Chapter 11

The Relativistic Quantum Point Particle

To prepare ourselves for quantizing the string, we study the light-cone gauge quantization of the relativistic point particle. We set up the quantum theory by requiring that the Heisenberg operators satisfy the classical equations of motion. We show that the quantum states of the relativistic point particle coincide with the one-particle states of the quantum scalar field. Moreover, the Schrödinger equation for the particle wavefunctions coincides with the classical scalar field equations. Finally, we set up light-cone gauge Lorentz generators.

11.1 Light-cone point particle

In this section we study the classical relativistic point particle using the light-cone gauge. This is, in fact, a much easier task than the one we faced in Chapter 9, where we examined the classical relativistic string in the light-cone gauge. Our present discussion will allow us to face the complications of quantization in the simpler context of the particle. Many of the ideas needed to quantize the string are also needed to quantize the point particle.

The action for the relativistic point particle was studied in Chapter 5. Let's begin our analysis with the expression given in equation (5.2.4), where an arbitrary parameter τ is used to parameterize the motion of the particle:

$$S = -m \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \, d\tau \,. \tag{11.1.1}$$

In writing the above action, we have set $c = 1$. We will also set $\hbar = 1$ when appropriate. Finally, the time parameter τ will be dimensionless, just as it was for the relativistic string. We can simplify our notation by writing

$$\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \eta_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\mu = \dot{x}^2\,. \tag{11.1.2}$$

Thinking of τ as a time variable and of the $x^\mu(\tau)$ as coordinates, the action S defines a Lagrangian L as

$$S = \int_{\tau_i}^{\tau_f} L\; d\tau\,, \qquad L = -m\sqrt{-\dot{x}^2}\,. \tag{11.1.3}$$

As usual, the momentum is obtained by differentiating the Lagrangian with respect to the velocity:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}\,. \tag{11.1.4}$$

The Euler-Lagrange equations arising from L are

$$\frac{dp_\mu}{d\tau} = 0\,. \tag{11.1.5}$$

To define the light-cone gauge for the particle, we set the coordinate x^+ of the particle proportional to τ:

$$\boxed{\text{Light-cone gauge condition:} \quad x^+ = \frac{1}{m^2}\, p^+ \tau\,.} \tag{11.1.6}$$

The factor of m^2 on the right hand side is needed to get the units to work. Now consider the + component of equation (11.1.4):

$$p^+ = \frac{m}{\sqrt{-\dot{x}^2}}\; \dot{x}^+ = \frac{1}{\sqrt{-\dot{x}^2}}\; \frac{p^+}{m}\,. \tag{11.1.7}$$

Cancelling the common factor of p^+, and squaring, we find the constraint

$$\dot{x}^2 = -\,\frac{1}{m^2}\,. \tag{11.1.8}$$

This result helps us simplify the expression (11.1.4) for the momentum:

$$p_\mu = m^2\, \dot{x}_\mu\,. \tag{11.1.9}$$

The appearance of m^2, as opposed to m, is due to our choice of unit-less τ. The equation of motion (11.1.5) then gives

$$\ddot{x}_\mu = 0\,. \tag{11.1.10}$$

Using (11.1.9) we can rewrite the constraint equation (11.1.8) as

$$p^2 + m^2 = 0\,. \tag{11.1.11}$$

Expanding in light-cone components,

$$-2p^+p^- + p^Ip^I + m^2 = 0 \quad \longrightarrow \quad p^- = \frac{1}{2p^+}(p^Ip^I + m^2)\,. \tag{11.1.12}$$

Having solved for p^-, equation (11.1.9) gives

$$\frac{dx^-}{d\tau} = \frac{1}{m^2}\,p^-\,, \tag{11.1.13}$$

which is integrated to find

$$x^-(\tau) = x_0^- + \frac{p^-}{m^2}\,\tau\,, \tag{11.1.14}$$

where x_0^- is a constant of integration. Equation (11.1.9) also gives $dx^I/d\tau = p^I/m^2$, which is integrated to give

$$x^I(\tau) = x_0^I + \frac{p^I}{m^2}\,\tau\,, \tag{11.1.15}$$

where x_0^I is a constant of integration. Note that the light-cone gauge condition (11.1.6) implies that $x^+(\tau)$ has no constant piece x_0^+.

The specification of the motion of the point particle is now complete. Equation (11.1.12) tells us that the momentum is completely determined once we fix p^+ and the components p^I of the transverse momentum $\vec{p}_T$. The motion in the x^- direction is determined by (11.1.14), once we fix the value of x_0^-. The transverse motion is determined by the $x^I(\tau)$, or the x_0^I, since we presume to know the p^I. For a symmetric treatment of coordinates versus momenta in the quantum theory, we choose the x^I as dynamical variables. Our independent dynamical variables for the point particle are therefore

$$\text{Dynamical variables:} \quad \Big(\,x^I, \quad x_0^-\,, \quad p^I\,, \quad p^+\Big)\,. \tag{11.1.16}$$

11.2 Heisenberg and Schrödinger pictures

Traditionally, there are two main approaches to the understanding of time evolution in quantum mechanics. In the Schrödinger picture, the state of a system evolves in time, while operators remain unchanged. In the Heisenberg picture, it is the operators which evolve in time, while the state remains unchanged. Of the two, the Heisenberg picture is more closely related to classical mechanics, where the dynamical variables (which become operators in quantum mechanics) evolve in time. Both the Schrödinger and the Heisenberg pictures will be useful in developing the quantum theories of the relativistic point particle and the relativistic string. Because we would like to exploit our understanding of classical dynamics in developing the quantum theories, we will begin by focusing on the Heisenberg picture.

Both the Heisenberg and the Schrödinger picture make use of the same state space. Whereas in the Heisenberg picture the state representing a particular physical system is fixed in time, in the Schrödinger picture the state of a system is constantly changing direction in the state space in a manner which is determined by the Schrödinger equation. Although we generally think of the operators in the Schrödinger picture as being time-independent, there are those which depend *explicitly* on time and therefore have time dependence. These operators are formed from time-independent operators and the variable t. For example, the position and momentum operators q and p are time-independent. But the operator $\mathcal{O} = q + p\,t$ has an explicit time dependence. If it has explicit time dependence, even the Hamiltonian $H(p, q; t)$ can be a time-dependent operator.

Now, as we move from the Schrödinger to the Heisenberg picture, we will encounter operators with two types of time dependence. As we noted earlier, Heisenberg operators have time dependence, but this time dependence can be both *implicit* and *explicit*. The Heisenberg-equivalent of a time-independent Schrödinger operator is said to have implicit time dependence. This implicit time dependence is due to our folding into the operators the time dependence which, in the Schrödinger picture, is present in the state. If a Heisenberg operator is explicitly time dependent it is because the explicit time dependence of the corresponding Schrödinger operator has been carried over.

