100 years of Relativity Conference

Renormalization Group in Curved Space and the Problem of Conformal Anomaly

Ilya Shapiro

DF-UFJF, Juiz de Fora, MG

Based on collaboration with:
Manuel Asorey (Univ. de Zaragoza, Spain)
Eduard Gorbar (Bogolyubov Inst., Ukraine)
Joan Solà, (Univ. de Barcelona, Spain)

Abstract.

Renormalization Group **(RG)** is a powerful method for investigating Effective Action **(EA)** of quantum fields in curved space.

The formalism of RG in curved space is well known since 1984, but the application to the problems of cosmology and black hole physics requires more knowledge and represents a new interesting field of study.

The formalism of QFT in curved space-time has been mainly developed in 70th and 80th. (see, e.g. books: Birrell & Davies , Fulling , Buchbinder, Odintsov & Sh.)

Renormalizable theory includes **class.action of vacuum** with, at least, following terms:

$$S_{vac} = S_{HE} + S_{HD}\,,$$

$$L_{HE} = -\frac{1}{16\pi G}\,R - \Lambda\,,$$

$$L_{HD} = a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\,,$$

where $C = C_{\mu\nu\alpha\beta}$ is the *Weyl tensor,* $E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the *Gauss-Bonnet* integrand.

G, Λ, $a_{1,2,3,4}$ are **vacuum parameters**. On quantum level they are renormalized and become running constants, that is

depend on the energy scale.

Formulation of RG in curved space

is based on the Minimal Subtraction $(\overline{\mathbf{MS}})$ scheme of renormalization.

B.L. Nelson & P. Panangaden, (1982, 1984).
I.L. Buchbinder et al, (1982-**1984**-1990)
L. Parker & D.J. Toms, (1983-1985)
Introduction:
Buchbinder, Odinstsov & Sh. Effective Action in Quantum Gravity, (IOPP, 1992).

Brief review of the formalism:

We assume that the theory is multiplicatively renormalizable - hence it includes vacuum terms and nonminimal $\int \xi R\varphi^2$-term in the scalar field sector.

$$S = S_{vac} + S_{min} + S_{non-min}\,.$$

The theory has quantized fields φ, ψ, A_μ;

external metric $g_{\mu\nu}$ (semiclassical approach);

couplings: g, Yukawa h, scalar f; masses; ξ and vacuum G, Λ, $a_{1,2,3,4}$ parameters.

The **EA is μ- independent.**

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta_P\frac{\partial}{\partial P} + \int\gamma_\Phi\,\frac{\delta}{\delta\Phi}\right\}\Gamma = 0,$$

where $$\beta_P(n) = \mu\frac{dP}{d\mu}, \qquad \gamma_\Phi(n) = \mu\frac{d\Phi}{d\mu}.$$

For global rescaling according to dimension

$$(\Phi, P, \mu, l) \;\rightarrow\; (\Phi\cdot e^{-d_\Phi\tau},\, P\cdot e^{-d_P\tau},\, \mu\cdot e^{\tau},\, l\cdot e^{-\tau})\,,$$

the EA $\Gamma =$ const.

Together with the RG equation this gives

$$\frac{d\Phi}{d\tau} = (\gamma_\Phi - d_\Phi)\Phi\,, \qquad \frac{dP}{d\tau} = \beta_P - Pd_P\,.$$

The global scaling $g_{\alpha\beta} \rightarrow g_{\alpha\beta}\cdot e^{-2\tau}, \quad \tau\rightarrow\infty$ means a short distance limit.

The main problem: RG is supposed to hold at high energies, while **at low energies** we expect a **decoupling of massive fields**.

How can we see this in curved space?

One has to go **beyond the universal RG** developed in 80-th.

The global scaling $g_{\alpha\beta} \to g_{\alpha\beta} \cdot e^{-2\tau}, \quad \tau \to \infty$ means a short distance limit.

In flat space it is equivalent to rescaling momenta $p^2 \to p^2 \cdot e^{2\tau}$.
In curved space $R \to R \cdot e^{2t}$, $\Box \to e^{2t} \cdot \Box$ etc.

RG requires caution! Inflation = rescaling?

$$g_{\alpha\beta} \longrightarrow g_{\alpha\beta} \cdot e^{H \cdot t}\,.$$

This **doesn't correspond to RG**, because R =**constant**.

The main problem: RG is supposed to hold at high energies, while **at low energies** we expect a **decoupling of massive fields**.

