Eleven-dimensional supergravity on a manifold with boundary

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Abstract

In this paper, we present a systematic analysis of eleven-dimensional supergravity on a manifold with boundary, which is believed to be relevant to the strong coupling limit of the \(E_8 \times E_8\) heterotic string. Gauge and gravitational anomalies enter at a very early stage, and require a refinement of the standard Green-Schwarz mechanism for their cancellation. This uniquely determines the gauge group to be a copy of \(E_8\) for each boundary component, fixes the gauge coupling constant in terms of the gravitational constant, and leads to several striking new tests of the hypothesis that there is a consistent quantum \(M\)-theory with eleven-dimensional supergravity as its low-energy limit.

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1. Introduction

In a previous paper [1], we proposed that the strong coupling limit of the ten-dimensional \(E_8 \times E_8\) heterotic string is eleven-dimensional \(M\)-theory compactified on \(\mathbb{R}^{10} \times S^1/\mathbb{Z}_2 = \mathbb{R}^{10} \times I\) (\(I\) is the unit interval), with the gauge fields entering via ten-dimensional vector multiplets that propagate on the boundary of space-time. This implies in particular that there must exist a supersymmetric coupling of ten-dimensional vector multiplets on the boundary of an eleven-manifold to the eleven-dimensional supergravity multiplet propagating in the bulk. The purpose of the present paper is to explore this coupling.

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In doing so, one comes quickly to a puzzle. The supergravity action in bulk is

$$-\frac{1}{2\kappa^2} \int_{M^{11}} d^{11}x \sqrt{g} R + \ldots ,$$

(1.1)

with $M^{11}$ being the eleven-dimensional space-time, and "\ldots" being the terms involving fermions and the bosonic three-form field. The supergauge action on the boundary is

$$-\frac{1}{4\lambda^2} \int_{M^{10}} d^{10}x \sqrt{g} \text{tr} F^2 + \ldots ,$$

(1.2)

where $M^{10}$ is the boundary (or a component of the boundary) of $M^{11}$, and $F$ is the field strength of the gauge fields that propagate on $M^{10}$. (For $E_8 \times E_8$, "\text{tr}" is as usual $1/30$ of the trace $\text{Tr}$ in the adjoint representation.) In the above formulas, $\kappa$ and $\lambda$ are the gravitational and gauge coupling constants. From those constants one can make a dimensionless number $\eta = \lambda^6 / \kappa^4$. The question is what determines the value of $\eta$. Note that there is no dilaton or other scalar whose expectation value controls the value of $\eta$. In fact, there is no scalar field at all in the theory, propagating either in bulk or on the boundary; in going to strong coupling, the dilaton of the perturbative heterotic string is reinterpreted as the radius of $S^1/\mathbb{Z}_2$.

Since string theory has no adjustable parameter corresponding to $\eta$, the strong coupling limit of the $E_8 \times E_8$ heterotic string, if it does have the eleven-dimensional interpretation proposed in our previous paper, must give a definite value for $\eta$. In fact, we will argue in this paper that by looking more precisely at gravitational and gauge anomalies (which were already used in the previous paper), one can determine $\eta$. We get

$$\eta = 128\pi^5 ,$$

(1.3)

or equivalently

$$\lambda^2 = 2\pi(4\pi\kappa^2)^{2/3} .$$

(1.4)

In the remainder of this introduction, we sketch the argument that will be used to determine $\eta$, and also sketch the other main qualitative results of this paper. The reason for presenting such a detailed sketch first is that the supergravity calculation that occupies the remainder of the paper is unavoidably rather complicated.

We recall that anomalies in ten dimensions are described by a formal twelve-form $I_{12}(R, F_1, F_2)$ that is a sixth-order homogeneous polynomial in the Riemann tensor $R$ and the field strengths $F_1$ and $F_2$ in the two $E_8$'s. It has the general form

$$I_{12}(R, F_1, F_2) = A(R) + B(R, F_1) + B(R, F_2) ,$$

(1.5)

where $A(R)$ is the contribution of the supergravity multiplet, and $B(R, F_i)$, for $i = 1, 2$, is the contribution of the gluinos of the $i^{th}$ $E_8$. In [1], we introduced
\[ \tilde{I}_{12}(R, F) = \frac{1}{2} A(R) + B(R, F), \] (1.6)

so that

\[ I_{12}(R, F_1, F_2) = \tilde{I}_{12}(R, F_1) + \tilde{I}_{12}(R, F_2). \] (1.7)

The idea here is that from the eleven-dimensional point of view, the gauge and gravitational anomaly is localized on the boundary of space-time, and the two terms on the right of (1.7) are the contributions of the two components of the boundary of \( \mathbb{R}^{10} \times I \). Of the two \( E_8 \)'s, the one propagating on a given boundary component is naturally the only one that contributes to the anomaly form of that component.

Anomaly cancellation for the perturbative heterotic string involves a factorization

\[ I_{12} = I_4 I_8, \] (1.8)

where \( I_4 = \text{tr} R^2 - \text{tr} F_1^2 - \text{tr} F_2^2 \) and \( I_8 \) is an eight-form given by a lengthy quartic polynomial in \( R \) and the \( F_i \). As was explained in [1], \( \tilde{I}_{12} \) has an analogous factorization

\[ \tilde{I}_{12} = \tilde{I}_4 \tilde{I}_8, \] (1.9)

with

\[ \tilde{I}_4(R, F) = \frac{1}{2} \text{tr} R^2 - \text{tr} F^2, \]

\[ \tilde{I}_8(R, F) = -\frac{1}{4} \tilde{I}_4(R, F)^2 + \left( -\frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2 \right). \] (1.10)

(This way of writing the formula for \( \tilde{I}_8 \), which was also noticed by M. Duff and R. Minasian, has a rationale that will become clear in Section 3.) It was proposed in [1] that this factorization of \( \tilde{I}_{12} \) would permit an extension of the Green-Schwarz anomaly cancellation mechanism to \( M \)-theory on eleven-dimensional manifolds with boundary.

The Green-Schwarz mechanism in ten dimensions depends on the existence in string theory of a two-form field \( B \) whose gauge-invariant field strength \( H \) obeys

\[ dH = I_4. \] (1.11)

Such an equation (with only the \( \text{tr} F^2 \) term in \( I_4 \)) holds even in the minimal ten-dimensional supergravity [2,3]. In addition, there are “Green-Schwarz interaction terms”, present in the string theory but not in the minimal low-energy supergravity theory, of the form

\[ \Delta L = \int B \wedge I_8. \] (1.12)

The combination of (1.11) and (1.12) gives a classical theory that is not gauge invariant, with an anomaly constructed from the twelve-form \( I_{12} = I_4 I_8 \). The minimal classical supergravity theory is gauge invariant because the anomalous fermion loops and the Green-Schwarz terms are both absent, and the string theory is gauge invariant because they are both present and the anomalies cancel between them.
Now let us discuss how the story will work in eleven dimensions. In doing so, and in most of this paper, we will use an orbifold approach in which we work on an eleven-manifold $M^{11}$ with a $\mathbb{Z}_2$ symmetry whose fixed points are of codimension one; alternatively, one can take the quotient and work on the manifold-with-boundary $X = M^{11}/\mathbb{Z}_2$, whose boundary points are the $\mathbb{Z}_2$ fixed points in $M^{11}$. In general, the formulation in terms of a manifold with boundary is convenient intuitively, and the orbifold formulation is convenient for calculation.