For example, when we pass from the Schrödinger to the Heisenberg picture, the time-independent Schrödinger operators q and p become $q(t)$ and $p(t)$, respectively. The Schrödinger commutator $[q, p] = i$ turns into the commutator

$$[\,q(t)\,,\,p(t)\,] = i\,. \tag{11.2.1}$$

Although $q(t)$ and $p(t)$ depend on time, their time dependence is implicit. If $\xi(t)$ is a Heisenberg operator arising from a time-independent Schrödinger operator, the time evolution of $\xi(t)$ is governed by

$$i\frac{d\xi(t)}{dt} = \Big[\,\xi(t)\,,\,H(p(t),q(t);t)\Big]\,. \tag{11.2.2}$$

Here $H(p(t),q(t);t)$ is the Heisenberg Hamiltonian corresponding to the possibly time-dependent Schrödinger Hamiltonian $H(p,q;t)$.

If $\mathcal{O}(t)$ is the Heisenberg operator which corresponds to an explicitly time-dependent Schrödinger operator, then the time evolution of $\mathcal{O}(t)$ is given by

$$i\,\frac{d\mathcal{O}(t)}{dt} = i\,\frac{\partial\mathcal{O}}{\partial t} + \Big[\mathcal{O}(t)\,,\,H(p(t),q(t);t)\Big]\,. \tag{11.2.3}$$

This equation reduces to (11.2.2) when the operator has no explicit time dependence. If the Hamiltonian $H(p(t),q(t))$ has no explicit time dependence, then we can use (11.2.2) with $\xi = H$, to find

$$\frac{d}{dt}H(p(t),q(t)) = 0\,. \tag{11.2.4}$$

In this case the Hamiltonian is a constant of the motion.

The discussion above is easily made very explicit when the Schrödinger Hamiltonian $H(p,q)$ is time independent. In this case a state $|\Psi\rangle$ at time $t=0$, evolves in time becoming, at time t,

$$|\Psi,t\rangle = e^{-iHt}|\Psi\rangle\,. \tag{11.2.5}$$

Quick Calculation 11.1. Confirm that $|\Psi,t\rangle$ satisfies the Schrödinger equation

$$i\frac{d}{dt}|\Psi,t\rangle = H|\Psi,t\rangle\,. \tag{11.2.6}$$

It is clear from (11.2.5) that the operator e^{iHt} brings time dependent states to rest:

$$e^{iHt}|\Psi,t\rangle = |\Psi\rangle\,. \tag{11.2.7}$$

If we act with this operator on the product $\alpha|\Psi, t\rangle$, where α is a Schrödinger operator, we find

$$e^{iHt}\alpha|\Psi, t\rangle = e^{iHt}\,\alpha\, e^{-iHt}|\Psi\rangle \equiv \alpha(t)|\Psi\rangle\,, \tag{11.2.8}$$

where $\alpha(t) = e^{iHt}\alpha\, e^{-iHt}$ is the Heisenberg operator corresponding to the Schrödinger operator α. This definition applies both if α has or does not have explicit time dependence. This simple relation ensures that if a set of Schrödinger operators satisfy certain commutation relations, the corresponding Heisenberg operators satisfy the same commutation relations:

Quick Calculation 11.2. If $[\alpha_1, \alpha_2] = \alpha_3$ holds for Schrödinger operators α_1, α_2, and α_3, show that $[\alpha_1(t), \alpha_2(t)] = \alpha_3(t)$ holds for the corresponding Heisenberg operators.

This result holds even if the Hamiltonian is time dependent (see Problem 11.2). It justifies the commutator in (11.2.1), noting that the constant right-hand side is not affected by the rule turning a Schrödinger operator into a Heisenberg operator.

11.3 Quantization of the point particle

We now develop a quantum theory from the classical theory of the relativistic point particle. We will define the relevant Schrödinger and Heisenberg operators, including the Hamiltonian, and describe the state space. All of this will be done in the light-cone gauge.

Our first step is to choose a set of time-independent Schrödinger operators. A reasonable choice is provided by the dynamical variables in (11.1.16):

$$\boxed{\text{Time-independent Schrödinger ops.}: \Big(x^I, \quad x_0^-\,, \quad p^I\,, \quad p^+\Big)\,.} \tag{11.3.1}$$

We could include hats to distinguish the operators from their eigenvalues, but this will not be necessary in most cases. We parameterize the trajectory of a point particle using τ, so the associated Heisenberg operators are:

$$\boxed{\text{Heisenberg ops.}: \Big(x^I(\tau), \quad x_0^-(\tau)\,, \quad p^I(\tau)\,, \quad p^+(\tau)\Big)\,.} \tag{11.3.2}$$

We postulate the following commutation relations for the Schrödinger operators:

$$\boxed{[\,x^I, p^J] = i\,\eta^{IJ}\,, \quad [\,x_0^-, p^+] = i\,\eta^{-+} = -i\,,} \tag{11.3.3}$$

with all other commutators set equal to zero. The first commutator is the familiar commutator of spatial coordinates with the corresponding spatial momenta (recall that $\eta^{IJ} = \delta^{IJ}$). The second commutator is well motivated, after all, x_0^- is treated as a spatial coordinate in the light-cone, and p^+ is the corresponding conjugate momentum. The second commutator, just as the first one, has an η carrying the indices of the coordinate and the momentum.

The Heisenberg operators, as explained earlier, satisfy the same commutation relations as the Schrödinger operators:

$$[\,x^I(\tau), p^J(\tau)] = i\,\eta^{IJ}\,, \quad [\,x_0^-(\tau), p^+(\tau)] = -i\,, \tag{11.3.4}$$

with all other commutators set equal to zero.

We have discussed the operators that correspond to the independent observables of the classical theory. But just as there are classical observables which depend on those independent ones, there are also quantum operators which are constructed from the set of independent Schrödinger operators, and time. These additional operators are $x^+(\tau), x^-(\tau)$ and p^-. The definitions of these operators are postulated to be the quantum analogues of equations (11.1.6), (11.1.14), and (11.1.12). These give us the operator equations

$$x^+(\tau) \equiv \frac{p^+}{m^2}\,\tau\,, \tag{11.3.5}$$

$$x^-(\tau) \equiv x_0^- + \frac{p^-}{m^2}\,\tau\,, \tag{11.3.6}$$

$$p^- \equiv \frac{1}{2p^+}\Big(p^I p^I + m^2\Big)\,. \tag{11.3.7}$$

Note that p^- is time-independent. Both $x^+(\tau)$ and $x^-(\tau)$ are time-dependent Schrödinger operators.