How can we see this in curved space?
One has to go **beyond the universal RG** developed in 80-th.

Physical RG in curved space!?

Main physical motivations:

- **The CC running.**
- **Modified Starobinsky model**
- **Effective Low-Energy Quantum Gravity**

The QED example (flat space):

The 1-loop vacuum polarization is

$$-\frac{e^2\,\theta_{\mu\nu}}{2\pi^2}\int_0^1 dx\,x(1-x)\,\ln\frac{m_e^2+p^2x(1-x)}{4\pi\mu^2},$$

where $\theta_{\mu\nu}=(p_\mu p_\nu - p^2 g_{\mu\nu})$, μ is the parameter of dim. regularization.

$\beta^{\overline{\text{MS}}}$ is $\frac{e}{2}\mu\frac{d}{d\mu}$ acting on the formfactor of $\theta_{\mu\nu}$

$$\beta_e^{\overline{\text{MS}}} = \frac{e^3}{12\pi^2}.$$

β_e in the physical **mass-dependent scheme:** subtract at $p^2=M^2$ and take $\frac{e}{2}M\frac{d}{dM}$.

The UV limit $(M \gg m_e)$: $\beta_e = \beta_e^{\overline{\text{MS}}}$.

The IR limit $(M \ll m_e)$:

$$\beta_e = \frac{e^3}{60\pi^2}\cdot\frac{M^2}{m_e^2} + \mathcal{O}\left(\frac{M^4}{m^4}\right).$$

Appelquist & Carazzone, (1975)

Compared to $\beta_e^{UV} = \beta_e^{\overline{\text{MS}}}$, in the IR there is a **suppression (decoupling)** $\sim p^2/m_e^2$.

Does **decoupling** take place in **curved space?**

The first works on decoupling in gravity:
Gorbar & Sh. JHEP 02,06(2003); 02(2004)

Massive scalar field:

$$S_s = \frac{1}{2}\int d^4x\, g^{1/2}\, \Big\{(\nabla\varphi)^2 + m^2\varphi^2 + \xi R\varphi^2\Big\}\,.$$

Euclidean Effective Action is

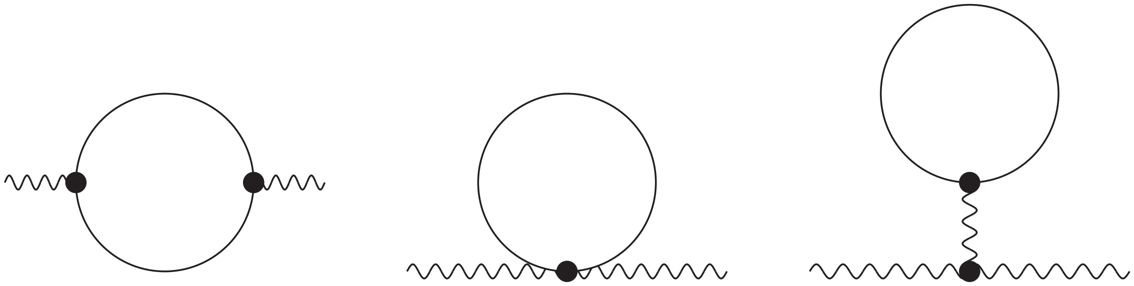
$$\Gamma[g_{\mu\nu}] = -\frac{1}{2}\,\mathrm{Tr}\,\ln\Big(-\nabla^2 + m^2 + \xi R\Big)\,.$$

Weak point: **No** covariant version of a mass-dependent renormalization scheme.

We can perform calculations only for the **linearized** gravity on the flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,.$$

Corrections to the graviton propagator:



The polarization operator must be compared to the tensor structure of the Lagrangians

$$L_{HE} = -\frac{1}{16\pi G}\,(R + 2\Lambda) \qquad \text{and}$$

$$L_{HD} = a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2 \,.$$

For the formfactors we find, e.g.

$$k_\Lambda = \frac{3m^4}{8\,(4\pi)^2}\,, \qquad k_R = \frac{m^2}{2\,(4\pi)^2}\,\tilde{\xi}\,,$$

$$k_1(a) = \frac{8A}{15\,a^4} + \frac{2}{45\,a^2} + \frac{1}{150}\,,$$

where $\tilde{\xi} = \xi - 1/6$,

$$A = 1 + \frac{1}{a}\ln\left|\frac{2-a}{2+a}\right|\,, \qquad a^2 = \frac{4\Box}{4m^2 - \Box}.$$

Result confirmed using covariant $\mathcal{O}(R^2)$
heat kernel solution
Avramidi, *Sov.J.Nucl.Phys.*49 (1989);
Barvinsky, Vilkovisky, *Nucl.Ph.*B282 (1990),

(properly generalized for a massive case).