Rather than a two-form $B$, the eleven-dimensional supergravity multiplet has a three-form field $C$ (denoted by $A^{(3)}$ in our previous paper [1]), whose field strength is a four-form $G$. In the absence of boundaries, $G$ obeys the usual Bianchi identity $dG = 0$. The analog of (1.11) will have to be a contribution to $dG$ supported at the $\mathbb{Z}_2$ fixed points. As $dG$ is a five-form, we will have to promote the four-form $I_4$ to a five-form supported on the fixed point set, so that it can appear as a correction to the Bianchi identity. To write such a five-form, one supposes that the fixed point set is defined locally by an equation $x^{11} = 0$, and one multiplies by the closed one-form $\delta(x^{11})dx^{11}$ to promote the four-form $I_4$ to a five-form. Thus, the eleven-dimensional analog of the ten-dimensional equation $dH = I_4$ will be an equation $dG = c\delta(x^{11})dx^{11}I_4$, with some constant $c$. In Section 2, we will determine the precise equation to be

$$dG_{11ijk} = -\frac{3\sqrt{2}}{2\pi} \left( \frac{\kappa}{4\pi} \right)^{2/3} \delta(x^{11}) \left( \text{tr} F_{[ij} F_{kl]} - \frac{1}{2} \text{tr} R_{[ij} R_{kl]} \right). \quad (1.13)$$

Here $F$ is of course the field strength of the gauge field propagating at $x^{11} = 0$, and $\text{tr} F_{[ij} F_{kl]} = \frac{1}{24} \text{tr} F_{ijkl} \pm$ permutations. Actually, in Section 2, we will see directly only the $\text{tr} F \wedge F$ term in (1.13); the $\text{tr} R \wedge R$ term is a sort of higher-order correction that we infer because it is needed for anomaly cancellation. (Analogously, in ten dimensions, the $\text{tr} F \wedge F$ term is required by supersymmetry, and the $\text{tr} R \wedge R$ term is an $O(\alpha')$ stringy correction needed for anomaly cancellation.)

At this stage the question is, what are the Green-Schwarz terms? In the familiar ten-dimensional story, because the Green-Schwarz terms are unconstrained by supersymmetry, the Green-Schwarz mechanism makes no general prediction (independent of anomalies or a detailed string model) about what $I_8$ should be. In eleven dimensions, the story will be quite different because the terms analogous to the Green-Schwarz terms are independently known. One of these terms is simply the $\int C \wedge G \wedge G$ interaction of eleven-dimensional supergravity. This term, discovered when the model was first constructed [8], has always seemed enigmatic because the rationale behind its apparently "topological" nature was not clear. We feel that the role of this term in canceling anomalies – we explain in Section 3 how $\int C \wedge G \wedge G$ comes to play the role of a Green-Schwarz term – removes some of the enigma.

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Following conventions in [2] which have become standard in eleven-dimensional supergravity, we define $G_{ijkl} = \partial_i C_{jkl} \pm 23$ terms, though the normalization is somewhat unusual. We also define $dG_{ijklem} = \partial_i G_{jklm} +$ cyclic permutations of $ijklm$.
The $\int C \wedge G \wedge G$ is the only "Green-Schwarz" interaction involved in canceling gauge anomalies, but to cancel also the gravitational anomalies requires an additional interaction. This is an eleven-dimensional interaction

$$\int_{M^{11}} C \wedge X_8(R), \quad (1.14)$$

with $X_8(R)$ an eight-form constructed as a quartic polynomial in the Riemann tensor. This interaction is known in two ways. (1) Upon dimensional reduction on $S^1$, it turns into a $B \wedge X_8$ interaction which can be computed as a one-loop effect in Type IIA superstring theory [7]. The one-loop calculation is exact since a dilaton dependence of the $B \wedge X_8$ coupling would spoil gauge invariance; because it is exact, it can be extrapolated to eleven dimensions and implies the existence of the interaction written in (1.14). (2) Alternatively, this coupling is needed to cancel one-loop anomalies on the five-brane world-volume and thus permit the existence of five-branes in the theory [4-6]. Happily, the two methods agree, with

$$X_8 = -\frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2. \quad (1.15)$$

As we will see in Section 3, it is no coincidence that the combination of $\text{tr} R^4$ and $(\text{tr} R^2)^2$ that appears here also entered in (1.10).

The fact that the terms analogous to Green-Schwarz terms are known independently of any discussion of space-time anomalies means that we get an \textit{a priori} prediction for $I_8$. (We have no \textit{a priori} prediction of $I_4$, as the coefficients in (1.13) will essentially be adjusted to make anomaly cancellation possible.) We regard the success of this prediction as a compelling confirmation that eleven-dimensional supergravity on a manifold with boundary is indeed related to ten-dimensional $E_8 \times E_8$ heterotic string theory as proposed in [1].

\textit{Classical and quantum consistency}

The details that we have just explained of how anomaly cancellation works in eleven dimensions have other implications for the structure of the theory.

The fact that, once one works on a manifold with boundary, some of the Green-Schwarz terms are present in the minimal supergravity Lagrangian means that the classical Lagrangian, including the vector supermultiplets on the boundary, is not gauge invariant. Thus, the theory with the supergravity multiplet in bulk and the vector multiplets on the boundary is only consistent as a quantum theory. The situation is rather different from perturbative string theory, where since the Green-Schwarz terms arise at the one-loop level, one has gauge invariance either classically (leaving out the anomalous chiral fermion loop diagrams and the effects of the Green-Schwarz terms) or quantum mechanically (including both of these).

The relation $\lambda^2 \sim \kappa^{4/3}$ between the gauge and gravitational couplings sheds a further light on this. It means that the gauge kinetic energy, of order $1/\lambda^2$, is a higher-order
correction, of relative order $\kappa^{2/3}$ compared to the gravitational action, which is of order $1/\kappa^2$. If one wants a fully consistent classical theory, one must ignore the gauge fields completely. Once one tries to include the supergauge multiplet, gauge invariance will fail classically (in relative order $\kappa^2$), and quantum anomalies are needed to compensate for this failure.

Since the classical theory with the gauge fields is not going to be fully consistent, one has to expect peculiarities in constructing it. In our analysis in Section 4, we certainly find such peculiarities. We will organize our discussion of the boundary interactions as an expansion in powers of $\kappa^{2/3}$. In order $\kappa^{2/3}$, things go smoothly, though the calculations are rather involved, roughly as in standard supergravity theories. Some novelties arise in order $\kappa^{4/3}$. In verifying invariance of the Lagrangian in that order, one has to cancel terms that are formally proportional to $\delta(0)$. The cancellation also involves adding to the Lagrangian new interactions (of relative order $\kappa^{4/3}$) proportional to $\delta(0)$. We interpret the occurrence of $\delta(0)$ terms in the Lagrangian and the supersymmetry variations of fields as a symptom of attempting to treat in classical supergravity what really should be treated in quantum $M$-theory. In a proper quantum $M$-theory treatment, there would presumably be a built-in cutoff that would replace $\delta(0)$ by a finite constant times $\kappa^{-2/9}$.

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Despite the infinities that arise in the construction, we hope and expect that the analysis of the anomalies is reliable. This should be analogous to the fact that anomalous loop diagrams can be reliably computed even in unrenormalizable effective theories, because the anomalies can be construed as an infrared effect and are independent of what cutoff one introduces.
Summary

To summarize, then, the lessons from our investigation, we will find that anomaly cancellation of the ten-dimensional heterotic string has an elegant eleven-dimensional interpretation that sheds light on properties of the anomaly twelve-form that were not needed before. This sharpens the eleven-dimensional interpretation of the strongly coupled $E_8 \times E_8$ heterotic string, fixing an otherwise unknown dimensionless parameter and adding to our confidence that the eleven-dimensional description is correct. The gauge anomalies that arise in the classical discussion also give an indication - and not the only one - that the theory only really makes sense at the quantum level.

