Global structure of supergravity domain wall space-times

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The supersymmetric supergravity domain wall space-times considered by Cvetic and Griffies and Rey are shown to be incomplete and extensions are provided, in detail in the case of type II walls. The extensions, which are not unique, are conformal to a portion to the Einstein cylinder with certain points removed. The presence of the domain wall may thus be said to close up the space. The extensions contain Cauchy horizons with zero surface gravity and provided one makes no identifications they are causal. The key to the extensions is the correct understanding of lorentzian horospheres. Because the domain walls are, in a certain sense, at the end of the universe there may be zero modes bound to the domain walls which are connected with the Dirac–Singleton representations of SO(3,2).

1. Introduction

The nature and stability of vacuum solutions of supergravity theories may prove essential for our understanding of the problem of the breaking of supersymmetry in superstring theories. In some interesting recent work on this subject Cvetic and Griffies and Rey [1–4] have presented three types of space-times which they interpret in terms of static domain walls separating different vacuum regions in supergravity theories. The purpose of the present paper is to point out that these space-times are geodesically incomplete, to discuss their global structures and to give a maximal extensions for them. The coordinates which arise naturally in the work of Cvetic, Griffies and Rey are fairly unfamiliar to most people. They are related to certain timelike hypersurfaces in anti de Sitter space-time which generalize the horospheres of hyperbolic geometry. These lorentzian horospheres have also arisen in earlier work on partially supersymmetric solutions of the Einstein equations with a negative cosmological term. A fairly detailed discussion of their properties has therefore also been provided. Towards the completion of the work reported here discussions with Cvetic revealed that she, Davis, Griffies and Soleng had come to broadly similar conclusions. Their recent preprint [5] concentrates on the case of type I walls separating a Minkowski space-time domain...
from an anti de Sitter domain. In this paper I will concentrate on type II walls which separate two different anti de Sitter domains. A point of contrast with their work is that all the extensions I consider are simply connected and have no closed timelike curves. It is of course possible to make identifications which result in closed timelike curves, just as one can pass from the universal covering space-time of anti de Sitter, CADS\(_4\) to its quotient ADS\(_4\).

2. Domain walls

Before describing the work of Cvetic, Griffies and Rey I wish to make a few general remarks on the gravitational fields of domain walls. Naively one might expect their gravitational field to be static, invariant under the three-dimensional Poincaré group acting parallel to the wall and, as a consequence, conformally flat. Moreover, by the local SO(2,1) Lorentz invariance within the domain wall itself one expects the active gravitational mass to be negative and the wall to be repulsive. In fact (as we shall see in detail shortly) the Einstein field equations imply that if the energy density of the matter is never negative then the metric cannot be static (i.e. invariant under the three-dimensional Poincaré group). However it is possible to resolve this paradox and maintain the local SO(2,1) Lorentz invariance on the domain wall itself by assuming that the metric is invariant under the action of the isometry group of three-dimensional de Sitter space-time, SO(3,1).

The metric is thus assumed to be of what one might call quasi-Robertson–Walker form:

\[ ds^2 = A^2(z) \left[ dz^2 + d\Omega_5^2 \right], \tag{2.1} \]

where \( d\Omega_5^2 \) is the metric on flat Minkowski three-space if \( k = 0 \) and it is the metric on three-dimensional de Sitter spacetime with unit radius of curvature if \( k = 1 \). The relevant Einstein equation is the Friedman equation with appropriate adjustments of signs made to take into account that the variable \( z \) is spacelike rather than timelike as in the usual case. For a scalar field source the equation is

\[ -A'^2 + k = -\frac{8\pi}{3} \left( \frac{1}{2} G_{AB}(\phi^C) \phi'^A \phi'^B - A^2 V(\phi^C) \right), \tag{2.2} \]

where the prime denotes differentiation with respect to \( z \) and the scalar fields \( \phi^C \) take their values in some target manifold \( M \) with metric \( G_{AB}(\phi^C) \). In the supergravity case \( \{M, G_{AB}(\phi^C)\} \) is a Kähler manifold.

Far from the wall the derivative terms can be ignored and \( \phi^C \) should be close to one of its vacuum values, i.e. near a critical point of the potential function \( V(\phi^C) \). Evidently this is inconsistent with the assumption of staticity, i.e. of three-dimen-
sional Poincaré invariance (for which with $k = 0$), if the potential function is everywhere non-negative. What is possible if the potential energy function is everywhere non-negative is that $k = 1$ and the metric is not globally static. It is SO(3,1) invariant. By analytically continuing a suitable timelike coordinate on three-dimensional de Sitter space-time one can obtain a positive definite, i.e. riemannian metric, invariant under SO(4).

However, if at the critical points of $V(\phi^C)$ the potential energy function $V(\phi^C)$ is negative Cvetic, Griffies and Rey point out that it is possible to have static walls, though as we shall see, the Killing field can become lightlike on an horizon.