The commutation relations involving the operators $x^+(\tau), x^-(\tau)$ and p^- are determined by the postulated commutation relations in (11.3.3), along with the defining equations (11.3.5) – (11.3.7). The decision to choose the operators in (11.3.1) as the independent operators of our quantum theory was very significant. For example, if we had chosen x^+ and p^- to be independent operators, we might have been led to write a commutation relation

$[x^+, p^-] = -i$. In our present framework, however, this quantity vanishes, since $[\,p^+, p^I] = 0$.

We have not yet determined the Hamiltonian H. Since p^- is the light-cone energy (see (2.5.11)), we expect it to generate x^+ evolution:

$$\frac{\partial}{\partial x^+} \quad \longleftrightarrow \quad p^- \,. \tag{11.3.8}$$

Although x^+ is light-cone time, we are parameterizing our operators with τ, so we expect H to generate τ evolution, which is related, but is not the same as x^+ evolution. Since $x^+ = p^+\tau/m^2$, we can anticipate that τ evolution will be generated by

$$\frac{\partial}{\partial \tau} = \frac{p^+}{m^2}\frac{\partial}{\partial x^+} \quad \longleftrightarrow \quad \frac{p^+}{m^2}\, p^- \,, \tag{11.3.9}$$

We therefore postulate the Heisenberg Hamiltonian

$$\boxed{H(\tau) = \frac{p^+(\tau)}{m^2}\; p^-(\tau) = \frac{1}{2m^2}\Big(p^I(\tau)p^I(\tau) + m^2\Big)\,.} \tag{11.3.10}$$

Note that $H(\tau)$ has no explicit time dependence. Equation (11.2.4) applies, and as a result, the Hamiltonian is actually time independent.

Let's now make sure that this Hamiltonian generates the expected equations of motion. First we check that H gives the correct time evolution of the Heisenberg operators (11.3.2) which arise from the time-independent Schrödinger operators. The equation governing the time evolution of those operators is (11.2.2). Let us begin with p^+ and p^I:

$$\begin{aligned} i\,\frac{dp^+(\tau)}{d\tau} &= \big[\,p^+(\tau)\,, H(\tau)\,\big] = 0\,, \\ i\,\frac{dp^I(\tau)}{d\tau} &= \big[\,p^I(\tau)\,, H(\tau)\big] = 0\,. \end{aligned} \tag{11.3.11}$$

Both of these commutators vanish because H is a function of $p^I(\tau)$ alone, and all the momenta commute. Equations (11.3.11) are good news, because the classical momenta p^+ and p^I are constants of the motion. This allows us to write $p^I(\tau) = p^I$ and $p^+(\tau) = p^+$. We now test the τ-development of the Heisenberg operator $x^I(\tau)$:

$$i\,\frac{dx^I(\tau)}{d\tau} = \Big[\,x^I(\tau)\,,\; \frac{1}{2m^2}\Big(p^Jp^J + m^2\Big)\Big] = i\,\frac{p^I}{m^2}\,. \tag{11.3.12}$$

Here, we have used $[x^I, p^J p^J] = [x^I, p^J]p^J + p^J[x^I, p^J] = 2i\, p^I$. Cancelling the common factor in (11.3.12) we find

$$\frac{dx^I(\tau)}{d\tau} = \frac{p^I}{m^2}\,. \tag{11.3.13}$$

This result is in accord with our classical expectations and allows us to write

$$x^I(\tau) = x_0^I + \frac{p^I}{m^2}\,\tau\,, \tag{11.3.14}$$

where x_0^I is an operator without any time dependence. Finally, we must examine $x_0^-(\tau)$. Since $x_0^-(\tau)$ commutes with $p^I(\tau)$,

$$i\,\frac{dx_0^-(\tau)}{d\tau} = \Big[\, x_0^-(\tau)\,,\, \frac{1}{2m^2}\Big(p^I p^I + m^2\Big)\Big] = 0\,. \tag{11.3.15}$$

As expected, this operator is a constant of the motion, and we can write $x_0^-(\tau) = x_0^-$. So as far as the operators in (11.3.1) are concerned, our ansatz for H functions properly as a Hamiltonian.

We now turn to the remaining operators $x^+(\tau)$, $x^-(\tau)$, and $p^-(\tau)$. Of these, $p^-(\tau)$ is a function of the p^I only and is therefore time independent. It is easy to to see that the commutator with H vanishes, so we have nothing left to check for this operator. The Heisenberg operators $x^+(\tau)$ and $x^-(\tau)$ both arise from Schrödinger operators with explicit time dependence, so we use (11.2.3) to calculate their time evolution. For example:

$$i\frac{dx^-(\tau)}{d\tau} = i\frac{\partial x^-}{\partial \tau} + \big[\, x^-(\tau)\,,\, H(\tau)\,\big]\,. \tag{11.3.16}$$

Since $x^-(\tau) \equiv x_0^- + p^-\tau/m^2$ and both x_0^- and p^- commute with the p^I, we see that $[x^-(\tau), H(\tau)] = 0$. Consequently,

$$\frac{dx^-(\tau)}{d\tau} = \frac{p^-}{m^2}\,, \tag{11.3.17}$$

which is the expected result. Similarly, since $x^+(\tau) = p^+\tau/m^2$, we find that $[x^+(\tau), H(\tau)] = 0$, and therefore

$$\frac{dx^+(\tau)}{d\tau} = \frac{\partial x^+}{\partial \tau} = \frac{p^+}{m^2}\,. \tag{11.3.18}$$

These computations show that our ansatz (11.3.10) for the Hamiltonian generates the expected equations of operator evolution.

Quick Calculation 11.3. We introduced x_0^I in (11.3.14) as a constant operator. Show that $dx_0^I/d\tau$ must be calculated by viewing x_0^I as an explicitly time-dependent Heisenberg operator defined by (11.3.14).