2. Correction to the Bianchi identity

Our eleven-dimensional conventions are as in [2]. We work with Lorentz signature $- + + \ldots +$. Vector indices will be written as $I, J, K$, and spinor indices as $\alpha, \beta, \gamma$. We introduce a frame field $e^m_I$ with the metric being $g_{IJ} = \eta_{mn}e^m_Ie^n_J$. The gamma matrices are $32 \times 32$ real matrices obeying $\{\Gamma_I, \Gamma_J\} = 2g_{IJ}$. One also defines $\Gamma^{I_1 \ldots I_n} = \Gamma^{I_1} \ldots \Gamma^{I_n} \equiv (1/n!) \Gamma^{IJ} \Gamma^I_{JK} \ldots \Gamma^J_{L} \pm$ permutations. Spinor indices are raised and lowered with a real antisymmetric tensor $C$ obeying $C_{a\beta} = -C_{\beta a}, C^{a\beta}C_{\beta\gamma} = \delta^a_{\gamma}$. In particular, by lowering an index in the gamma matrix $\Gamma^\alpha_{I\beta}$ one gets a symmetric tensor $\Gamma^\alpha_{I\beta} = \Gamma_{I\beta}$. All spinors will be Majorana spinors; the symbol $\bar{\psi}_a$ is simply defined by $\bar{\psi}_a = C_{a\beta}\psi^\beta$.

The supergravity multiplet consists of the metric $g$, the gravitino $\psi_{Ia}$, and a three-form $C$ (with field strength $G$, normalized as in a previous footnote). The supergravity Lagrangian, up to terms quartic in the gravitino (which we will not need), is [8]

$$L_s = \frac{1}{k^2} \int d^{11}x \sqrt{g} \left( -\frac{1}{2} R - \frac{1}{2} \bar{\psi}_I \Gamma^{IJK} D_J \psi_K - \frac{1}{48} G_{IJKL} G^{IJKL} ight. $$

$$\left. - \frac{\sqrt{2}}{192} \left( \bar{\psi}_I \Gamma^{IJKLMN} \psi_N + 12 \bar{\psi}^J \Gamma^{JLM} \psi^M \right) G_{JKLM} ight. $$

$$\left. - \frac{\sqrt{2}}{3456} \epsilon_{I_1 I_2 \ldots I_{11}} C_{I_1 I_2 I_3} G_{I_4 \ldots I_{10}} G_{I_5 \ldots I_{11}} \right). \tag{2.1}$$

We work in 1.5 order formalism: the spin connection $\Omega$ is formally regarded as an independent variable, and eventually set equal to the solution of the bulk equations of motion. The Riemann tensor is the field strength constructed from $\Omega$.

The transformation laws of local supersymmetry read

$$\delta e^m_I = \frac{1}{2} \eta \Gamma^m \psi_I,$$

$$\delta C_{IJK} = -\frac{\sqrt{2}}{8} \eta \Gamma_{IJK} \psi_K,$$

$$\delta \psi_I = D_I \eta + \frac{\sqrt{2}}{288} \left( \Gamma^{JKLM} - 8 \delta^I_I \Gamma^{KLM} \right) \eta G_{JKLM} + \ldots. \tag{2.2}$$
(The ... are three-fermi terms in the transformation law of $\psi$, often absorbed in a
definition of "supercovariant" objects; we will not need them.)

We suppose that there is a $\mathbb{Z}_2$ symmetry acting on $M^{11}$, with codimension one fixed
points. We let $M^{10}$ be a component of the fixed point set; we will study the physics near
$M^{10}$. We suppose that the fields are required to be invariant under the $\mathbb{Z}_2$; this means that
we could pass to the manifold-with-boundary $X = M^{11}/\mathbb{Z}_2$ (with boundary $M^{10}$), but
that will not be particularly convenient. If $M^{10}$ is defined locally by an equation $x^{11} = 0,$
$x^{11}$ being one of the coordinates (and the $\mathbb{Z}_2$ acting by $x^{11} \rightarrow -x^{11}$), then (with an
appropriate lifting of the $\mathbb{Z}_2$ action to spinors and the three-form) the supersymmetries
that commute with the $\mathbb{Z}_2$ action are generated by spinor fields $\eta$ on $M^{11}$ that obey

$$\Gamma^{11} \eta = \eta \quad \text{at} \quad x^{11} = 0.$$  \hspace{1cm} (2.3)

$\mathbb{Z}_2$ invariance of the gravitino means that

$$\Gamma^{11} \psi_A = \psi_A, \quad A = 1, \ldots, 10,$$

$$\Gamma^{11} \psi_{11} = -\psi_{11}. \hspace{1cm} (2.4)$$

As in (2.4), we will use $A, B, C, D = 1, \ldots, 10$ for indices tangent to $M^{10}$. For the
three-form $C$, because it is odd under parity [this follows from the CGG interaction in
(2.1)], $\mathbb{Z}_2$ invariance means that $C_{BCD} = 0$ at $x^{11} = 0$. A gauge-invariant statement that
follows from this is that

$$G_{ABCD} = 0 \quad \text{at} \quad x^{11} = 0, \quad \hspace{1cm} (2.5)$$

or in other words, the pull-back of the differential form $G$ to $M^{10}$ vanishes. We will
eventually find a sort of modification of this statement in order $\kappa^{2/3}$.

The vector supermultiplets, which propagate on $M^{10}$, consist of the $E_8$ gauge field $A$
(with field strength $F_{CD} = \partial_C A_D - \partial_D A_C + [A_C, A_D]$) and fermions (gluinos) $\chi$ in
the adjoint representation, obeying $\Gamma^{11} \chi = \chi$. The minimal Yang-Mills Lagrangian is

$$L_{YM} = -\frac{1}{\lambda^2} \int_{M^{10}} d^{10}x \sqrt{g} \, tr \left( \frac{1}{4} F_{AB} F^{AB} + \frac{1}{2} \chi \Gamma^A D_A \chi \right). \hspace{1cm} (2.6)$$

(Here $d^{10}x \sqrt{g}$ is understood as the Riemannian measure of $M^{10}$, using the restriction to
$M^{10}$ of the metric on $M^{11}$.) In (2.6), $\lambda$ is the gauge coupling constant. We will ultimately
see that $\lambda \sim \kappa^{2/3}$, so that $L_{YM}$ is of order $\kappa^{2/3}$ relative to $L_S$. The supersymmetry
transformation laws are

$$\delta A^a_A = \frac{1}{2} \eta \Gamma_A \chi^a,$$

$$\delta \chi^a = -\frac{1}{4} \Gamma^{AB} F_{AB}^a \eta. \hspace{1cm} (2.7)$$

We have here made explicit an index $a = 1, \ldots, 248$ labeling the adjoint representation
of $E_8$. We define an inner product by $X^a X^a = \text{tr} \, X^2 = \frac{1}{30} \text{Tr} \, X^2$, with $\text{Tr}$ the trace in the
adjoint representation.
We wish to add additional interactions to the above and modify the supersymmetry transformation laws so that \( L_S + L_{YM} + \ldots \) will be locally supersymmetric. The first steps are as follows. Let \( T_{YM} \) and \( S_{YM} \) be the energy-momentum tensor and supercurrent of the supergauge multiplet. As in any coupling of matter to supergravity, the variation of \( L_{YM} \) under local supersymmetry contains terms \( D_A \bar{\eta} S^A_{YM} \), reflecting the fact that \( L_{YM} \) is only invariant under (2.7) if \( \eta \) is covariantly constant, and \( \bar{\eta} \psi / T_{YM} \), coming from the variation of \( L_{YM} \) under a local supersymmetry transformation of the metric. To cancel these variations, it is necessary – as usual in supergravity – to add an interaction \( \bar{\psi} S_{YM} \).

In the case at hand, this interaction is

\[
L_1 = -\frac{1}{4\lambda^2} \int_{M^{10}} d^{10}x \sqrt{g} \bar{\psi}_A \Gamma^{BC} \Gamma^A F^a_{BC} \chi^a. \tag{2.8}
\]

A small calculation shows that the variation of \( L_1 \) under local supersymmetry cancels the \( D \bar{\eta} X F \) and \( \eta \psi X D X \) variations of \( L_{YM} \), and also cancels part of the \( \eta \psi F^2 \) variation. The uncanceled variation turns out, after some gamma matrix gymnastics, to be

\[
\Delta = -\frac{1}{16\lambda^2} \int_{M^{10}} d^{10}x \sqrt{g} \bar{\psi}_A \Gamma^{ABCD} F^a_{BC} F^a_{DE} \eta. \tag{2.9}
\]

Rather as in the coupling of the ten-dimensional vector multiplet to ten-dimensional supergravity [2], there is no way to cancel this variation by adding to the Lagrangian additional matter couplings. A peculiar mixing of the supergravity and matter multiplets is needed.