3. Thin Minkowski domain walls

In this section we will illustrate the general theory of the last section by the particular example of the thin Minkowski domain wall [6,7]. The main point is to understand the global structure of the space-time because there are many resemblances to the anti de Sitter case. The metric outside such a thin Minkowski domain wall is flat. The global structure of the space-time is that of two copies of the interior of a hyperboloid in Minkowski space-time glued across their common boundary. The manifold is time symmetric, spatially compact and contains cosmological event horizons.

Because the metric outside a thin domain wall is (in the thin-wall approximation) locally flat the apparent gravitational repulsion is entirely due to a global effect analogous to but different from that arising in the theory of cosmic strings. Each side of the domain wall is isometric with the interior of a timelike hyperboloid of constant proper radius in flat Minkowski space-time. The two portions of Minkowski space-time are identified across their common boundary – the hyperboloid. The resulting space-time is locally flat but because the domain wall accelerates with respect to the inertial frames in each portion, freely falling world lines appear to be repelled away from it. The interior of a hyperboloid is spatially compact and has proper radius

$$R = \frac{1}{2\pi G\sigma},$$

where $\sigma$ is the mass per unit area of the domain wall. The space-time is therefore spatially compact. At formation (when the domain wall is momentarily at rest) the universe consists of two flat euclidean balls of radius $R$ identified across their common boundary. The spatial radius increases with Minkowski time $X^0$,

$$\left(R^2 + (X^0)^2\right)^{1/2}.$$
To verify the remarks in the last paragraph it is sufficient to note that in the thin-wall approximation the metric of a domain wall is given locally by [3,4]

\[ ds^2 = e^{-2a|z|}(dz^2 - dt^2) + e^{2a(t-|z|)}(dx^2 + dy^2) \]  (3.3)

with \( a = 2\pi G \sigma \).

The two sides of the wall correspond to \( z \leq 0 \) and \( z \geq 0 \) respectively. The discontinuity in the \( z \)-derivative of the metric corresponds to a distributional stress-tensor of the form

\[ T_{\mu\nu} = \sigma \delta(z) \text{ diag}(1, -1, -1, 0). \]  (3.4)

One may check directly by calculating the geodesics of the metric (3.4) that the coordinate chart \( \{t, z, x, y\} \) is geodesically incomplete. The simplest way to complete it is to note that on either side of the wall one may introduce coordinates \( (X^0, X^1, X^2, X^3) \) by

\[ X^1 = x \exp[a(t - |z|)], \]  (3.5)
\[ X^2 = y \exp[a(t - |z|)], \]  (3.6)
\[ X^3 + X^0 = \frac{1}{a} \exp[a(t - |z|)], \]  (3.7)
\[ X^3 + X^0 = \frac{1}{a} \exp[-a(t + |z|)] - a(x^2 + y^2)\exp[a(t - |z|)], \]  (3.8)

the inverse of which is

\[ x = X^1/(X^3 - X^0), \]  (3.9)
\[ y = X^2/(X^3 - X^0), \]  (3.10)
\[ \frac{1}{a^2} \exp[-2a|z|] = (X^3)^2 - (X^0)^2 + (X^1)^2 + (X^2)^2, \]  (3.11)
\[ \exp[at] = (X^2 - X^0)/\sqrt{(X^3)^2 + (X^2)^2 + (X^2)^2 - (X^0)^2}. \]  (3.12)

As the notation suggests, the coordinates \( (X^0, X^2, X^2, X^3) \) are inertial coordinates for Minkowski space in which the metric (3.3) is manifestly flat, i.e. eq. (3.3) becomes

\[ ds^2 = -(dX^0)^2 + (dX^3)^2 + (dX^1)^2 + (dX^2)^2. \]  (3.13)
From eqs. (3.5)–(3.12) it is now completely clear that the metric form (3.3) is incomplete. The coordinate \((t, z, x, y)\) cover only that part of Minkowski space for which

\[
X^2 > X^0, \tag{3.14}
\]

\[
0 < (X^3)^2 + (X^1)^2 + (X^2)^2 - (X^0)^2 < \frac{1}{a^2}. \tag{3.15}
\]

In other words they cover that part of the interior of the hyperboloid which is to the past of the null plane

\[
X^3 - X^0 = 0, \tag{3.16}
\]

and outside the null cone

\[
(X^0)^2 = (X^3)^2 + (X^1)^2 + (X^2)^2. \tag{3.17}
\]

To obtain a complete space-time we must first add the portions of the interior of the hyperboloid to the future of the null plane and inside the null cone. If we then glue an identical copy (representing the other side of the domain wall) we shall obtain a complete spacetime. In fact one does not need the calculations of refs. [6,7] to verify that we have a solution of Einstein's equations representing a domain wall since it is immediately obvious that the familiar Israel [8] juncture conditions are satisfied. By Lorentz invariance the second fundamental form of the hyperboloid is obviously proportional to the induced metric. The discontinuity in the second fundamental form is also clearly twice the second fundamental form so (3.4) is satisfied.

Widrom [9] has attempted to go beyond the thin-wall approximation. His solution appears to be entirely consistent with our picture.

