states give rise to \((p - q)\) of scalar fields, since the index \(r\) does not correspond to a world-volume direction on the D\(q\)-brane. All other states are necessarily massive since they have \(N^\perp \geq 1\), and this together with \(p > q\) implies that \(M^2 > 0\). In particular, we do not find massless gauge fields.
Problems

Problem 14.1. A Dp-brane with orientifolds.

This problem can be viewed as a sequel to Problem 13.6. We study here the effects of orientifolds on open strings.

The space-filling O25-plane truncates the spectrum down to the set of states invariant under a twist that acts on all string coordinates (see final comments in Problem 13.6). When we have a Dp-brane, the twist operator acts on the open string coordinates as follows

\[ \Omega X^a(\tau, \sigma) \Omega^{-1} = X^a(\tau, \pi - \sigma), \]
\[ \Omega X^i(\tau, \sigma) \Omega^{-1} = X^i(\tau, \pi - \sigma). \]

(1)

(2)

As usual, assume that \( \Omega x_0^- \Omega^{-1} = x_0^- \), and \( \Omega p^+ \Omega^{-1} = p^+ \).

(a) Give the action of the twist operator on the oscillators \( \alpha_n^a \) and \( \alpha_n^i \). What is the expected twist action on \( \alpha_n^- \)? Does it work out?

(b) Assume that the ground states \( |p^+, \vec{p}\rangle \) are twist invariant. Find the states of the theory for \( N^\perp \leq 2 \). As you will see, some massless states survive. Interpret these states along the lines of the discussion below (14.2.30).

Replace the O25-plane by an Op-plane coincident with the Dp-brane at \( \bar{x}^a = 0 \). Let \( \Omega_p \) denote the operator for which this theory keeps only the states with \( \Omega_p = +1 \).

(c) How should equations (1) and (2) change when \( \Omega \) is replaced by \( \Omega_p \)? Give the \( \Omega_p \) action on the oscillators \( \alpha_n^a \) and \( \alpha_n^i \).

(d) Describe the full spectrum of the theory as a simple truncation of the Dp-brane spectrum. You will find no massless scalars in this case. What does this suggest regarding possible motions of the Dp-brane?

Problem 14.2. String products and orientation reversing symmetries.

Equation (14.3.22) tells how open string sectors combine under interactions. The same product notation can be used for strings. By

\[ |A\rangle \ast |B\rangle, \]

we mean the string state that is obtained when a string in state \( |A\rangle \) interacts with a string in state \( |B\rangle \). The string product must obey the rule of sectors: the state
in (1) must belong to the sector $[A] * [B]$, where $[A]$ and $[B]$ denote the sectors
where string states $|A\rangle$ and $|B\rangle$ belong, respectively.

Use pictures of strings $A$ and $B$ to motivate the equations

$$\Omega(|A\rangle * |B\rangle) = (\Omega|B\rangle) * (\Omega|A\rangle),$$

(2)

$$\Omega_p(|A\rangle * |B\rangle) = (\Omega_p|B\rangle) * (\Omega_p|A\rangle).$$

(3)

Here $\Omega$ is string orientation reversal and $\Omega_p$ is orientifolding (orientation reversal
plus reflection about a set of coordinates).

**Problem 14.3.** $N$ coincident Dp-branes and orientifolds.

Let $N$ coincident Dp-branes coincide with an Op-plane, all of them located
at $\bar{x}^a = 0$. The orientifolding symmetry $\Omega_p$, as usual, includes reflection of the
coordinates normal to the orientifold and orientation reversal of strings. Assume
that the reflection of coordinates leaves each of the Dp-branes invariant (as opposed
to mapping them into each other).

(a) Explain why it is reasonable to postulate that

$$\Omega_p|p^+, \vec{p}; [ij]\rangle = |p^+, \vec{p}; [ji]\rangle.$$

What are the ground states of the theory? How many are there?