When one verifies the local supersymmetry of the eleven-dimensional supergravity Lagrangian \( L_S \), it is necessary among other things to check the cancellation of the \( \eta GD\psi \) and \( D\bar{\psi}G\psi \) terms. In this verification, it is necessary to integrate by parts and use the Bianchi identity \( dG = 0 \). To cancel \( \Delta \), one must modify the Bianchi identity to read

\[
dG_{11ABCD} = -3\sqrt{2} \frac{k^2}{\lambda^2} \delta(x^{11}) F^q_{[AB} F^q_{CD]}. \tag{2.10}
\]

This correction to the Bianchi identity adds an extra variation of \( L_S \) that precisely cancels \( \Delta \).

Much as in the analogous story in ten dimensions, (2.10) implies that the three-form \( C \) is not invariant under Yang-Mills gauge transformations. To determine the gauge transformation law of \( C \), it is convenient to solve the modified Bianchi identity by introducing

\[
\omega_{BCD} = \text{tr} \left( A_B (\partial_C A_D - \partial_D A_C) + \frac{2}{3} A_B [A_C, A_D] + \text{cyclic permutations of } B, C, D \right). \tag{2.11}
\]

\(^4\)This occurs when one varies the interaction \( \bar{\psi}_A \Gamma^{IJJKLM} \psi_N G^{JKLM} \) with \( \delta \bar{\psi}_A \sim D_I \bar{\eta} \). To cancel other variations, one must integrate by parts so that the \( D_I \) acts on \( \psi_N \) instead of on \( \eta \). The integration by parts gives a term proportional to \( dG^{IJJKLM} \).
Thus
\[ \partial_A \omega_{BCD} + \text{cyclic permutations} = 6 \text{ tr } F_{[AB}F_{CD]} . \] (2.12)

The Bianchi identity can then be solved by modifying the definition of \( G_{11ABC} \), the new definition being
\[ G_{11ABC} = (\partial_{11} C_{ABC} \pm 23 \text{ permutations}) + \frac{\kappa^2}{\sqrt{2} \lambda^2} \delta(x^{11}) \omega_{ABC} . \] (2.13)

Under an infinitesimal gauge transformation \( \delta A^a_A = -D_A e^a \), \( \omega \) transforms by
\[ \delta \omega_{ABC} = \partial_A (\text{ tr } e_{BC}) + \text{cyclic permutations of } A, B, C , \] (2.14)

so gauge invariance of \( G_{11ABC} \) holds precisely if the three-form \( C \) transforms under gauge transformations by
\[ \delta C_{11AB} = -\frac{\kappa^2}{6 \sqrt{2} \lambda^2} \delta(x^{11}) \text{ tr } e_{AB} . \] (2.15)

A correction to the supersymmetry transformation law of \( C_{11BC} \) is also necessary. It can be determined by requiring that the supersymmetry variation of \( G_{11ABC} \) be gauge-invariant (otherwise this variation gives gauge non-invariant, uncancellable terms in the supersymmetry variation of the Lagrangian) and is
\[ \delta G_{11ABC} = \frac{\kappa^2}{\sqrt{2} \lambda^2} \delta(x^{11}) \left( \frac{1}{\eta^A} \chi^a \delta e_{AB} + \text{cyclic permutations of } A, B, C \right) . \] (2.16)

Boundary behavior

There is another sense in which we can "solve the Bianchi identity". We can ask, compatibly with the equation of motion, what can be the behavior near \( x^{11} = 0 \) of a \( G \) field that obeys the corrected Bianchi identity found above, which was
\[ dG_{11ABCD} = -\frac{\kappa^2}{\lambda^2} \delta(x^{11}) F^a_{[AB}F^a_{CD]} . \] (2.17)

How can \( dG \) acquire such a delta function? \( G \) itself cannot have a delta function at \( x^{11} = 0 \), as that would not be compatible with the equations of motion. However, as \( G \) is odd under \( x^{11} \rightarrow -x^{11} \), it is natural for \( G_{ABCD} \) to have a step function discontinuity at \( x^{11} = 0 \), giving a delta function in \( dG \). In fact, \( G_{ABCD} \) must have a jump at \( x^{11} = 0 \) given precisely by
Here $\epsilon(x^{11})$ is 1 for $x^{11} > 0$ and $-1$ for $x^{11} < 0$; the ... are terms that are regular near $x^{11} = 0$ and therefore (since $G$ is odd under $x^{11} \rightarrow -x^{11}$) vanish at $x^{11} = 0$. This is the behavior required by the modified equations of motion and Bianchi identity.

This discontinuity means that $G_{ABCD}$ does not have a well-defined limiting value as $x^{11} \rightarrow 0$. However, $G^2$ has such a limit, which moreover is determined by (2.19) in terms of the gauge fields at $x^{11} = 0$.

There is another interesting way to think about (2.19). In this paper we are working "upstairs" on a smooth eleven-manifold $M^{11}$, and requiring $Z_2$ invariance. It is natural conceptually (though sometimes less convenient computationally) to work "downstairs" on the manifold-with-boundary $X = M^{11}/Z_2$. In that case, it is not natural to add a correction to $dG$ supported at the boundary of $X$ (that is, at $x^{11} = 0$). More natural is to impose a boundary condition that has the same effect. Assuming that one identifies $X$ with the portion of $M^{11}$ with $x^{11} > 0$, the requisite boundary condition is simply

$$G_{ABCD}|_{x^{11}=0} = -\frac{3 \kappa^2}{\sqrt{2} \lambda^2} F^a_{[AB} F^a_{CD]}.$$  

(If one identifies $X$ with the $x^{11} < 0$ portion of $M^{11}$, one would want the opposite sign in (2.20).) The idea here is that, since $dG = 0$, the integration by parts explained in the footnote just before (2.10) no longer picks up a delta function term, but (since there is now a boundary) it does pick up a boundary term that has the same effect. 5

Thus, in working downstairs on $X$, $G$ has a well-defined boundary value given by (2.20) (or the same expression with opposite sign if one picks orientations oppositely). In working upstairs on $M$, $G$ does not quite have a well-defined value at $x^{11} = 0$, but $G^2$ does.

3. Analysis of anomalies

The most important conclusions of the last section are the gauge transformation law (2.15) for the three-form $C$, and the formula (2.19) for the behavior of $G$ near $M^{10}$. We will now put these together to get an eleven-dimensional view of gauge and gravitational anomalies.

The idea is that (2.15) is analogous to the gauge transformation law $\delta B \sim \text{tr} \epsilon F$ for the two-form $B$ of string theory, and (2.19) will turn the "Chern-Simons interaction" $\int C \wedge G \wedge G$ of eleven-dimensional supergravity into a Green-Schwarz term.