Our final step in constructing the quantum theory of the point particle is to develop the state space. The states are labeled by the eigenvalues of a maximal set of commuting operators. For the set of operators we have introduced in (11.3.1), a maximal commuting subset can include only one element from the pair (x^-, p^+), and one element from each of the pairs (x^I, p^I). Because it is convenient to work in momentum space, we will work with the operators p^+ and p^I. So we write the states as

$$\boxed{\text{States of the quantum point particle:} \quad |p^+, \vec{p}_T\rangle\,,} \tag{11.3.19}$$

where p^+ is the eigenvalue of the p^+ operator, and $\vec{p}_T$ is the transverse momentum, the components of which are the eigenvalues of the p^I operators:

$$\widehat{p}^{\,+}|p^+, \vec{p}_T\rangle = p^+|p^+, \vec{p}_T\rangle\,, \quad \widehat{p}^{\,I}|p^+, \vec{p}_T\rangle = p^I|p^+, \vec{p}_T\rangle\,. \tag{11.3.20}$$

In light of (11.3.7), these equations imply

$$\widehat{p}^{\,-}|p^+, \vec{p}_T\rangle = \frac{1}{2p^+}\Big(p^I p^I + m^2\Big)|p^+, \vec{p}_T\rangle\,. \tag{11.3.21}$$

To write a Schrödinger equation for the point particle, we consider time-dependent states. These are formed as time-dependent superpositions of the basis states in (11.3.19):

$$|\Psi, \tau\rangle = \int dp^+ d\vec{p}_T\, \psi(\tau, p^+, \vec{p}_T)|p^+, \vec{p}_T\rangle\,. \tag{11.3.22}$$

Since p^+ and $\vec{p}_T$ are continuous variables, an integral is necessary. To produce a general τ-dependent superposition, we introduced the arbitrary function $\psi(\tau, p^+, \vec{p}_T)$. In fact, this function is the momentum-space wavefunction associated to the state $|\Psi, \tau\rangle$. Indeed, with dual bras $\langle p^+, \vec{p}_T|$ defined to satisfy

$$\langle p^{+\prime}, \vec{p}_T{}'|\, p^+, \vec{p}_T\rangle = \delta(p^{+\prime} - p^+)\, \vec{\delta}\,(\vec{p}_T{}' - \vec{p}_T)\,, \tag{11.3.23}$$

we see that

$$\langle p^+, \vec{p}_T|\, \Psi, \tau\rangle = \psi(\tau, p^+, \vec{p}_T)\,. \tag{11.3.24}$$

The Schrödinger equation for the state $|\Psi, \tau\rangle$ is

$$i\frac{\partial}{\partial\tau}|\Psi, \tau\rangle = H|\Psi, \tau\rangle\,. \tag{11.3.25}$$

Using the state in (11.3.22) and the Hamiltonian in (11.3.10), we find

$$\int dp^+ d\vec{p}_T\,\Big[i\frac{\partial}{\partial\tau}\psi(\tau, p^+, \vec{p}_T) - \frac{1}{2m^2}\Big(p^I p^I + m^2\Big)\psi(\tau, p^+, \vec{p}_T)\Big]\,|p^+, \vec{p}_T\rangle = 0\,. \tag{11.3.26}$$

Since the basis vectors $|p^+, \vec{p}_T\rangle$ are all linearly independent, the expression within brackets must vanish for all values of the momenta:

$$i\frac{\partial}{\partial\tau}\,\psi\,(\tau, p^+, \vec{p}_T) = \frac{1}{2m^2}\Big(p^I p^I + m^2\Big)\psi(\tau, p^+, \vec{p}_T)\,. \tag{11.3.27}$$

We recognize this equation as a Schrödinger equation for the momentum-space wavefunction $\psi(\tau, p^+, \vec{p}_T)$. We have thus developed a theory of the quantum point particle.

11.4 Quantum particle and scalar particles

The states of the quantum point particle given in (11.3.19) may remind you of the one-particle states (10.4.34) in the quantum theory of the scalar field. This is actually a fundamental correspondence:

There is a natural identification of the quantum states of a relativistic point particle of mass m with the one-particle states of the quantum theory of a scalar field of mass m:

$$\boxed{\;|p^+,\, \vec{p}_T\rangle \quad\longleftrightarrow\quad a^\dagger_{p^+,\, p_T}|\Omega\rangle\,.\;} \tag{11.4.1}$$

The identification is possible because the labels of the point particle states correspond to the labels of the creation operators which generate the one-particle states of the scalar quantum field theory. The correspondence between the quantum point particle and the quantum scalar field theory can be extended from the state space to the operators that act on the state spaces. The quantum point particle theory has operators p^+, p^I, and p^-, and so does the quantum field theory, as shown in (10.4.35). If we identify the state spaces using (11.4.1), then the two sets of operators give the same eigenvalues. This makes the identification natural.

The above observations lead us to conclude that the states of the quantum point particle and the one-particle states of the scalar field theory are indistinguishable. Because it contains creation operators that can act multiple times on the vacuum state, the scalar field theory has multiparticle states that did not arise in our quantization of the point particle. Indeed, there are no creation operators in the theory of the quantum point particle. Because it provides a natural description of multi-particle states, the scalar field theory can be said to be a more complete theory.

How could we have anticipated that the one-particle states of a quantum *scalar field theory* would match those of the quantum point particle? The answer is quite interesting: the Schrödinger equation for the quantum point particle wavefunctions has the form of the classical field equation for the scalar field. More precisely:

There is a canonical correspondence between the quantum point particle wavefunctions and the classical scalar field, such that the Schrödinger equation for the quantum point particle wavefunctions, becomes the classical field equation for the scalar field.

One element of this correspondence is the classical field equation for the scalar field. In light-cone gauge, this equation takes the form (10.3.16):

$$\Big(i\,\frac{\partial}{\partial\tau} - \frac{1}{2m^2}(p^Ip^I + m^2)\Big)\,\phi(\tau, p^+, \vec{p}_T) = 0\,. \tag{11.4.2}$$

This differential equation is first order in τ. The other element in the correspondence is this Schrödinger equation (11.3.27). The two equations are identical once we identify the wavefunction $\psi(\tau, p^+, \vec{p}_T)$ and the scalar field $\phi(\tau, p^+, \vec{p}_T)$:

$$\boxed{\psi(\tau, p^+, \vec{p}_T) \quad \longleftrightarrow \quad \phi(\tau, p^+, \vec{p}_T)\,.} \tag{11.4.3}$$

This is the claimed correspondence.

The quantization of the point particle is an example of *first quantization*. In first quantization, the coordinates and momenta of classical mechanics are turned into quantum operators and a state space is constructed. Generically, the result is a set of one-particle states. *Second quantization* refers to the quantization of a classical field theory, the result of which is a quantum field theory with field operators and multi-particle states. Our analysis allows us to see how second quantization follows after first quantization. A first-quantization of the classical point-particle mechanics gives one-particle states. We then re-interpret the Schrödinger equation for the associated wavefunctions as the classical field equation for a scalar field. A second-quantization, this time of the classical field theory, gives us the set of multi-particle states.

So far we have only quantized the *free* relativistic point particle. All quantum states, including the multi-particle ones obtained by second quantization, represent free particles. How do we get interactions between the particles? Such processes are included in the scalar field theory by adding interaction terms to the action. So far, all the terms that we have included are quadratic in the fields. The interaction terms include three or more fields. Since the quantum point particle state space does not include multi-particle states, the description of interactions in the language of first quantization is not straightforward. On the other hand, in the framework of quantum field theory interactions are dealt with very naturally.