(b) Describe the full open string spectrum of the theory in terms of the spectrum
of a single Dp-brane. Check that for $N = 1$ you reproduce the result of
Problem 14.1, part (d).

**Problem 14.4.** Separated Dp-branes and an Op-plane.

We have learned that an orientifold acts as a kind of mirror. If we are to
have D-branes that do not coincide with an orientifold, there must be mirror D-
branes at the reflected points. Therefore, to analyze the theory of Dp-branes off
an orientifold Op-plane we begin with $N$ Dp-branes and $N$ mirror Dp-branes at
the reflected positions. We must then define the orientifold action on all the states
of the theory of $2N$ Dp-branes. Finally, we use this action to truncate down to
the invariant states, obtaining in this way the states of the orientifold theory.

Consider the situation illustrated in Figure 14.4, where we show the configu-
ration as seen in a plane spanned by two coordinates normal to the branes and
the orientifold. The $N$ Dp branes are labelled $1, 2, \ldots, N$, and the mirror images
are labeled $\bar{1}, \bar{2}, \ldots, \bar{N}$. Two strings are exhibited: one in the $[24]$ sector and the
other in the $[1\bar{1}]$ sector.
(a) Show the two strings obtained by the orientifold symmetry. Since the arguments $p^+, \vec{p}$ of the ground states are always present, let’s omit them for brevity. The ground states are of four types:

\[ |[ij]\rangle, \quad |[i\bar{j}]\rangle, \quad |[\bar{i}j]\rangle, \quad |[\bar{i}\bar{j}]\rangle. \tag{1} \]

Each class contains $N^2$ ground states since $i$ and $j$ run from 1 to $N$, and $\bar{i}$ and $\bar{j}$ run from $\bar{1}$ to $\bar{N}$. Define an expected action of $\Omega_p$ on the ground states in (1). Show that your choice satisfies $\Omega_p^2 = 1$ acting on the ground states.

(b) What are the possible interactions between strings in the four types of sectors built on the states (1)? Write your answers using the notation of (14.3.22).

(c) It is a fact about string interactions that the string product of ground states gives states that have a component along a ground state. Thus, for example,

\[ |[i\bar{j}]\rangle \ast |[\bar{j}k]\rangle = |[i\bar{k}]\rangle + \ldots. \tag{2} \]

Write the other possible ground state products. Test the consistency of your definition of $\Omega_p$ by acting with $\Omega_p$ on both sides of the equations giving ground state products. To act on products use equation (3) of Problem 14.2.

(d) Find the $\Omega_p$ action on the $\alpha^n_I$ oscillators using $\Omega_pX^I(\tau, \sigma)\Omega_p^{-1} = X^I(\tau, \pi - \sigma)$. Since the strings are stretched along or have nonzero values for the
CHAPTER 14. D-BRANES AND GAUGE FIELDS

$x^a$ coordinates an equation of the type $\Omega_p X^a(\tau, \sigma) \Omega_p^{-1} = -X^a(\tau, \pi - \sigma)$ cannot be fully implemented. Using (14.2.13) for $X^a$, for example, we would need $\Omega_p \bar{x}^a \Omega_p^{-1} = -\bar{x}^a$, which cannot hold since the $\bar{x}^a$ are numbers. A legal derivation of the $\Omega_p$ action on the $\alpha^n_a$ oscillators can be obtained by requiring $\Omega_p \dot{X}^a(\tau, \sigma) \Omega_p^{-1} = -\dot{X}^a(\tau, \pi - \sigma)$. Verify that for any arbitrary product $R$ of oscillators of both types

$$\Omega_p R \Omega_p^{-1} = (-1)^{N^\perp} R, \quad (3)$$

where $N^\perp$ is the total number of $R$.