We recall that the $CGG$ interaction is, to be precise, a term

5 In working on $X = M^{11}/Z_2$ instead of $M^{11}$, one should replace the $1/\kappa^2$ in (2.1) by $2/\kappa^2$, because one is integrating (2.1) over a space of half the volume. This factor of 2 goes into verifying the normalization of (2.20).
The variation of $W$ under an arbitrary variation of $C$ is therefore

$$\delta W = -\frac{\sqrt{2}}{1152\kappa^2} \int \epsilon_{M_1 M_2 \ldots M_{11}} \delta C_{M_1 M_2 M_3} G_{M_4 \ldots M_7} G_{M_8 \ldots M_{11}}.$$

(3.2)

Given that $C$ is not invariant under gauge transformations, neither is $W$. Using (2.15) for the gauge variation of $C$, we get for the gauge variation of $W$

$$\delta W = -\frac{1}{2304\kappa^2} \int \epsilon_{M_1 M_2 \ldots M_{10}} \epsilon^a_{M_1 M_2} F^a_{M_3 M_4} F^b_{M_5 M_6} F^c_{M_7 M_8} F^d_{M_9 M_{10}}.$$

(3.3)

To proceed further, we need the value of $G^2$ at $x^{11} = 0$. This is given by (2.19) in the orbifold approach or equivalently by the boundary condition (2.20) if one works “downstairs”. Either way, one gets

$$K^4 \int \epsilon_{M_1 M_2 \ldots M_{10}} \epsilon^a_{M_1 M_2} F^a_{M_3 M_4} F^b_{M_5 M_6} F^c_{M_7 M_8} F^d_{M_9 M_{10}}.$$

(3.4)

So, as promised, the classical theory is not gauge invariant. There is no way to cure this at the classical level. The only recourse is to quantum anomalies. The anomalous variation of the effective action $\Gamma$ for ten-dimensional Majorana-Weyl fermions in an arbitrary representation of a simple gauge group is

$$\delta \Gamma = \frac{1}{2} \frac{1}{(4\pi)^5} \int \epsilon_{M_1 M_2 \ldots M_{10}} \epsilon^a_{M_1 M_2} F^a_{M_3 M_4} F^b_{M_5 M_6} F^c_{M_7 M_8} F^d_{M_9 M_{10}}.$$

(3.5)

with $\text{Tr}$ being the trace in the fermion representation. The case that we are interested in is that the gauge group is $E_8$ and the fermions are in the adjoint representation. In that case, one has the wonderful and unique (to $E_8$) identity $\text{Tr} W^6 = (\text{Tr} W^2)^3/7200$ (and likewise $\text{Tr} eF^5 = \text{Tr} eF (\text{Tr} F^2)^2/7200)$. If furthermore we write, as is customary, $\text{tr} W^2 = \text{Tr} W^2/30$, then $\text{Tr} W^6 = 15/4 (\text{tr} W^2)^3$. In this case, therefore, the quantum anomaly (3.5) can be written

$$\delta \Gamma = \frac{15}{8(4\pi)^5} \int \epsilon_{M_1 M_2 \ldots M_{10}} \epsilon^a_{M_1 M_2} \epsilon^b_{M_3 M_4} \epsilon^c_{M_5 M_6} \epsilon^d_{M_7 M_8} \epsilon^e_{M_9 M_{10}}.$$

(3.6)

It therefore has the right structure to cancel (3.4) (recall that the metric on the Lie algebra was defined by $e^a F^a = \text{tr} eF$).

Implementing this cancellation, we learn finally that, as promised in the introduction, the gauge coupling is related to the gravitational coupling by

$$\lambda^2 = 2\pi \left(4\pi \kappa^2\right)^{2/3}.$$

(3.7)
One might have expected that the analogs of the Green-Schwarz terms in the present discussion would be boundary interactions, that is interactions supported at $x^{11} = 0$. This is not the case, as we have seen. In fact, given a gauge variation of $C$ proportional to $\delta(x^{11})$, the possible resulting gauge variation of a boundary interaction would necessarily be proportional to $\delta(0)$. Thus, the “Green-Schwarz terms” must be bulk interactions; this goes for the “Chern-Simons” $CGG$ term and other terms discussed below.

3.1. Extension to gravitational anomalies

We determined the gauge coupling by canceling the purely gauge anomalies at the boundary of the eleven-dimensional world. We would now like to include also the gravitational and mixed anomalies.

From the above discussion, the anomaly four-form $\hat{I}_4$ is the four-form that appears (multiplied by $\delta(x^{11}) dx^{11}$) in the Bianchi identity for $G$. In our work so far, we have seen only a $\text{tr} F^2$ term in $\hat{I}_4$, but in view of the known form of the ten-dimensional anomalies, the actual structure must be $\text{tr} F^2 - \frac{1}{2}\text{tr} R^2$. Thus, the modified Bianchi identity (2.10) should be replaced by

$$dG_{11ABCD} = -3\sqrt{2} \frac{\kappa^2}{\lambda^2} \delta(x^{11}) \left( F_{[AB}^a F_{CD]}^a - \frac{1}{2} \text{tr} R_{[AB} R_{CD]} \right), \quad (3.8)$$

and the formula (2.19) for the behavior near $x^{11} = 0$ should correspondingly be replaced by

$$G_{ABCD} = -\frac{3}{\sqrt{2}} \frac{\kappa^2}{\lambda^2} \varepsilon(x^{11}) \left( F_{[AB}^a F_{CD]}^a - \frac{1}{2} \text{tr} R_{[AB} R_{CD]} + \ldots \right). \quad (3.9)$$

There is also a corresponding local Lorentz transformation law of (3.9), analogous to the $E_6$ gauge transformation law (2.15).

The $\text{tr} R^2$ terms in these formulas are not required by the low-energy supergravity, but (since they are needed for anomaly cancellation, given the structure of the one-loop chiral anomalies), they must be present in the full $M$-theory. The situation is presumably analogous to what is seen for the perturbative heterotic string, where the $\text{tr} R^2$ terms in the analogous formulas arise as corrections of order $\alpha'$. Note that the $\text{tr} R^2$ correction will appear in (3.8) and (3.9) with the same coefficient, since (3.9) is deduced from (3.8) by reasoning that was explained above.

Having understood how $\hat{I}_4$ enters in eleven dimensions, we would like now to understand the origin of $\hat{I}_8$, or equivalently to complete our understanding of the Green-Schwarz terms. We have already found one of the Green-Schwarz terms above – the long-familiar “Chern-Simons” interaction of eleven-dimensional supergravity. This particular interaction gives a contribution to $\hat{I}_8$ that is a multiple of $\hat{I}_4$, since the boundary behavior of $G$ is $G \sim \hat{I}_4$, as we have seen. The other Green-Schwarz terms will have to be bulk interactions, as explained at the end of the last subsection, and more precisely will have to be interactions of the form...
\begin{equation}
\mathcal{I} = \int_{M^{11}} C \wedge (a \text{tr} R^4 + b (\text{tr} R^2)^2),
\end{equation}

these being the terms that have the right sort of gauge and local Lorentz variations to cancel chiral anomalies. Note that it is impossible to add to (3.10) terms that directly involve $F$, since the gauge fields propagate only on $M^{10}$. It is also impossible for $F$-dependence to arise indirectly from the behavior of $G$ near $M^{10}$, since (3.10) is independent of $G$; the $C \wedge G \wedge G$ has already been taken into account (with a coefficient known from low-energy supergravity), and a term $C \wedge G \wedge \text{tr} R^2$ is not possible, as it would violate the parity symmetry of $M$-theory.

$\mathcal{I}$ will contribute to $\hat{I}_8$ a term that involves $R$ only, so we get the striking prediction that $\hat{I}_8$ is a multiple of $\hat{I}_4^2$ plus an eight-form constructed only from $R$. At this point, it is helpful to note that

\begin{equation}
\hat{I}_8 = -\frac{1}{4} (\text{tr} F^2 - \frac{1}{2} \text{tr} R^2)^2 + \left(-\frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2\right),
\end{equation}

and thus has the expected form. This is a satisfying test of $M$-theory, as this structure of $\hat{I}_8$ has no known rationale in perturbative string theory.

Actually, we can be more precise, since the interaction (3.10) is known (at least up to an overall multiplicative constant; fixing this constant requires a more precise comparison of the normalizations of string theory and $M$-theory or a precise knowledge of the two-brane and five-brane tensions in $M$-theory). As we explained in the introduction, the interaction (3.10) is known up to a constant multiple either from comparison to a one-loop calculation for Type IIA superstrings [7] or from anomaly cancellation for eleven-dimensional five-branes [4–6]. Either way, one finds that (3.10) is a multiple of

\begin{equation}
\frac{\sqrt{2}}{(4\pi)^3(4\pi\kappa^2)^{1/3}} \int_{M^{11}} C \wedge \left(-\frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2\right).
\end{equation}

Thus, at least the relative coefficient of $\text{tr} R^4$ and $(\text{tr} R^2)^2$ agrees with the “experimental” structure of $\hat{I}_8$. This is again a real test of $M$-theory, since there is no perturbative string theory reason for this to work. The structure of (3.10) is deduced either via Type IIA perturbation theory or anomaly cancellation for eleven-dimensional five-branes, and any known rationale for comparing the anomaly polynomial of the perturbative heterotic string to either of these involves $M$-theory.