It is of some interest to examine the global structure of the space-time in more detail. The Carter–Penrose diagram is evidently obtained by gluing together two portions of the diagram for Minkowski space-time. The most important point is that observers on either side of the domain wall eventually go out of causal contact, i.e. there are cosmological event horizons [10]. This might lead to interesting thermal effects. It is important to note however that the cosmological event horizons are not Killing horizons and thus have no definite temperature. In fact the horizon on one side is just the future light cone of the point \((X^0, X^1, X^2, X^3) = (0, 0, 0, 0)\). The generators do not coincide with the orbits of any Killing field in the four-dimensional Poincaré group. In fact, the generators coincide with an orbit of the dilation subgroup of the conformal group. As a consequence, the area of the horizon increases as \(4\pi((X^1)^2 + (X^2)^2 + (X^3)^2)\) and is not constant as it would be if the event horizon were static.
Although there are event horizons in this space-time there are no Cauchy horizons. In fact from the causal point of view the space-time is similar to de Sitter space-time in being globally hyperbolic, with topology \( S^3 \times \mathbb{R} \).

4. The work of Cvetic, Griffies and Rey

All three types of space-times presented by Cvetic and Griffies and Rey [1–4] the metric is static, invariant under the three-dimensional Poincaré group acting parallel to the wall and conformally flat. The metric is given locally by

\[
\text{d}s^2 = A^2 \{ \text{d}z^2 + \text{d}x^2 + \text{d}y^2 - \text{d}t^2 \},
\]

where \( A = A(z) \) is a positive function only of \( z \), and the coordinates \( \{ t, x, y \} \) are allowed to range between \(-\infty \) and \(+\infty \). What distinguishes the three cases is the behaviour of the function \( A(z) \) and the range it adopts for the coordinate \( z \).

For type I space-times \(-\infty < z < +\infty \) while \( A(z) \) starts from a constant positive value near \(-\infty \) and decreases monotonically to zero as \( z \to +\infty \), such that \( zA(z) \) tends to a positive constant. In fact, in what follows the detailed behaviour of \( A(z) \) is not important, only its behaviour at the ends of the interval over which it is defined.

For type II space-times \(-\infty < z < +\infty \) as well but now \( A(z) \) decreases from a maximum value, which may with no loss of generality be taken to be at \( z = 0 \), in both positive and negative directions in such a way that \( |z|A(z) \) tends to a positive constant as \( |z| \to \infty \). The constant need not be the same for positive and negative values of \( z \). An especially interesting case is when \( A(z) \) is a symmetric function, \( A(z) = A(-z) \).

Finally, for type III space-times, \( z \) has a finite lower limit, which may with no loss of generality be taken to be at \( z = 0 \), so \( 0 < z < +\infty \) and \( A(z) \) decreases from an infinite value at \( z = 0 \) and tends to zero as \( z \to +\infty \) in such a way that \( \lim zA(z) \) is an (in general different) positive constant at the two ends of the interval \( 0 < z < +\infty \) of the interval.

To interpret this behaviour Cvetic and Griffies and Rey [1–4] note that if

1. \( A = 1 \) we obtain Minkowski space-time, while if
2. \( A = 1/z \) we obtain (one half of) anti de Sitter space-time of unit radius. More generally, if \( V_{\text{vac}} \) is a critical value of the function \( V(\phi^C) \), then an exact solution is given by

\[
A = \left( \frac{-8\pi V_{\text{vac}}}{3} \right)^{1/2} \frac{1}{z}. \tag{4.2}
\]

Therefore type I space-times were viewed as domain walls interpolating between
the flat Minkowski vacuum and an anti de Sitter vacuum while type II space-times were viewed as domain walls interpolating between two different anti de Sitter vacua, possibly with different radii. Type III space-times were also viewed as domain walls interpolating between two different anti de Sitter vacua. These interpretations received further support by a consideration of the behaviour of the scalar fields and their potentials which provide the energy–momentum tensor of the space-times. At either end of the interval the scalar field tends to a vacuum value, i.e. to a critical value of the potential energy function. In the supergravity case the potential energy function \( V(\phi^C) \) has a negative second derivative at its critical point, or more strictly the matrix of second partial derivatives or hessian,

\[
\frac{\partial^2 V}{\partial \phi^A \partial \phi^B},
\]

is not positive semi-definite. The eigenvalues of the hessian matrix, i.e. the squared masses \( m_i^2, \ i = 1, 2, \ldots \ \dim M \) are negative but the vacuum is nevertheless marginally stable since they attain the lower bound obtained by Breitenlohner and Freedman [11] \( m_i^2 \geq -9a^2/4 \), where \( a \) is the radius of curvature above which anti de Sitter space-time is stable against perturbations of a scalar field with potential which is convex upwards.