Similar expressions were obtained for massive **fermions and vectors.** (JHEP 06-2003).

In the **mass-dep. scheme** for Weyl term

$$\beta_1 = -\frac{1}{(4\pi)^2}\left(\frac{1}{18a^2} - \frac{1}{180} - \frac{a^2-4}{6a^4}A\right).$$

Then

$$\beta_1^{UV} = -\frac{1}{(4\pi)^2}\frac{1}{120} + \mathcal{O}\left(\frac{m^2}{p^2}\right) = \beta_1^{\overline{MS}} + \mathcal{O}\left(\frac{m^2}{p^2}\right),$$

$$\beta_1^{IR} = -\frac{1}{1680\,(4\pi)^2}\cdot\frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right),$$

Appelquist & Carazzone Th. for **gravity!**

For the $\int \Box R$ - term coefficient a_3 :

$$\beta_3^{UV} = -\frac{1}{180(4\pi)^2} + \mathcal{O}\left(\frac{m^2}{p^2}\right),$$

IR limit shows AC-like decoupling

$$\beta_3^{IR} = -\frac{1}{1260\,(4\pi)^2}\,\frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right).$$

In a models with **broken SUSY**
β_3 **changes sign** between **UV** and **IR**
due to the decoupling of sparticles.

Consider the **conformal anomaly** – a typical phenomenon for the **massless** fields.

As an example, consider scalar field

$$L = \frac{1}{2}\left\{(\nabla\varphi)^2 + m^2\varphi^2 + \left(\tilde{\xi} + \frac{1}{6}\right)R\varphi^2\right\}.$$

As we already know

$$\bar{\Gamma}^{(1)} = \int \frac{d^4x\sqrt{g}}{2(4\pi)^2}\left\{\frac{m^4}{2}\left(\frac{1}{\varepsilon}+\frac{3}{2}\right) + m^2R\tilde{\xi}\left(\frac{1}{\varepsilon}+1\right)\right.$$

$$\left.+\frac{1}{2}C\left[\frac{1}{60\varepsilon}+k_1(a)\right]C + R\left[\frac{\tilde{\xi}^2}{2\varepsilon}+k_4(a)\right]R\right\},$$

where

$$\frac{1}{\varepsilon} = \frac{1}{n-4} + \ln\left(\frac{4\pi\mu^2}{m^2}\right).$$

In the $m = 0,\, \xi = 1/6$ limit we obtain

$$-\frac{1}{12\cdot 180(4\pi)^2}\int d^4x\, g^{1/2}\, R^2,$$

fitting perfectly with the conformal anomaly obtained by point-splitting (Christensen, 78), ζ (Cristley & Dowker, 76; Hawking, 77) and other methods

$$< T_\mu^\mu >= \frac{1}{180(4\pi)^2}\Box R + \dots .$$

This coincidence holds for ϕ and A_μ.
However, in the dimensional regularization the result is different (Duff, 77).

The standard statements are:

- The dimensional regularization and other regularizations (e.g. point-splitting) differ in the coefficient α' of the $\Box R$ term.

•• α' may be changed by adding a finite $\int d^4x\sqrt{g}R^2-$ term.

$$-\frac{2}{\sqrt{g}}\,g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\int d^4x\sqrt{-g}\,R^2=12\,\Box R\,.$$

Hence α' is arbitrary. **Is all this correct?**

Dimensional regularization:

The renormalized 1-loop Eff. Action is

$$\Gamma^{(1)}_{\text{ren}}\;=\;S_{vac}+\bar{\Gamma}^{(1)}+\Delta S_{vac}\,,$$

- S_{vac} is the classical conformal action,
- $\bar{\Gamma}^{(1)}$ non-renormalized quantum correction (divergent, conf. invariant & non-local),
- ΔS_{vac} is the local counterterm.

Anomaly: $$T=<T^\mu_\mu>=-\frac{2}{\sqrt{g}}\,g_{\mu\nu}\frac{\delta\Delta S_{vac}}{\delta g_{\mu\nu}}\,.$$

Indeed

$$\frac{\delta}{\delta g_{\mu\nu}} \int d^n x \sqrt{g}\, (\Box R) \,\equiv\, 0\,.$$

In the dim.reg. the $\int \Box R$-type counterterm does not contribute to the $\Box R$ term in T.