11.5 Light-cone momentum generators

Since the point particle Lagrangian L in (11.1.3) depends only on τ-derivatives of the coordinates, it is invariant under the translations

$$\delta x^\mu(\tau) = \epsilon^\mu \,, \tag{11.5.1}$$

with ϵ^μ constant. The conserved charge associated to this symmetry transformation is the momentum p_μ of the particle. This follows from (8.2.9) and (11.1.4).

What happens to conserved charges in the quantum theory? They become quantum operators with a remarkable property: they generate, via commutation, a quantum version of the symmetry transformation that gave rise to them classically!

This property is most apparent if we use a framework where the manifest Lorentz invariance of the classical theory is preserved in the quantization. This is *not* the framework we have used to quantize the point particle. In light-cone gauge quantization, the x^0 and x^1 coordinates of the particle are afforded special treatment. This hides the Lorentz invariance of the theory from plain view. We will not discuss fully the Lorentz covariant quantization of the point particle. A few remarks will suffice for our present purposes. The covariant quantization of the string is discussed in some detail in Chapter 21.

In the Lorentz covariant quantization of the point particle, we have Heisenberg operators $x^\mu(\tau)$ and $p^\mu(\tau)$. Note that even the time coordinate $x^0(\tau)$ becomes an operator! The commutation relations are

$$[\, x^\mu(\tau), p^\nu(\tau)] = i\, \eta^{\mu\nu}\,, \tag{11.5.2}$$

as well as

$$[\, x^\mu(\tau), x^\nu(\tau)] = 0 \quad \text{and} \quad [\, p^\mu(\tau), p^\nu(\tau)] = 0\,. \tag{11.5.3}$$

Equation (11.5.2) is reasonable. The indices match, which ensures consistency with Lorentz covariance. Moreover, when μ and ν take spatial values, the commutation relations are the familiar ones. We already know that (11.5.2) is not consistent with the light-cone gauge commutators of section 11.3. We saw there that $[x^+(\tau), p^-(\tau)] = 0$, while (11.5.2) would predict a nonzero result. An equality of two objects carrying Lorentz indices can be used letting the indices run over the light-cone values $+, -$, and I. The equation $R^{\mu\nu} = S^{\mu\nu}$ gives, for example, $R^{+-} = S^{+-}$ (see Problem 10.3). As a result, equation (11.5.2), indeed gives $[x^+(\tau), p^-(\tau)] = i\eta^{+-} = -i$.

Let us now check that the operator $p^\mu(\tau)$ generates translations. More precisely, we check that $i\epsilon_\rho\, p^\rho(\tau)$ generates the translation (11.5.1):

$$\delta x^\mu(\tau) = [\, i\epsilon_\rho\, p^\rho(\tau)\,, x^\mu(\tau)\,] = i\epsilon_\rho\, (-i\eta^{\rho\mu}) = \epsilon^\mu\,. \tag{11.5.4}$$

This is an elegant result, but it is by no means clear that it carries over to our light-cone gauge quantization. We must find out if the light-cone gauge momentum operators generate translations.

For this purpose, we expand the generator $i\epsilon_\rho\, p^\rho(\tau)$ in light-cone components:

$$i\epsilon_\rho\, p^\rho(\tau) = -i\epsilon^-\, p^+ - i\epsilon^+\, p^- + i\epsilon^I\, p^I\,. \tag{11.5.5}$$

We have dropped the τ arguments from the momenta because they are τ-independent. Note that in here, p^- is given by (11.3.7). Let's test (11.5.4) with $\epsilon^I \neq 0$, and $\epsilon^+ = \epsilon^- = 0$:

$$\delta x^\mu(\tau) = \, i\epsilon^I[\, p^I\,, x^\mu(\tau)\,]\,. \tag{11.5.6}$$

We would expect that $\delta x^J(\tau) = \epsilon^J$ and that $\delta x^+(\tau) = \delta x^-(\tau) = 0$. All these expectations are realized. Choosing $\mu = J$, and using the commutator

(11.3.4) we find $\delta x^J(\tau) = \epsilon^J$. To compute the action on $x^+(\tau)$ and $x^-(\tau)$, we must use their definitions:

$$x^+(\tau) = \frac{p^+}{m^2}\tau\,, \qquad x^-(\tau) = x_0^- + \frac{p^-}{m^2}\tau\,. \tag{11.5.7}$$

Recalling that p^I commutes with all momenta and with x_0^-, we confirm that $\delta x^+(\tau) = \delta x^-(\tau) = 0$.

Quick Calculation 11.4. Test (11.5.4) with $\epsilon^- \neq 0$ and $\epsilon^+ = \epsilon^I = 0$. To do this compute $\delta x^\mu(\tau) = -i\epsilon^-[\,p^+\,, x^\mu(\tau)\,]$. Confirm that $\delta x^-(\tau) = \epsilon^-$ and that all other coordinates are not changed.

It remains to see if p^- generates the expected translations. Since p^- is a nontrivial function of other momenta, there is some scope for complications! This time we consider the transformations that are generated using (11.5.4) with $\epsilon^+ \neq 0$ and $\epsilon^- = \epsilon^I = 0$:

$$\delta x^\mu(\tau) = \,-i\epsilon^+[\,p^-\,, x^\mu(\tau)\,]\,. \tag{11.5.8}$$

The naive expectation $\delta x^+(\tau) = \epsilon^+$ is not realized: choosing $\mu = +$ and using (11.5.7) we see that

$$\delta x^+(\tau) = -i\epsilon^+\Big[\,p^-\,, p^+\frac{\tau}{m^2}\Big] = 0\,. \tag{11.5.9}$$

Not only is $x^+(\tau)$ left unchanged, but the other components, which naively should be left unchanged, are not:

$$\delta x^I(\tau) = -i\epsilon^+\Big[\,p^-, x^I(\tau)\Big] = -i\epsilon^+\frac{1}{2p^+}\,(-2ip^I)\; = -\epsilon^+\frac{p^I}{p^+}\,, \tag{11.5.10}$$

$$\delta x^-(\tau) = -i\epsilon^+\Big[\,p^-, x_0^- + \frac{p^-}{m^2}\tau\Big] = -i\epsilon^+[\,p^-, x_0^-] = -\epsilon^+\frac{p^-}{p^+}\,. \tag{11.5.11}$$

In these calculations only one step requires some explanation. How do we find $[\,p^-, x_0^-]$? The only reason p^- does not commute with x_0^- is that p^- depends on p^+. In fact, what we need to know is the commutator $[x_0^-, 1/p^+]$. This can be done as follows:

$$\begin{aligned}\Big[\,x_0^-\,,\,\frac{1}{p^+}\Big] &= x_0^-\,\frac{1}{p^+} - \frac{1}{p^+}x_0^- = \frac{1}{p^+}\,p^+x_0^-\frac{1}{p^+} - \frac{1}{p^+}x_0^-p^+\frac{1}{p^+}\\ &= \frac{1}{p^+}\;[\,p^+\,, x_0^-\,]\,\frac{1}{p^+} = \frac{i}{{p^+}^2}\,.\end{aligned} \tag{11.5.12}$$