Notice that the coefficient $-1/4$ of the $\hat{I}_4^2$ term in $\hat{I}_8$ is a matter of convention; it could be shifted by a scaling $\hat{I}_4 \rightarrow u \hat{I}_4$, $\hat{I}_8 \rightarrow u^{-1} \hat{I}_8$, without affecting the factorization $\hat{I}_{12} = \hat{I}_4 \hat{I}_8$. Modulo this imprecision in the definition of $\hat{I}_8$, we have from $M$-theory a complete a priori prediction for $\hat{I}_8$, which amounts to a prediction for three numbers ($\hat{I}_8$ is a linear combination of four monomials $(\text{tr} F^2)^2$, $\text{tr} F^2 \text{tr} R^2$, $(\text{tr} R^2)^2$, and $\text{tr} R^4$, but the coefficient of one monomial can be scaled out as just explained). There are therefore three predictions, of which we have here verified two; verification of the last prediction requires a more precise comparison of different conventions, as noted in the last paragraph.
4. Construction of the Lagrangian

In this section, we will proceed with additional steps in the construction of the locally supersymmetric Lagrangian. The formula $\lambda^2 = 2\pi(4\pi\kappa^2)^{2/3}$ obtained in the last section is of some conceptual interest in organizing the computation. It shows that the theory has only one natural length scale, given by $\kappa^{2/9}$. Moreover, on dimensional grounds, the decomposition of the boundary interactions in terms with more and more matter fields is an expansion in powers of $\kappa$. The leading boundary interactions (the minimal Lagrangian $L_{YM}$ of the gauge multiplet and terms related to it by supersymmetry) are of order $\kappa^{2/3}$ relative to the gravitational action. Formally, the construction of the locally supersymmetric classical action appears to be an expansion in integral powers of $\kappa^{2/3}$. Other exponents must arise in the actual quantum $M$-theory, since we will run into infinities which, when cut off in the quantum theory, must on dimensional grounds give anomalous powers of $\kappa$.

There are two principal goals of the rather complicated computation performed in this section:

1. To add to our confidence that the supersymmetric coupling of the vector multiplet on $M^{10}$ to the supergravity multiplet on $M^{11}$ does exist, by working out the classical construction of this coupling to the extent that it makes sense.

2. To exhibit the limits of the classical construction (beyond what is evident from the discussion of anomalies in Section 3) by showing how infinities arise in order $\kappa^{4/3}$.

In the computation, one can be guided to a certain extent by the ten-dimensional coupling of the vector and supergravity multiplets [2,3], to which our discussion must reduce at low energies in the appropriate limit. This gives clues to many of the terms that must be added to the Lagrangian and transformation laws. On the other hand, in the computation one definitely meets terms (involving $D_{117}$, for instance) that vanish upon dimensional reduction to ten dimensions but must be canceled to achieve local supersymmetry in eleven dimensions. Thus, the existence of the coupling we are constructed (and again, we believe that it only exists in full at the quantum level) goes well beyond ten-dimensional considerations.

We will carry out the computation in three stages: (i) first we complete the construction of the Lagrangian in order $\kappa^{2/3}$; (ii) then we look at some terms in order $\kappa^{4/3}$; (iii) finally we look systematically at all four-fermi variations in order $\kappa^{2/3}$. It might seem illogical to put (ii) before (iii). We have done this because (ii) is much simpler than (iii), and also gives an easy way to determine some of the transformation laws that are needed in (iii).

Of course, we cannot hope for a full determination of the structure. Apart from requiring a much fuller knowledge of the quantum mechanics of $M$-theory than one has, the full structure is presumably non-polynomial, like the $\alpha'$ expansion of perturbative string theory. Once one reaches a sufficiently high order in $\kappa$, one would require among other things a more complete knowledge of the low-energy expansion of $M$-theory in bulk (including higher derivative interactions) in order to proceed.
4.1. Some new interactions

The boundary interactions (that is, interactions supported at $x^{11} = 0$) that we discussed in Section 2 are the minimal super Yang-Mills action and the supercurrent coupling

$$L_0 = \frac{1}{2\pi(4\pi\kappa^2)^{2/3}} \int d^{10}x \sqrt{g} \left( \frac{1}{4} \text{tr} F_{AB} F^{AB} + \frac{1}{2} \text{tr} \bar{\chi} \Gamma^A D_A \chi + \frac{1}{4} \bar{\psi}_A \Gamma^{BC} \Gamma^A F_{BC}^a \chi^a \right).$$

(4.1)

These terms are all of order $\kappa^{2/3}$ compared to the supergravity action. There is precisely one more boundary interaction of the same order. To find it, one can look at the terms of order $F \chi G \eta$ in the supersymmetry variation of the Lagrangian. One source of such terms comes from the variation of the supercurrent interaction in (4.1) with $\delta\psi \sim G \eta$. Another source comes as follows. We found in Section 2 a correction (2.17) to the supersymmetry variation of $G_{\alpha\beta\gamma}$.

The $G_{\alpha\beta\gamma}$ term in the bulk supergravity action therefore picks up a new variation supported at $x^{11} = 0$; this term is again proportional to $F \chi G \eta$. These terms by themselves do not cancel. After a moderately lengthy computation, one finds that to cancel them one must add a new boundary interaction,

$$L_1 = \frac{\sqrt{2}}{96\pi(4\pi\kappa^2)^{2/3}} \int d^{10}x \sqrt{g} \bar{\chi}^a \Gamma^{ABC} \chi^a G_{\alpha\beta\gamma}. \quad (4.2)$$

This term – and the verification that the $F \chi G \eta$ terms cancel – is quite similar to an analogous term and verification in the ten-dimensional supergravity/Yang-Mills coupling.

This actually completes the construction of the Lagrangian in order $\kappa^{2/3}$ and verification of local supersymmetry up to four-fermi terms, whose analysis we postpone to the next subsection. Instead we turn to something that is of conceptual interest and still relatively simple.

In (4.2), we see an interaction in which $G_{\alpha\beta\gamma}$ is evaluated on $M^{10}$, that is at $x^{11} = 0$. On the other hand, in (2.17), we found a term in the supersymmetry variation of $G_{\alpha\beta\gamma}$ that is proportional to $\delta(x^{11})$. If we combine the two, that is if we vary $L_1$ according to (2.17), we get a result proportional to $\delta(0)$. This presumably should be interpreted as a linear divergence that is cut off somehow in the quantum $M$-theory. For our present purposes, though, we will be pragmatic, and without worrying about precisely what $\delta(0)$ means, we will attempt to formally cancel the $\delta(0)$ terms.

Obviously, to do this we need more sources of $\delta(0)$ terms. Since the term we want to cancel is proportional to $\chi \chi \chi F \eta$, there are two sources of terms that might cancel it. We could add to the gravitino variation an extra term $\delta\psi_A \sim \delta(x^{11}) \chi \chi \chi \eta$. When combined with the $\chi F \psi$ interaction in $L_0$, it gives another term of the general form $\delta(0) \chi^3 F \eta$. Finally, one could add to the Lagrangian a term $\delta(0) \chi^4$, which will again have a variation of the desired form. After another moderately long calculation, one
finds that the new terms required in the gravitino variation are
\[ \delta \psi_A = -\frac{1}{576\pi} \left( \frac{\kappa}{4\pi} \right)^{2/3} \delta(x^{11}) \bar{\chi}^a \Gamma_{BCD} \chi^a \left( \Gamma_A^{BCD} \eta - 6 \delta^{B}_{A} \Gamma^{CD} \eta \right) , \] (4.3)
and that the new interaction required is
\[ L_x = -\frac{\delta(0)}{96(4\pi)^{10/3} \kappa^{2/3}} \int_{M^{10}} d^{10}x \sqrt{g} \bar{\chi}^a \Gamma^{ABC} \chi^a \bar{\chi}^b \Gamma^{ABC} \chi^b . \] (4.4)
The \( \delta(0) \) presumably means that in the quantum theory this interaction is really of order \( \kappa^{-8/9} \), that is, of order \( \kappa^{10/9} \) relative to the original supergravity action.