The $\Box R$ term comes from

$$2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} \int d^n x \sqrt{g}\, \frac{C^2(4)}{n-4}\bigg|_{n\to 4} \;=\; (C^2 - \frac{2}{3}\Box R).$$

Hence $\quad T = -(2\beta_1/3)\cdot \Box R \qquad$ (Duff, 77).

But **why we have to choose**

$$\Delta S_{vac} = \int \sqrt{g}\, C^2(4) \qquad \textbf{??}$$

E.g. for $C^2(n)$ there is no $\Box R$ term**!**

For $C^2(n+\gamma\cdot[n-4])$ we meet $\alpha' \sim \gamma$.

Therefore:

- Dimensional. reg. leavs α' **arbitrary**.

- No contradiction with other regularizations!

- This arbitrariness is $\equiv$ adding $\int R^2$ **to the classical action.**

Can we observe an arbitrariness in α' in some other regularization?

Consider covariant Pauli-Villars reg. known from YM & SC (Slavnov et al, Asorey et al).

PV impies introducing massive auxiliary fields with distinct Grassmann parity which cancel all (quartic, quadratic, log.) divergences in a covariant way

$$S_{\text{reg}} = \sum_{i=0}^{N} \int d^4x\sqrt{g}\,\Big\{(\nabla\varphi_i)^2 + (\xi_i R + m_i^2)\varphi_i^2\Big\}\,.$$

The physical scalar field $\varphi \equiv \varphi_0$ is conformal $\xi = 1/6$, $m_0 = 0$ and bosonic $s_0 = 1$.

The Pauli-Villars regulators φ_i $(i = 1, \dots, N)$ are massive $m_i = \mu_i M \neq 0$ and can have either bosonic $s_i = 1$ or fermionic statistics $s_i = -2$.

After UV limit $M \to \infty$ we arrive at the vacuum Eff. Action. The calculation is based on our result for the EA of the massive scalar.

We assume that the Pauli-Villars regulators might have non-conformal couplings $\xi_i \neq \frac{1}{6}$.

The regularized effective action is

$$\bar{\Gamma}^{(1)}_{\text{reg}} = \lim_{\Lambda\to\infty} \sum_{i=0}^{N} s_i \bar{\Gamma}^{(1)}_{\text{i}}\left(m_i, \xi_i, \Lambda\right) ,$$

where Λ is an auxiliar momentum cut-off.

All divergences in the ultraviolet cut-off Λ are **canceled** out due to the PV conditions

$$\sum_{i=1}^{N} s_i = -1 ; \qquad \text{(quartic divs.)}$$

$$\sum_{i=1}^{N} s_i \mu_i^2 = \sum_{i=1}^{N} s_i \left(\xi_i - \frac{1}{6}\right) = 0 ; \qquad \text{(quadratic)}$$

$$\sum_{i=1}^{N} s_i \mu_i^4 = \sum_{i=1}^{N} s_i \left(\xi_i - \frac{1}{6}\right)^2 = 0 ; \qquad \text{(log.)}$$

A simple solution of these equations is

$$s_{1,2,3,4,5} = (1, 4, -2, 2, -2) ,$$

$$\mu^2_{1,2,3,4,5} = (4, 3, 1, 3, 4) , \qquad \xi_i = \mu_i^2 + 1/6 .$$

The conformal anomaly in the covariant PV regularization is given by

$$T = \frac{1}{(4\pi)^2}\Big[\frac{1}{180}E - \frac{1}{120}C^2 + \Big(12\delta - \frac{1}{180}\Big)\,\Box R\Big],$$

where

$$\delta = \sum_{i=1}^{N} s_i\,(\,\xi_i - \frac{1}{6}\,)^2\,\ln\,\mu_i^2\,,$$

Again, we meet an ambiguity due to the $\int\sqrt{g}R^2$-term to the *classical* vacuum action. The only way to fix this ambiguity is through a special renormalization condition.

However, as we know from the literature *S.J. Hathrell, Ann.Phys.* **139**;**142** *(1982).* in the interacting theory the $\int\sqrt{g}R^2$-type counterterm emerges anyway at higher loops and, therefore, this extra condition does not lead to essential modification of the theory structure.