5. Geodesic incompleteness

The question I want to address in this section is the geodesic completeness. The case of the thin Minkowski domain wall illustrates how important this is. As Cvetic and Griffies and Rey correctly point out (see footnote in ref. [2]) the distance along the spacelike geodesic curves of constant \( t, x, y \) diverges as \( |z| \to \infty \). However this does not mean that the local form (4.1) is geodesically complete with respect to timelike and null geodesics. In fact it is not. If \( \lambda \) is an affine parameter for the world line of a particle of mass \( m \), conserved energy \( E \) and conserved transverse momentum \( p \) conjugate to \( t \) and \( x = (x, y) \) respectively one finds from the geodesic equations that

\[
d\lambda = \frac{A^2 \, dz}{\sqrt{E^2 - p^2 - m^2 A^2}}, \tag{5.1}
\]

\[
dt = \frac{E \, dz}{\sqrt{E^2 - p^2 - m^2 A^2}}, \tag{5.2}
\]

\[
dx = \frac{p \, dz}{\sqrt{E^2 - p^2 - m^2 A^2}}, \tag{5.3}
\]
If we set \( E = 0, \ p = 0 \) in (5.1) with \( m^2 \) negative we recover the result of Cvetic et al. [2] that the affine parameter \( \lambda \) diverges logarithmically as \( |z| \to \infty \) for those spacelike geodesics which lie in surfaces of constant \( t \). However, if neither \( E \) nor \( p \) vanishes, irrespective of whether the geodesic is spacelike, timelike or null, the affine parameter \( \lambda \) converges to a finite limit as \( |z| \to \infty \) and so it follows from (5.1) that the coordinate chart \{\( z, t, x, y \)\} cannot cover the entire space-time since world lines can reach both \( z = +\infty \) and \( z = -\infty \) in finite affine parameter \( \lambda \). The situation is reminiscent of the case of the thin Minkowski-domain walls analysed above for which the chart \{\( z, t, x, y \)\} is also incomplete. One therefore anticipates that the complete spacetime will be globally non-trivial. This expectation will be borne out later.

6. Horospheric coordinates

To see what is happening we return to the exact anti de Sitter metric which has

\[
A = \frac{1}{z}. \tag{6.1}
\]

The coordinates \{\( t, z, x, y \)\} are the generalization of horospherical coordinates in hyperbolic riemannian geometry to the lorentzian case of interest here [13]. The intrinsically flat hypersurfaces of constant \( z \) are the generalization to our case of the horospheres of hyperbolic geometry \(^*\). If one regards anti de Sitter space-time as a quadric \( \mathcal{S} \) in \( \mathbb{R}^{4,2} \) with coordinates \( Y^0, Y^1, Y^2, Y^3, Y^4 \) and metric \( \text{diag}(-1, +1, +1, +1, -1) \),

\[
(Y^0)^2 + (Y^4)^2 - (Y^1)^2 - (Y^2)^2 - (Y^3)^2 = 1, \tag{6.2}
\]

they are the intersections of the quadric \( \mathcal{S} \) with a one parameter family of null hyperplanes

\[
Y^3 + Y^4 = 1/z. \tag{6.3}
\]

The quadric \( \mathcal{S} \) is parameterized in terms of the coordinates \{\( t, z, x, y \)\} by (6.3) and

\[
Y^0 = t/z, \tag{6.4}
\]

\[
Y^1 = x/z, \tag{6.5}
\]

\[
Y^2 = y/z, \tag{6.6}
\]

\(^*\) The definition of conventional horospheres in hyperbolic geometry will be given later.
\[ Y^4 - Y^3 = z + \left( x^2 + y^2 - r^2 \right)/z. \] (6.7)

It is clear from symmetry of the problem that the horospheres are hypersurfaces of constant mean extrinsic curvature, indeed, their second fundamental form \( K_{ij} \) is proportional to the intrinsic metric \( g_{ij} \) on each horosphere:

\[ K_{ij} \propto g_{ij}. \] (6.8)

Reversing the sign of \( z \) in eqs. (6.3)–(6.7) reverses the signs of all of the coordinates \((Y^0, Y^1, Y^2, Y^3, Y^4)\) which is the antipodal map on the anti de Sitter quadric \( \mathcal{E} \). Thus in order to cover all of anti de Sitter space-time, rather than the quotient of anti de Sitter space-time by the antipodal map, one must consider at least two horospheric charts, one with \( z \) positive and the other with \( z \) negative. Vanishing \( z \) corresponds to the spacelike infinity of anti de Sitter space-time. On the other hand, the horosphere corresponding to \(|z| \to \infty\) and which, strictly speaking is not covered by our two charts, is a perfectly regular null hypersurface \( \mathcal{N} \) in anti de Sitter spacetime. It is in fact a degenerate Killing horizon of the Killing vector field \( \partial /\partial t \). Amongst other things this implies that both the expansion and shear of the null geodesic generators of \( \mathcal{N} \) must vanish.

7. Conformal embeddings

Both anti de Sitter space-time and the domain-wall space-time are conformally flat. According to a theorem of Schmidt [12] the Einstein static universe is the maximal conformally flat space-time, in that it is simply connected and conformally complete. Thus one expects to be able to imbed every simply connected conformally flat space-time, including these two examples, into it.