(e) Describe the orientifold spectrum in terms of the spectrum of a single $D_p$-brane. For this consider an arbitrary product $R$ of oscillators and build the general states

$$\sum_{ij} \left( r_{ij} R[[ij]] + r_{i\bar{j}} R[[i\bar{j}]] + r_{\bar{i}j} R[[\bar{i}j]] + r_{\bar{i}\bar{j}} R[[\bar{i}\bar{j}]] \right), \quad (4)$$

where $r_{ij}, r_{i\bar{j}}, r_{\bar{i}j},$ and $r_{\bar{i}\bar{j}}$ are a set of four $N$ by $N$ matrices. Find the conditions that $\Omega_p$ invariance imposes on these matrices. There are two cases to consider, depending on the number $N^\perp$ of $R$. You should find that for $N^\perp$ odd there are $N(2N - 1)$ linearly independent states in (4). For $N^\perp$ even there are $N(2N + 1)$ linearly independent states.

Since the gauge fields arise from $N^\perp = 1$ states, there are $N(2N - 1)$ of them. This is the number of entries in a $2N$ by $2N$ antisymmetric matrix. The interacting theory of such gauge fields is an $SO(2N)$ Yang-Mills gauge theory ($SO$ stands for special orthogonal). The $SO(10)$ gauge theory, for example, can be used to build a grand unified theory of the strong and electroweak interactions.

**Problem 14.5. Separated $D_p$-branes and a different $O_p$-plane.**

The brane setup here is that of Problem 14.4. In part (a) of that problem, you defined a simple action of $\Omega_p$ on the ground states $[ij], [i\bar{j}], [\bar{i}j],$ and $[\bar{i}\bar{j}]$. Find an alternative $\Omega_p$ action where some of the relations have a minus sign: $\Omega_p [[\ldots]] = \pm [[\ldots]]$. Test its consistency by verifying that $\Omega_p^2 = 1$ on ground states, and that the products of ground states are compatible with $\Omega_p$ action, as you tested in part (c) of Problem 14.4. Determine the spectrum in this variant orientifold theory.

The gauge fields arise from $N^\perp = 1$, and you will find that there are $N(2N + 1)$ of them. The interacting theory of such gauge fields is an $USp(2N)$ Yang-Mills gauge theory ($USp$ stands for unitary symplectic).
Problem 14.6. *DN string coordinates.*

In section 14.4 we considered strings stretching from a Dp- to a Dp-brane, focusing on coordinates $X^r$ satisfying ND boundary conditions. Consider now strings stretching from the Dq- to the Dp-brane. For such strings, the coordinates $X^r$ are of DN type.

(a) Write the boundary conditions satisfied by the $X^r$ coordinates and use them to derive a mode expansion along the lines of our result (14.4.15) for a ND coordinate.

(b) Find also the equations that replace (14.4.17). Explain briefly why the mass-squared formula (14.4.32) needs no modification.

(c) If we let $\sigma \rightarrow \pi - \sigma$ in (14.4.15) we get automatically a function with DN boundary conditions. Compare with the mode expansion you found in (a), and explain why the Hermiticity properties are consistent.

Problem 14.7. *Strings in a configuration with a Dp-brane and a D25-brane.*

Consider the full state space of open string theory in a configuration with a Dp-brane and a D25-brane. Assume $1 \leq p \leq 24$. For each sector of the theory, give the $M^2$ operator, and examine explicitly the states arising in the two lowest levels indicating the types of fields they correspond to and where these fields live.


We study here a configuration of two D22-branes. One of them, henceforth called brane 1, is defined by $x^{25} = x^{23} = x^{22} = 0$. The other one henceforth called brane 2, is defined by $x^{24} = x^{23} = x^{22} = 0$.

Draw a picture of the brane configuration as it appears in the $(x^{24}, x^{25})$ plane. Construct a table, such as table 14.1, adding more columns to include all coordinates, and more rows to include all sectors. For each sector of the theory, give the $M^2$ operator and examine explicitly the states arising in the two lowest levels indicating the types of fields they correspond to and where these fields live.

A basic point: if $x^a$ is a Dirichlet direction for a configuration of two intersecting D-branes, the $x^a$ coordinates of the two D-branes must be the same. Explain why.