We focussed here on a particular four-fermi term of relative order \( \kappa^{4/3} \) because it enabled us to exhibit in a simple fashion the "divergences" that appear in trying to construct the classical Lagrangian. In the next section, we look systematically at the four-fermi terms of order \( \kappa^{2/3} \).

4.2. Four-fermi terms in order \( \kappa^{2/3} \)

To this order, the structure of the Lagrangian and the supersymmetry variation of the fields can be determined by canceling terms \( \sim \chi \chi \bar{\psi} \eta \) with one covariant derivative acting on one of the fermi fields. There are two natural classes of such terms, depending on whether the derivative is normal to the boundary, or acts along the boundary.

First consider the class of terms containing the norm derivative, i.e. terms proportional to \( D_{11} \eta \) or \( D_{11} \psi_A \). Such terms clearly vanish upon the dimensional reduction to ten dimensions: from the point of view of a ten-dimensional low-energy observer, they only contain contributions from massive Kaluza-Klein modes of \( \psi_A \) and \( \eta \) that decouple from the low-energy modes as the radius of the eleventh dimension goes to zero. In the present case, however, the cancellation of such terms is not automatic, and will help us determine some new additions to the Lagrangian.

What are the possible sources of terms proportional to \( D_{11} \psi_A \) or \( D_{11} \eta \)? One source of such terms is generated by the correction (4.3) to the variation of \( \psi_A \) determined in the previous subsection. Since this correction is proportional to the delta function localized at the boundary, it will generate boundary four-fermi terms proportional to \( D_{11} \psi_A \) when applied to the bulk kinetic term of the gravitino. Given (4.3), this variation generates just one term, equal to
\[ \frac{1}{64\pi(4\pi \kappa^2)^{2/3}} \int_{M^{10}} d^{10}x \sqrt{g} \bar{\chi}^a \Gamma^{ABC} \chi^a \bar{\eta} \Gamma_{AB} D_{11} \psi_C . \] (4.5)
This term cancels exactly against a similar term that comes from the bulk variation of \( G_{ABC} \) in the interaction term \( L_1 \). No other terms with \( D_{11} \psi_A \) appear in the supersymmetry variation of the Lagrangian at this order.

\footnote{Here \( \delta(x^{11}) \) is understood as the delta function that transforms as a scalar under diffeomorphisms; this involves an implicit power of the 11-11 component of the frame field \( e_{11}^{11} \), and is required to match the transformation properties of \( \delta \psi_A \).}
As to the terms with $D_{11}\eta$, they have two sources among the terms already present in the Lagrangian. First of all, the bulk variation of $G_{ABC}{}^{11}$ in $L_1$ produces a term proportional to $\chi \Gamma^{ABC} \chi \overline{\psi}_A \Gamma_{BC} D_{11}\eta$. This term can only be canceled if we introduce a new interaction,

$$L_2 = \frac{1}{64\pi(4\pi\kappa^2)^{2/3}} \int_{M^{10}} d^{10}\chi \sqrt{g} \chi^a \Gamma^{ABC} \chi^a \overline{\psi}_A \Gamma_{BC} \psi_{11}. \quad (4.6)$$

(When $L_2$ is varied, $\delta \psi_A \sim D_A \eta$ leads to a $\chi \chi \psi_{11} D_A \eta$ term; this term cancels against the term of the same form that comes from the variation of $G_{ABC}{}^{11}$ in $L_1$. In addition, the variations of the bulk supergravity action $L_S$ and of $L_1$ give terms of the form $\chi \chi \eta D_A \psi_{11}$; happily, these terms cancel against each other.)

Another source of terms proportional to $D_{11}\eta$ is the variation of the spin connection in the gaugino kinetic term. This point requires a further explanation. In our treatment of eleven-dimensional supergravity in bulk, we have adopted the 1.5 order formalism, which means that the spin connection is first treated as an independent variable and set equal to the solution of its bulk equations of motion at the end of the calculation. One then need not worry about the supersymmetry variation of the spin connection, which vanishes by the equations of motion. In including the boundary interactions, we prefer to continue to use the “bulk” formula for the spin connection. This means that when the spin connection appears in boundary interactions, its supersymmetry variation must be included. (One could avoid this by extending the 1.5 order formalism to incorporate boundary corrections to the spin connection determined by the equations of motion, but we did not find that approach simpler.)

In practice, to the order we will calculate, the spin connection only appears in the gaugino kinetic term. Its variation produces an additional term proportional to $D_{11}\eta$ (plus other terms we will consider later). Canceling this $D_{11}\eta$ term requires a new term in the Lagrangian,

$$L_3 = \frac{1}{64\pi(4\pi\kappa^2)^{2/3}} \int_{M^{10}} d^{10}\chi \sqrt{g} \chi^a \Gamma_{ABC} \chi^a \overline{\psi}_D \Gamma^{DABC} \psi_{11}. \quad (4.7)$$

Now we will show that no other $\psi_{11}$ dependent four-fermi terms are generated in the Lagrangian at this order in $\kappa$, beyond those given by (4.6) and (4.7). To see this, we will proceed as follows. Supersymmetry variation of such additional four-fermi terms would produce additional terms proportional to $D_{11}\eta$. Notice that these $D_{11}\eta$ terms could only be canceled if there is a three-fermi correction to the variation of the gaugino, $\delta' \chi \sim \chi \psi_{11} \eta$, and the gaugino kinetic term is varied. We will prove our claim that no new $\psi_{11}$-dependent four-fermi terms arise in the Lagrangian at this order, by showing that there are no $\chi \psi_{11} \eta$ corrections to the supersymmetry variation of $\chi$. Upon variation of the gaugino kinetic term, such corrections would produce terms of the form

$$\chi \Gamma_A D_B \chi \overline{\psi}_{11} \ldots \eta \quad \text{and} \quad \chi \Gamma_{ABCDE} D_F \chi \overline{\psi}_{11} \ldots \eta. \quad (4.8)$$
(Here \ldots denotes all possible combinations of $\Gamma$ matrices.) There is no other possible source of such terms; a simple calculation shows that their cancellation requires the supersymmetry variation of $\chi^a$ to be independent of $\psi_1$, thus completing our argument.

Having canceled all terms with $D_{11}\psi$ and $D_{11}\eta$, we can determine the rest of the structure at this order in $\kappa$ by looking at cancellations of $\chi^a\chi^a \eta$ terms where now the ten-dimensional derivative $D_A$ acts on one of the four fermions. First we determine the correction to the supersymmetry variation of $\chi^a$, by canceling terms of the form

$$\chi^a D_B \chi^a \eta \ldots \psi_C \quad \text{and} \quad \chi^a D_{FG} \chi^a \eta \ldots \psi_G.$$  \hspace{1cm} (4.9)

Terms of this structure must cancel by themselves, since chirality and Fermi statistics do not allow one to use integration by parts to move the derivative away from the gauginos. There are two obvious sources of such terms: the variation of $e_A^a$ in the gaugino kinetic term, and the variation of $F_{AB}^a$ in the supercurrent coupling of $L_0$. As these do not cancel, one has to look for another source of such terms. We can add a correction, $\delta' \chi^a \sim \chi \psi_1 \eta$, to the supersymmetry variation of the gauginos. This correction will produce terms of the required form (4.9) from the variation of the gaugino kinetic term, and the precise form of $\delta' \chi^a$ will be determined from the cancellation of these terms. \(^7\) After a tedious calculation, one obtains

$$\delta' \chi^a = \frac{1}{64} \left( 7 (\bar{\psi}_A \Gamma_B \eta) \Gamma^{AB} \chi^a + 9 \left( \bar{\psi}_A \Gamma^A \eta \right) \chi^a - \frac{1}{2} \left( \bar{\psi}_A \Gamma_{BCD} \eta \right) \Gamma^{ABCD} \chi^a \right)$$

$$- \frac{5}{2} \left( \bar{\psi}^A \Gamma_{ABC} \eta \right) \Gamma^{BC} \chi^a + \frac{1}{24} \left( \bar{\psi}^A \Gamma_{ABCD} \eta \right) \Gamma^{BCDE} \chi^a. \hspace{1cm} (4.10)$$

The correction (4.10) to the supersymmetry variation of $\chi^a$ can be simplified considerably by the Fierz rearrangement formula, leading to

$$\delta' \chi^a = \frac{1}{4} \left( \bar{\psi}_A \Gamma_B \chi^a \right) \Gamma^{AB} \eta.$$ \hspace{1cm} (4.11)

This is exactly what one would have expected from the requirement that the total supersymmetry transformation of $\chi^a$ be "supercovariant". This also explains why no $\psi_1$-dependent corrections to the supersymmetry variation of $\chi^a$ arise – when varied, such terms would produce terms with $D_{11}\eta$, and supercovariance of the total supersymmetry variation of the gauginos would be spoiled.