Quick Calculation 11.5. Use (11.5.12) to show that

$$[\, x_0^- \,,\, p^- \,] = i\,\frac{p^-}{p^+}\,. \tag{11.5.13}$$

Equations (11.5.9), (11.5.10), and (11.5.11) show that p^- does not generate the expected transformations. What happened? It turns out that p^- actually generates both a translation and a reparameterization of the world-line of the particle. We know that the particle action is invariant under changes of parameterization $\tau \to \tau'(\tau)$. When we described symmetries in Chapter 8, however, we exhibited them as changes in the dynamical variables of the system. A change in parameterization can also be described in that way. Writing $\tau \to \tau' = \tau + \lambda(\tau)$, with λ infinitesimal, we note that the plausible change

$$x^\mu(\tau) \;\to\; x^\mu(\tau + \lambda(\tau)) = x^\mu(\tau) + \lambda(\tau)\partial_\tau x^\mu(\tau)\,, \tag{11.5.14}$$

leads us to write

$$\delta x^\mu(\tau) = \lambda(\tau)\partial_\tau x^\mu(\tau)\,. \tag{11.5.15}$$

We claim that these are *symmetries* of the point particle theory. Actually, the variation (11.5.15) does not leave the point particle Lagrangian invariant. The Lagrangian changes into a total τ-derivative (Problem 11.4), and this, in fact, suffices to have a symmetry (Problem 8.5).

Let's now show that p^- generates a translation plus a reparameterization. The expected translation was $\delta x^+ = \epsilon^+$. On the other hand, from (11.5.15), a reparameterization of x^+ gives $\delta x^+ = \lambda\partial_\tau x^+$. Bearing in mind (11.5.9), the expected translation plus the reparameterization give zero variation, so,

$$0 = \epsilon^+ + \lambda\partial_\tau x^+(\tau) = \epsilon^+ + \lambda\,\frac{p^+}{m^2} \quad\longrightarrow\quad \lambda = -\frac{m^2}{p^+}\epsilon^+\,. \tag{11.5.16}$$

The reparameterization parameter λ turns out to be a constant. We can now use this result to "explain" the transformations (11.5.10) and (11.5.11) that p^- generates on x^I and on x^-. For these coordinates there is no translation, but the reparameterization still applies. Therefore,

$$\delta x^I(\tau) = \lambda\,\partial_\tau x^I(\tau) = -\frac{m^2}{p^+}\epsilon^+\,\frac{p^I}{m^2} = -\epsilon^+\frac{p^I}{p^+}\,, \tag{11.5.17}$$

$$\delta x^-(\tau) = \lambda\,\partial_\tau x^-(\tau) = -\frac{m^2}{p^+}\epsilon^+\,\frac{p^-}{m^2} = -\epsilon^+\frac{p^-}{p^+}\,, \tag{11.5.18}$$

in perfect agreement with the transformations generated by p^-. We can also understand why p^- does not change x^+. If x^+ had been changed by a constant ϵ^+, the new x^+ coordinate would not satisfy the light-cone gauge condition whereby x^+ is just proportional to τ. In fact, p^- generates a translation plus the compensating transformation needed to preserve the light-cone gauge condition! That transformation turned out to be a reparameterization of the world-line.

One final remark about momentum operators. The Lorentz covariant momentum operators that we used to motivate our analysis generate simple translations and commute among each other. It follows directly that, using light-cone *coordinates*, the operators $p^\pm = (p^0 \pm p^1)/\sqrt{2}$ and the transverse p^I all commute. The light-cone *gauge* momentum operators we discussed above are completely different objects. They had an intricate action on coordinates, and p^- was defined in terms of the transverse momenta and p^+. Nevertheless, all the light-cone gauge momentum operators still commute. They obey the same commutation relations that the covariant operators do when expressed using light-cone coordinates.

11.6 Light-cone Lorentz generators

In section 8.5 we determined the conserved charges that are associated with the Lorentz invariance of the relativistic string Lagrangian. Similar charges exist for the relativistic point particle. As we found in (8.5.1), the infinitesimal Lorentz transformations of the point particle coordinates $x^\mu(\tau)$ take the form

$$\delta x^\mu(\tau) = \epsilon^{\mu\nu} x_\nu(\tau)\,, \tag{11.6.1}$$

where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ are a set of infinitesimal constants. The associated Lorentz charges are given by

$$M^{\mu\nu} = x^\mu(\tau)p^\nu(\tau) - x^\nu(\tau)p^\mu(\tau)\,, \tag{11.6.2}$$

as you may have derived in Problem 8.2. These charges are conserved classically. The quantum charges are expected to generate Lorentz transformations of the coordinates. Again, it is straightforward to see this using the operators of Lorentz-covariant quantization. In this case, the quantum charges are given by (11.6.2) with $x^\mu(\tau)$ and $p^\mu(\tau)$ taken to be the Heisenberg operators introduced earlier and satisfying the commutation relations (11.5.2) and (11.5.3). Both $x^\mu(\tau)$ and $p^\mu(\tau)$ are Hermitian operators. The Lorentz charges $M^{\mu\nu}$ are Hermitian as well:

$$(M^{\mu\nu})^\dagger = p^\nu(\tau)x^\mu(\tau) - p^\mu(\tau)x^\nu(\tau) = M^{\mu\nu}\,, \tag{11.6.3}$$

since the two constants induced by rearranging the coordinates and momenta back to the original form cancel out.

Quick Calculation 11.6. Show that

$$[\, M^{\mu\nu}\,,\, x^\rho(\tau)\,] = i\,\eta^{\mu\rho}\,x^\nu(\tau) - i\,\eta^{\nu\rho}\,x^\mu(\tau)\,. \tag{11.6.4}$$

This commutator helps us check that the quantum Lorentz charges generate Lorentz transformations:

$$\begin{aligned}\delta x^\rho(\tau) &= \Big[-\frac{i}{2}\,\epsilon_{\mu\nu}M^{\mu\nu}\,,\, x^\rho(\tau)\Big],\\ &= \frac{1}{2}\,\epsilon_{\mu\nu}\Big(\eta^{\mu\rho}\,x^\nu(\tau) -\, \eta^{\nu\rho}\,x^\mu(\tau)\Big)\,,\\ &= \frac{1}{2}\,\epsilon^{\rho\nu}x_\nu(\tau) + \frac{1}{2}\,\epsilon^{\rho\mu}\,x_\mu(\tau) = \epsilon^{\rho\nu}x_\nu(\tau)\,.\end{aligned} \tag{11.6.5}$$

Equation (11.6.4) can be used in light-cone *coordinates* by simply using light-cone indices. For example,

$$[M^{-I}, x^+(\tau)] = i\eta^{-+}\,x^I(\tau) - i\eta^{I+}x^-(\tau) = -i\,x^I(\tau)\,, \tag{11.6.6}$$

since $\eta^{I+} = 0$. The operator M^{-I} here is a Lorentz-covariant generator expressed in light-cone coordinates. It is *not* a light-cone gauge Lorentz generator. Those we have not yet constructed.