The Einstein static universe, or the Einstein cylinder, is the universal covering space of “compactified identified Minkowski space-time” and this may be regarded as the set of null rays in \( \mathbb{R}^{4,2} \). These lie on the light cone \( \mathcal{C} \) given by

\[ (Y^0)^2 + (Y^4)^2 - (Y^1)^2 - (Y^2)^2 - (Y^3)^2 - (Y^5)^2 = 0, \] (7.1)

with \((Y^0, Y^1, Y^2, Y^3, Y^4, Y^5)\) and \(\lambda Y^0, \lambda Y^1, \lambda Y^2, \lambda Y^3, \lambda Y^4, \lambda Y^5\) identified for all real non-zero values of \(\lambda\). The rays in \(\mathcal{C}\) satisfying \((Y^5)^2 = 1\) are the points of anti de Sitter space-time modulo the antipodal map. This correspondence is in fact a conformal one. Explicitly we set

\[ Y^0 = \sec \rho \sin \eta, \] (7.3)

\[ Y^4 = \sec \rho \cos \eta, \] (7.4)
The metric of anti de Sitter space-time then becomes

$$ds^2 = \Omega^2 \{-d\eta^2 + d\rho^2 + \sin^2\rho (d\theta^2 + \sin \theta^2 d\phi^2)\}, \quad (7.7)$$

where

$$\Omega = \frac{1}{\cos \rho}. \quad (7.8)$$

The metric inside the braces in (7.7) is that of the Einstein static universe which has topology $\mathbb{R} \times S^3$ and the allowed range of coordinates is $0 \leq \rho < \pi$, $-\infty < \eta < +\infty$, $0 \leq \phi < 2\pi$ and $0 \leq \theta < \pi$. Since the conformal factor $\Omega$ in (7.8) blows up at $\rho = \pi/2$ the allowed range for $\rho$ for the universal covering space of anti de Sitter space-time (CADS$_4$) is $0 \leq \rho < \pi/2$, just half of the Einstein cylinder. To obtain anti de Sitter space-time one must restrict $\eta$ to lie between 0 and $2\pi$. To obtain anti de Sitter space-time modulo the antipodal map one must identify the points $(\eta, \rho, \theta, \phi)$ with the points $(\eta + \pi, \rho, \pi - \theta, \phi + \pi)$.

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### 8. Horospheres in the Einstein universe

We now look at how the lorentzian horospheres embed into the Einstein universe. In terms of the coordinates $\{\eta, \rho, \theta, \phi\}$ we have

$$z^{-1} = \tan \rho \cos \theta + \sec \rho \cos \eta, \quad (8.1)$$

or

$$z = \frac{\cos \rho}{\sin \rho \cos \theta + \cos \eta}. \quad (8.2)$$

As claimed above, spacelike infinity (i.e. $\rho = \pi/2$) corresponds to $z = 0$, whereas $|z| \to \infty$ on an infinite number of images of the regular null hypersurface $\mathcal{N}$. Note that from (8.1) it follows that $z$ jumps from infinitely large positive values to infinitely large negative values as one crosses the Killing horizon $\mathcal{N}$.

The horospheres appear as time dependent two surfaces moving across the spatial sections of the Einstein static cylinder in a periodic fashion. To visualize
them it is helpful to project stereographically the three-sphere with one point removed onto flat euclidean three-space with coordinates
\[
(\bar{x}, \bar{y}, \bar{z}) = \left(2 \tan \left(\frac{\rho}{2}\right) \sin \theta \cos \phi, 2 \tan \left(\frac{\rho}{2}\right) \sin \theta \sin \phi, 2 \tan \left(\frac{\rho}{2}\right) \cos \theta\right).
\]

The metric of anti de Sitter space-time then becomes
\[
\text{d}s^2 = -\left(\frac{4 + \bar{r}^2}{4 - \bar{r}^2}\right) \text{d}\eta^2 + \frac{\text{d}\bar{x}^2}{(4 - \bar{r}^2)^2}.
\]

The spatial cross sections of anti de Sitter space-time are isometric to hyperbolic three-space and are confined to the interior of the sphere about the origin of radius 2.

In hyperbolic geometry this sphere is sometimes called “the absolute”. From the point of view of the conformal geometry it is a sphere which is left (absolutely) invariant by those conformal transformations of \(S^3\) which coincide with the isometry group of hyperbolic three-space. Historically the term arose in projective geometry where it is a the quadric which must be left (absolutely) invariant by those projective transformations of \(P_3(\mathbb{R})\) (i.e. linear transformations of \(\mathbb{R}^4\)) which reduce to Lorentz transformations. From the classical projective viewpoint the points outside the absolute may be identified with three-dimensional de Sitter space-time modulo the antipodal map. I shall not be using the projective viewpoint in this article, since I wish to view the points outside the sphere of radius 2 as another copy of hyperbolic three-space but I shall retain the use of the word absolute to denote the two-dimensional spatial boundary of hyperbolic three-space or its three-dimensional evolution in time.