Given the correction (4.10) to the supersymmetry variation of the gauginos, the terms that remain to be determined at this order in $\kappa$ are:

1. The correction to the supersymmetry variation of $\psi_1$; on the basis of chirality and Fermi statistics, this correction can only be proportional to $(\bar{\psi}^a \Gamma_{ABC} \chi^a) \Gamma^{ABC} \eta$.

\(^7\) In this and some of the following calculations, we need a Fierz rearrangement formula for chiral ten-dimensional fermions. All rules follow from the expansion of the product of two fermions $\xi$ and $\zeta$ on $M^{10}$ that obey $\Gamma_{11} \xi = \xi$ and $\Gamma_{11} \zeta = \zeta$:

$$\xi^a \bar{\xi}_b = - \frac{1}{32} \left( \xi^a \Gamma \bar{\xi}_b \right) \Gamma^A \alpha^a_{\beta} - \frac{1}{3} \left( \xi^a \Gamma_{ABC} \bar{\xi}_b \right) \Gamma^{ABC} \alpha^a_{\beta} + \frac{1}{120} \left( \xi^a \Gamma_{ABCD} \bar{\xi}_b \right) \Gamma^{ABCD} \alpha^a_{\beta}.$$
(2) Coefficients of all possible $\chi \chi \psi_A \psi_B$ terms in the Lagrangian; there are exactly four possible inequivalent terms of this structure. We will see momentarily that these additional four-fermi terms do appear in the Lagrangian.

We start by canceling terms $\sim \bar{\chi} \Gamma^{ABC} \chi \bar{\psi}_D \ldots \eta$ with $D_E$ on one of the four fermions. If the derivative is on one of the gauginos, we can now use integration by parts to move it to either $\psi_A$ or $\eta$. This leaves us with two classes of terms to cancel – one with $D_A \psi_B$, and one with $D_A \eta$. The $\chi \chi \eta D \psi$ terms do not get any contribution from the so far undetermined $\chi \chi \psi_A \psi_B$ terms in the Lagrangian, since at this order those will only contribute to $\chi \chi D \eta \psi$ terms. Hence, we can use cancellation of the $\chi \chi \eta D \psi$ terms to determine the correction to the supersymmetry variation of $\psi_{11}$; another lengthy calculation leads to \(^8\)

$$\delta' \psi_{11} = \frac{1}{576 \pi} \left( \frac{\kappa}{4 \pi} \right)^{2/3} \delta(x^{11}) \left( \bar{\chi} \Gamma^{ABC} \chi \right) \Gamma_{ABC} \eta. \quad (4.12)$$

Once $\delta' \psi_{11}$ has been determined, we can go on and calculate the $\bar{\chi} \Gamma^{ABC} \chi \bar{\psi} \ldots D_E \eta$ terms; their cancellation will determine the coefficients of the remaining four-fermi terms in the Lagrangian. (As in the case of the $\chi \chi \eta D \psi$ terms, there will be a non-zero contribution from the variation of the spin connection in the gaugino kinetic term.) After some additional algebra, one obtains

$$L_4 = \frac{1}{256 \pi (4 \pi \kappa^2)^{2/3}} \int d^{10} x \sqrt{g} \bar{\chi} \Gamma^{ABC} \chi \left( \frac{3}{2} \bar{\psi}_A \Gamma_B \psi_C - \bar{\psi}_A \Gamma_{BCD} \psi^D \right)$$

$$- \frac{1}{2} \bar{\psi}_D \Gamma_{ABC} \psi^D - \frac{12}{6} \bar{\psi}^D \Gamma_{DABCE} \psi^E). \quad (4.13)$$

This completes the construction of the boundary Lagrangian to order $\kappa^{2/3}$, which is thus equal to the sum $L = L_0 + L_1 + L_2 + L_3 + L_4$, with the individual terms given by (4.1), (4.2), (4.6), (4.7) and (4.13).

We could stop our discussion here; instead, however, one simple point seems worth making. It turns out that the four-fermi terms that we found at order $\kappa^{2/3}$ are exactly those implied by supercovariance to this order in $\kappa$, and can therefore be absorbed into the definition of supercovariant objects. This allows us to summarize the structure of all boundary terms in the Lagrangian at order $\kappa^{2/3}$ as constructed in this section, in the following succinct formula:

$$L = \frac{1}{2 \pi (4 \pi \kappa^2)^{2/3}} \int d^{10} x \sqrt{g} \left( -\frac{1}{4} \text{tr} F_{AB} F^{AB} - \frac{1}{2} \text{tr} \bar{\chi} \Gamma^A D_A (\hat{\Theta}) \chi \right.$$

$$- \frac{1}{8} \bar{\psi}_A \Gamma^{BC} \Gamma^A \left( F^a_{BC} + F^a_{BC} \right) \chi^a + \frac{\sqrt{2}}{48} \bar{\chi} \Gamma^{ABC} \chi^a \hat{G}_{ABC11} \right); \quad (4.14)$$

\(^8\) The only subtlety here is related to the cancellation of terms $\bar{\chi} \Gamma^{ABC} \chi^a \eta_1 A D_B \psi_C$, which gets a contribution from the variation of the spin connection in the gaugino kinetic term; recall the discussion of the 1.5 order formalism above.
Here the supercovariant spin connection $\tilde{\Omega}^{mn}_{A}$, Yang-Mills field strength $\tilde{F}^a_{AB}$, and field strength $\tilde{G}^{ABC}_{11}$ are given by

$$\tilde{\Omega}^{ABC} = \Omega^{ABC} + \frac{1}{2}\bar{\psi}^D \Gamma_{DABC}\psi^E - \frac{1}{4}\bar{\psi}^D \Gamma_{DABC}\psi_{11},$$
$$\tilde{F}^a_{AB} = F^a_{AB} - \bar{\psi}_{[A} \Gamma_{B]} \chi^a,$$
$$\tilde{G}^{ABC}_{11} = G^{ABC}_{11} + 3\sqrt{2} \left( \bar{\psi}_{[A} \Gamma_{BC]} \psi_{11} - \bar{\psi}_{[A} \Gamma_{B} \psi_{C]} \right).$$ (4.15)

(In accord with the version of the 1.5 order formalism used in this paper, the spin connection $\Omega^{ABC} \equiv e_m^A e_c^A \Omega^{mn}_A$ in (4.15) is a composite of $e_A^m$ and $\psi_A$, and solves the bulk equations of motion.)

Hence, we see that – just as in the case of pure eleven-dimensional supergravity [8] – no further four-fermi terms are generated at order $\kappa^{2/3}$ beyond those required by eleven-dimensional supercovariance. Of course, at higher orders in $\kappa$ we encounter additional four-fermi terms that are not explained in this way – the first example of such terms is the term $L_x$ of (4.4), which is quartic in the gauginos and appears at relative order $\kappa^{4/3}$.

References