Given a set of quantum operators, it is interesting to calculate their commutators. In quantum mechanics, for example, you learned that the components L_x, L_y, and L_z of the angular momentum satisfy a set of commutation relations ($[L_x, L_y] = iL_z$, and others) that define the Lie algebra of angular momentum. The momentum operators p^μ considered earlier define a very simple Lie algebra; they all commute. We would like to know what is the commutator of two Lorentz generators. The computation takes a few steps (see Problem 11.5). Using equation (11.6.4), and a similar equation for $[M^{\mu\nu}\,,\, p^\rho]$, one finds that the commutator can be written as a linear combination of four Lorentz generators:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu}\,. \tag{11.6.7}$$

This result defines the Lorentz Lie algebra. Equation (11.6.7) must be satisfied by the analogous operators $M^{\mu\nu}$ of any Lorentz-invariant quantum theory. If it is not possible to construct such operators, the theory is not Lorentz-invariant. This will be crucial to our quantization of the string, for requiring that (11.6.7) holds imposes additional restrictions, which have significant physical consequences.

Quick Calculation 11.7. Since $M^{\mu\nu} = -M^{\nu\mu}$, the left hand side of (11.6.7) changes sign under the exchange of μ and ν. Verify that the right-hand side also changes sign under this exchange.

We can now use (11.6.7) to determine the commutators of Lorentz charges in light-cone *coordinates*. The Lorentz generators are given by

$$M^{IJ}\,, \quad M^{+I}\,, \quad M^{-I}\,, \;\text{ and }\; M^{+-}\,. \tag{11.6.8}$$

Consider, for example the commutator $[M^{+-}, M^{+I}]$. To use (11.6.7) notice the structure of its right-hand side: each η contains one index from each of the generators in the left-hand side. For $[M^{+-}, M^{+I}]$, the only way to get a nonvanishing η is to use the $-$ from the first generator and the $+$ from the second generator. The nonvanishing term is the second one on the right hand side of (11.6.7), and we find

$$[M^{+-}, M^{+I}] = -i\,\eta^{-+} M^{+I} = iM^{+I}\,. \tag{11.6.9}$$

Similarly,

$$[M^{-I}, M^{-J}] = 0\,. \tag{11.6.10}$$

Here η must use the I and J indices, but then the other two indices must go into M giving us M^{--}, which vanishes by antisymmetry.

So far, we have considered the covariant Lorentz charges in light-cone *coordinates*. We must now find Lorentz charges for our light-cone *gauge* quantization of the particle. Our earlier discussion of the momenta suggests that we really face three questions:

(1) How are these charges going to be defined?

(2) What kind of transformations will they generate?

(3) Which commutation relations will they satisfy?

In the remaining of this section we will explore question (1) in detail. Before doing so, let's give brief answers to questions (2) and (3), leaving further analysis of these questions to Problems 11.6 and 11.7. The light-cone gauge Lorentz generators are expected to generate Lorentz transformations of coordinates and momentum, but in some cases, these transformations will be accompanied by reparameterizations of the world-line. Regarding (3), the light-cone gauge Lorentz generators will satisfy the same commutation relations that the covariant operators in light-cone coordinates do. This establishes that Lorentz symmetry holds in the light-cone theory of the quantum point particle. The success of the construction is not obvious *a priori*. It is not clear that the reduced set of light-cone gauge operators suffices to construct quantum Lorentz charges that generate Lorentz transformations (plus other transformations) and satisfy the Lorentz algebra.

The simplest guess for the light-cone gauge generators is to use light-cone coordinates in the covariant formula (11.6.2) and then replace $x^+(\tau)$ and p^- using their light-cone gauge definitions in (11.3.5) and (11.3.7). Let's try this prescription with M^{+-}:

$$\begin{aligned} M^{+-} &\stackrel{?}{=} x^+(\tau)\, p^-(\tau) - x^-(\tau)\, p^+(\tau)\,, \\ &\stackrel{?}{=} \frac{p^+\tau}{m^2}\, p^- - \Big(x_0^- + \frac{p^-}{m^2}\tau\Big)p^+\,, \\ &\stackrel{?}{=} -\, x_0^-\, p^+\,. \end{aligned} \tag{11.6.11}$$

Since x_0^- and p^+ are τ-independent, so too is M^{+-}. We have a minor complication, however. The operator M^{+-} is not Hermitian: $(M^{+-})^\dagger - M^{+-} = [x_0^-, p^+] \neq 0$. This failure of Hermiticity illustrates how the use of the light-cone gauge can affect basic properties of operators. The covariant Lorentz generators were automatically Hermitian, the light-cone gauge generators are not. We are therefore motivated to define a Hermitian M^{+-} as

$$M^{+-} = -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-)\,. \tag{11.6.12}$$

We take this to be the light-cone gauge Lorentz generator M^{+-}.

The most complicated of all generators is M^{-I}. It is also the most interesting one as well. The prescription used for M^{+-} this time gives

$$\begin{aligned} M^{-I} &\overset{?}{=} x^-(\tau)\, p^I - x^I(\tau)\, p^- \,, \\ &\overset{?}{=} \Big(x_0^- + \frac{p^-}{m^2}\tau\Big) p^I - \Big(x_0^I + \frac{p^I \tau}{m^2}\Big) p^- \,, \\ &\overset{?}{=} x_0^-\, p^I - x_0^I\, p^- \,. \end{aligned} \tag{11.6.13}$$

As before, the τ dependence vanishes, but we are left with a complicated result, since p^- is a nontrivial function of the other momenta. We define M^{-I} as the Hermitian version of the operator obtained above:

$$M^{-I} \equiv x_0^- p^I - \frac{1}{2}\left(x_0^I\, p^- + p^-\, x_0^I\right). \tag{11.6.14}$$

If the light-cone gauge Lorentz charges are to satisfy the Lorentz algebra we must have

$$[\, M^{-I}\,,\, M^{-J}\,] = 0\,, \tag{11.6.15}$$

as we noted in (11.6.10). Does M^{-I}, as defined by (11.6.14), satisfy this equation? The answer is yes, as you will see for yourself in Problem 11.6. This result is necessary to ensure Lorentz invariance of the quantum theory.