Since
\[
z = \frac{4 - \bar{r}^2}{(4 + \bar{r}^2) \cos \eta + 4\bar{z}};
\]

surfaces of constant \(z\) are spheres in the \(\{\bar{x}, \bar{y}, \bar{z}\}\) coordinates centred at
\[
\left(0, 0, \frac{-2z}{1 + z \cos \eta}\right)
\]

with radius
\[
\frac{2\sqrt{2 + z^2 \sin^2 \eta}}{1 + z \cos \eta}.
\]
The sphere \( z = 0 \) corresponds to spacelike infinity of anti de Sitter space-time and coincides with the sphere of radius 2 centred at the origin, i.e. to the absolute. At each moment of conformal time \( \eta \), the Killing horizon \( \mathcal{H} \) is given by \( \| z \| \to \infty \) has the appearance of a spherical wavefront of radius

\[
2 \tan \eta
\]

centred on

\[
\left( 0, 0, -\frac{2}{\cos^2 \eta} \right).
\]

Each spherical wave front intersects the absolute orthogonally. Positive values of \( z \) lie inside both the absolute and the wave front. The coordinate \( z \) jumps from \( +\infty \) to \( -\infty \) as one crosses the wave front. The coordinate \( z \) also changes sign as one crosses the absolute. Thus for \( \eta/\pi \not\in \mathbb{Z} \) the three-sphere \( \eta = \) constant is partitioned into four regions, two in which \( z \) is positive and two in which it is negative. The boundaries of these regions are the absolute and the wave front. The wave fronts start off at \( \eta = 0 \) as the point \((0, 0, -2)\) on the absolute (so in fact no point of the spatial section \( \eta = 0 \) is covered by the horospheric coordinates) and expand so that at \( \eta = \pi/2 \) they have infinite radius and correspond to the plane \( \tilde{z} = 0 \). The wave fronts then shrink down on the point \((0, 0, 2)\) on the absolute at \( \eta = \pi \). The motion then reverses itself, they expand back outwards from \((0, 0, 2)\) occupying the plane \( \tilde{z} = 0 \) at \( \eta = 3\pi/2 \) and finally re-focus onto the original point \((0, 0, -2)\) after the elapse of one complete period. The complete system of spheres intersects any plane through the \( \tilde{z} \) axis in a set of “circles of Appolonius”.

9. Horospheres in Robertson–Walker coordinates

It is well known that a portion of anti de Sitter space-time may be cast in Robertson–Walker \( k = -1 \) form and it is of interest to describe the horospheres in those terms. If one sets

\[
Y^0 = \sin \tau, \tag{9.1}
\]
\[
Y^4 + Y^3 = \cos \tau/u, \tag{9.2}
\]
\[
Y^4 - Y^3 = \cos \tau(u + (x^2 + y^2)/u), \tag{9.3}
\]
\[
Y^1 = \cos \tau x/u, \tag{9.4}
\]
\[
Y^2 = \cos \tau y/u, \tag{9.5}
\]
then the anti de Sitter metric becomes
\[ ds^2 = -d\tau^2 + \cos^2 \tau \left( \frac{dx^2 + dy^2 + du^2}{u^2} \right). \] (9.6)

This metric does not cover all of anti de Sitter space-time because of the coordinate singularities at \( \tau = \pm \pi/2. \)

The spatial metric inside the brace in (9.6) is just the metric of hyperbolic three-space in conventional horospheric coordinates \( \{x, y, u\}, \) the two-surfaces of constant \( u \) being horospheres. The horospheres are a set of mutually equidistant surfaces each of which is intrinsically flat. They are thus in some ways the analogue of a set of parallel two-planes in ordinary euclidean three-space. There is no simple analogue in the geometry of the three-sphere Horospheric coordinates with \(-\infty < x < \infty, -\infty < y < \infty \) and \( u > 0 \) are almost complete for hyperbolic three-space. Spatial infinity or the absolute is given by \( u = 0. \) This is the analogue of the representation of hyperbolic two-space as the upper half of the complex plane.

Our course these two-dimensional horospheres are present in any Robertson–Walker \( k = -1 \) metric. In the special case of the anti de Sitter metric we have in addition lorentzian horospheres, i.e. the hypersurfaces surfaces of constant \( z \) whose form may be read off from the relation
\[ 1/z = \cos \tau/u. \] (9.10)

As is geometrically obvious, they intersect the surfaces of constant cosmic time \( \tau \) in two-dimensional horospheres. Spatial infinity, \( z = 0 \) corresponds to the absolute \( u = 0. \) The null hypersurface \( \mathcal{N} \) given by \( |z| \to \infty \) is not included in the \( \{\tau, x, y, u\} \) coordinate chart because to avoid the coordinate singularities at \( \tau = \pm \pi/2 \) we must restrict \( u \) to the interval \( -\pi/2 < \tau < \pi/2. \)

10. Supersymmetric gravitational waves in anti de Sitter space-time

In the absence of a cosmological term, gravitational waves are usually described in terms so-called pp-wave space-times. These space-times admit the existence of a covariantly constant Weyl spinor which can serve as a Killing spinor for the \( N - 1 \) Poincaré supergravity theory. These Ricci-flat solutions are the only ones admitting Killing spinors. If there is a negative cosmological term, then it may be shown [14–17] that the analogous solution are generalization of pp-waves. This generalization cannot be made (at least not in a straightforward way) if the cosmological term is positive. These metrics are the only Einstein metrics admitting Killing spinors of the \( N = 1 \) theory. Interestingly their construction is related to horospheres.
The metrics have the form
\[ ds^2 = \Omega^2 \{- du \ dv + dz^2 + dy^2 - H(u, z, y) \ du^2 \}, \]  
(10.1)
where
\[ \Omega = 1/z^2 \]  
(10.2)
and the \( u \)-dependence of \( H \) is arbitrary save that it satisfies
\[ \frac{\partial^2 H}{\partial z^2} - 2 \frac{\partial H}{\partial z} + \frac{\partial^2 H}{\partial y^2} = 0. \]  
(10.3)
The general solution is of the form
\[ H = \text{Re} z^2 \frac{\partial}{\partial z} \left( \frac{f(u, z + iy)}{z} \right), \]  
(10.4)
where the function \( f(u, z + iy) \) of one real and one complex variable is arbitrary save that it be holomorphic in the second variable.

The case \( H = 0 \) reduces to anti de Sitter space-time in horospheric coordinates with \( u = t - x \) and \( v = t + x \). Quite generally if \( u = \) constant we obtain null hypersurfaces which are the wave fronts of the gravitational wave. The metric on the null surfaces is degenerate. If one thinks of \( iz - y \) as the upper half plane it becomes the Poincaré metric and the curves \( z = \) constant are horocycles and \( z = 0 \) is the boundary of two-dimensional hyperbolic space. It seems that there may be some as yet unclear connection between supersymmetry and horospheres and horocycles. In that connection it is perhaps interesting to recall that horocycles and horospheres have a group-theoretical interpretation and also play an important role in group representation theory and harmonic analysis on some symmetric spaces.

11. Type II domain walls

It is now fairly obvious how to obtain the extensions of the type II supergravity domain wall spacetimes. One simply attempts to extend the metric and the scalar field \( \phi^A(x) = \phi^A(z) \) to as much of the Einstein cylinder as one can. This has a chance of working because in the original incomplete spacetime region covered by the coordinates \( \{t, x, y, z\} \) the metric (4.1) is of the form (7.7) but where now instead of (7.8) the conformal factor \( \Omega \) is given by
\[ \Omega = \frac{1}{\sin \rho \cos \theta + \cos \eta} \left( \frac{1}{\tan \rho \cos \theta + \sec \rho \cos \eta} \right). \]  
(11.1)
Such extensions though are in effect Cauchy horizons and as such cannot be unique. An additional source of non-uniqueness is the possibility of making identifications. If one chooses not to do this all extensions will be simply connected and will not contain any closed timelike or null curves (because the Einstein cylinder contains no such curves). Making identifications will, in general, lead to closed timelike curves. This is completely analogous to the usual situation with respect to anti de Sitter space-time $\text{ADS}_4$ and for example its universal covering space-time $\text{CADS}_4$.

There is some ambiguity in how the extension is carried out. This ambiguity may be illustrated by the Penrose-type diagram of the two-dimensional lorentzian space-time obtained by projecting onto the orbits of the Killing vectors $\partial/\partial x$ and $\partial/\partial y$. In other words, one regards the space-time as being plane symmetric. The original chart $\{t, z\}$ is a diamond whose sides are at 45 degrees to the horizontal and the wall the vertical diagonal. The right-hand sides are at $z = +\infty$ and the left-hand walls at $z = -\infty$. The sides are Cauchy horizons. One must add on further diamonds to obtain a completion and check that the resulting metric is regular. The corners of the diamonds are especially important because the chart $\{t, z, x, y\}$ breaks down there in a far more vicious way than it does on the Cauchy horizons at which $|z| = \infty$. The only safe procedure is to use a set of good global coordinates. Such a set is provided by the Einstein cylinder coordinates $\{\eta, \rho, \theta, \phi\}$ provided allowance is also made for the fact that these break down at the angular singularities $\theta = 0$ and $\theta = \pi$.

The following remarks owe much to a conversation with Stephen Hawking. From the diamond representation it is clear that two possibilities are either

(i) to translate the diamonds through one period along a directions null directions at 45 degrees or

(ii) to reflect them about their sides.

In case (i) the cosmological constant and scalar field will in general be discontinuous across the Cauchy horizon while in case (ii) they will be continuous.