When we quantize the string, the calculation of $[\, M^{-I}\,,\, M^{-J}\,]$ will be fairly complicated. But the answer is very interesting. It turns out that the commutator is zero only if the string lives in a particular spacetime dimension and, furthermore, only if the definition of mass is changed in such a way that we can find massless gauge fields in the spectrum of the open string! String theory is such a constrained theory that it is only Lorentz invariant for a fixed spacetime dimensionality.

Problems

Problem 11.1. *Equation of motion for Heisenberg operators.*

Assume the Schrödinger Hamiltonian $H(p,q)$ is time independent. In this case the time-independent Schrödinger operator ξ yields a Heisenberg operator $\xi(t) = e^{iHt}\xi e^{-iHt}$. Show that this operator satisfies the equation

$$i\frac{d\xi(t)}{dt} = \Big[\,\xi(t)\,,\; H(p(t),q(t))\Big]\,.$$

This computation proves that equation (11.2.2) holds for time-independent Hamiltonians.

Problem 11.2. *Heisenberg operators and time-dependent Hamiltonians.*

When the Schrödinger Hamiltonian $H = H(p,q;t)$ is time dependent, time evolution of states is generated by a unitary operator $U(t)$:

$$|\Psi, t\rangle = U(t)|\Psi\rangle\,, \tag{1}$$

where $U(t)$ bears some nontrivial relation to H. Here $|\Psi\rangle$ denotes the state at zero time.

(a) Use the Schrödinger equation to show that

$$i\frac{dU(t)}{dt} = HU(t)\,. \tag{2}$$

Let $U \equiv U(t)$, for brevity. Since U^{-1} acting on $|\Psi, t\rangle$ gives a time-independent state, considerations similar to those given for (11.2.8) lead us to define the Heisenberg operator corresponding to the Schrödinger operator α as

$$\alpha(t) = U^{-1}\alpha\, U\,. \tag{3}$$

(b) Let ξ be a time-independent Schrödinger operator, and $\xi(t)$ the corresponding Heisenberg operator, defined using (3). Show that

$$i\frac{d\xi(t)}{dt} = \Big[\,\xi(t)\,,\; H(p(t),q(t);t)\Big]\,.$$

This computation proves that equation (11.2.2) holds for time-dependent Hamiltonians.

(c) If $[\alpha_1, \alpha_2] = \alpha_3$ holds for arbitrary Schrödinger operators α_1, α_2, and α_3, show that $[\alpha_1(t), \alpha_2(t)] = \alpha_3(t)$ holds for the corresponding Heisenberg operators.

Problem 11.3. *Classical dynamics in Hamiltonian language.*

Consider a classical phase space (q,p), a trajectory $(q(t),p(t))$, and an observable $v\,(q(t),p(t);\,t)$. From the standard rules of differentiation,

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial p}\frac{dp}{dt} + \frac{\partial v}{\partial q}\frac{dq}{dt}\,. \tag{1}$$

With the Poisson bracket defined as

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\,, \tag{2}$$

show that

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \{v, H\}\,. \tag{3}$$

Comparing this result to (11.2.3) we see the parallel between the time evolution of a general operator $\mathcal{O}$ and the classical Hamiltonian evolution of an observable v in phase space.

To derive (3) you need the classical equations of motion in Hamiltonian language. These can be obtained by demanding that

$$\int dt\Big(p(t)\dot{q}(t) - H(p(t), q(t); t)\Big)\,,$$

be stationary for independent variations $\delta q(t)$ and $\delta p(t)$.

Problem 11.4. *Reparameterization symmetries of the point particle.*

Show that the variation $\delta x^\mu(\tau) = \lambda(\tau)\partial_\tau x^\mu(\tau)$ induces a variation δL of the point particle Lagrangian that can be written as

$$\delta L(\tau) = \partial_\tau\Big(\lambda(\tau)L(\tau)\Big)\,. \tag{11.6.16}$$

This proves that the reparameterizations δx^μ are symmetries of the point particle theory, in the sense defined in Problem 8.5. Show, however, that the charges associated to these reparameterization symmetries vanish. When λ is τ-independent, the reparameterization is an infinitesimal constant τ translation. The conserved charge is then the Hamiltonian. Show directly that the Hamiltonian defined canonically from the point particle Lagrangian vanishes.

Problem 11.5. *Lorentz generators and Lorentz algebra.*

In this problem we consider the Lorentz-covariant charges (11.6.2).

(a) Calculate the commutator $[\, M^{\mu\nu}\, ,\, p^\rho]$.

(b) Calculate the commutator $[\, M^{\mu\nu}\, ,\, M^{\rho\sigma}]$ and verify that (11.6.7) holds.

(c) Consider the Lorentz algebra in light cone *coordinates*. Give

$$[\, M^{\pm I}\, ,\, M^{JK}]\,, \quad [\, M^{\pm I}\, ,\, M^{\mp J}]\,, \quad [M^{+-}\, ,\, M^{\pm I}]\,, \quad \text{and} \quad [\, M^{\pm I}\, ,\, M^{\pm J}]\,.$$

Problem 11.6. *Light-cone gauge commutator* $[M^{-I}, M^{-J}]$ *for the particle.*

The purpose of the present calculation is to show that

$$[\, M^{-I}\, ,\, M^{-J}\,] = 0\,. \tag{1}$$

(a) Verify that the light-cone gauge operator M^{-I} takes the form

$$M^{-I} = (x_0^- p^I - x_0^I p^-) + \frac{i}{2}\frac{p^I}{p^+}\,. \tag{2}$$

Set up now the computation of (1) distinguishing the two kinds of terms in (2). Calculate the contributions to the commutator from mixed terms and from the last term.

(b) Complete the computation of (1) by finding the contribution from the first term in the right hand side of (2).

Problem 11.7. *Transformations generated by the light-cone gauge Lorentz generators* M^{+-} *and* M^{-I}*.*

(a) Calculate the commutators of M^{+-}, defined in (11.6.12), with the light-cone coordinates $x^+(\tau), x^-(\tau)$, and $x^I(\tau)$. Show that M^{+-} generates the expected Lorentz transformations of these coordinates.

(b) Calculate the commutators of M^{-I} with the light-cone coordinates $x^+(\tau), x^-(\tau)$, and $x^J(\tau)$. Show that M^{-I} generates the expected Lorentz transformations together with a compensating reparameterization of the world-line. Calculate the parameter λ for this reparameterization.

(c) Calculate $[M^{-I}\, ,\, p^+]$. Combine this result with your result for the commutator of M^{-I} with $x^+(\tau)$ to argue that the generator M^{-I} preserves the light-cone gauge condition.