The first case (i) corresponds to using eq. (11.1) over a large a part of the Einstein cylinder as possible. If the function $A(z)$ has the properties stated, the function $\Omega$ defined by (11.1) is bounded and non-vanishing and on almost all of the Einstein cylinder. However the conformal factor $\Omega$ blows up the focal points of the Killing horizon where the light cone of a point on the absolute reconverges. It also blows up at the original vertex. Thus it is necessary to remove the points $(\eta, \rho, \theta) = (0, \pi/2, \pi)$ and $(\eta, \rho, \theta) = (\pi, \pi/2, 0)$ together with their periodically repeated images with $\eta$ advanced by $2\pi$. The conform together with their periodically repeated images with $\eta$ advanced by $2\pi$. The conformal factor will in general not be continuous, and hence the metric will not be continuous, across the wave fronts since the value of the cosmological constant, i.e. of $\lim |z| A(z)$ will depend on the sign of $z$ as $|z| \to \infty$. It should be noted that even in those cases when the limits are the same so that $\Omega$ and hence the metric is continuous the scalar
field $\phi^A(x) = \phi^A(z)$ cannot be continuous, since it takes its value in two different vacua at infinite values of $z$ depending upon its sign. The space-time thus contains a shock wave across which the cosmological constant jumps from one negative value to another negative value. Since the null hypersurfaces across which the Ricci tensor jumps have zero expansion and shear, it follows from Penrose's theory of null shocks [18] that there is no distributional Ricci curvature on the shock and hence no distributional energy–momentum tensor. In fact one can always glue together two copies of anti de Sitter space-time, but with different cosmological constants, across a null hypersurface with zero shear and expansion. Near the wave front this extension looks exactly like that.

The alternative, and in many ways more natural way (ii) of extending the original static coordinate chart is such that both the cosmological constant and the scalar field are continuous. For the extension (i) described above the antipodal map is not an isometry,

$$\Omega^2(\rho, \theta, \phi, \eta) \neq \Omega^2(\rho, \pi - \theta, \phi + \pi, \eta + \pi).$$ (11.2)

The alternative procedure (ii) is to extend $\Omega$ so that the antipodal map is an isometry. That is to demand that

$$\Omega^2(\rho, \theta, \phi, \eta) = \Omega^2(\rho, \pi - \theta, \phi + \pi, \eta + \pi).$$ (11.3)

This procedure will ensure that both the cosmological constant and the scalar field are continuous. Roughly speaking the space-time (ii) looks like two different copies of the universal covering spacetime of anti de Sitter spacetime $\text{CADS}_4$, i.e. two halves of the Einstein cylinder with, in general, different cosmological constants glued by a domain wall across their boundaries or absolutes. Since covariance under the antipodal map plays such a significance in the physics of anti de Sitter spacetime it is perhaps not too surprising to see it differentiating the two cases here.

12. Periodic space-times

Although the metric we have constructed is locally static in terms of the $\{t, z, x, y\}$ coordinates in terms of the coordinates $\{\eta, \rho, \theta, \phi\}$, it is not static at all but rather periodic in the time coordinate $\eta$ with the anti de Sitter period. Space-times which are periodic in time are rather unusual and a number of theorems show that they cannot occur as solutions of the Einstein equations if the energy–momentum tensor enjoys certain positivity conditions.

For example consider a space-time diffeomorphic to $\mathbb{R} \times \Sigma$, where $\Sigma$ is a closed
spacelike three-manifold whose metric varies periodically with a time coordinate \( \eta \). The volume \( V(\eta) \) of \( \Sigma \) is therefore a periodic function of \( \eta \). The function \( V(\eta) \) must therefore have at least one minimum in each time cycle. The second derivative of \( V(\eta) \) is given by integrating the Raychaudhuri equation over \( \Sigma \). It turns out to equal

\[
- \int_{\Sigma} \left( R_{\alpha\beta} t^\alpha t^\beta + \sigma_{\mu\nu} \sigma^{\mu\nu} \right),
\]

where \( t^\alpha \) is the unit timelike normal of the three-surface \( \Sigma \) and \( \sigma_{\mu\nu} \) is the trace-free part of its second fundamental form. The strong energy condition states that

\[
R_{\alpha\beta} t^\alpha t^\beta \geq 0
\]

for all timelike vectors \( t^\alpha \). If it holds, it follows from (12.2) that the volume \( V(\eta) \) cannot have a minimum and hence that the metric cannot be periodic in \( \eta \). With a little bit more work one can rule out the case when the second derivative of \( V(\eta) \) vanishes by looking at the fourth derivative [19].

The strong energy condition is violated in the present case by the negative potential energy of the scalar field and so the theorem whose proof we sketched in the last paragraph does not apply. Moreover, because of the missing points the global geometry is not quite the same.

13. Conclusions

It is rather interesting that just as in the Minkowski case the effect of the domain wall is to make the spatial sections of the solutions close up on themselves. In other words, it compactifies the space. However, at the level we have studied it here, this is not a dynamical phenomenon, since the domain wall is assumed to be in existence for all time. It would be interesting to study the formation of domain walls and their subsequent effect on the geometry and topology of space.

Another interesting question is whether there are any zero-modes trapped on the domain wall. Since the domain wall is located on the Einstein cylinder at the spacelike infinity of anti de Sitter space-time it is tempting to speculate that these zero-modes may be related to the Dirac-Singleton representations of the group \( SO(3,2) \) which appear to be localized in some sense at infinity [20].

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References