No question of how temperature transforms under boosts

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Motivation

- The problem of answering the question of "how temperature transforms under boosts" has been with us since Einstein's 1907 article.
- This question represented one of Einstein's worries toward the end of his life, as captured in an Einstein-von Laue correspondence of 1952-53.
- After almost a century we steel find articles favoring opposite answers: Aldrovandi and Gariel (1992) VS. Fenech and Vigier (1996).
- The commemoration of the first centennial of the Special Relativity Theory, seems to be a good time to reach a consensus about this question and sketch a pedagogical review which lay the ghost raised by this puzzle.

The Planck-Ott Controversy

With respect to an observer moving with relative velocity v a body is hotter or cooler than with respect to e co-moving observer?

- According to Einstein, Planck, Tolman, Pauli, and von Laue, among others (1907 to 1961),

$$T' = T\sqrt{1 - v^2}$$

- According to Ott (1963) and Arzeliès (1965), for instance,

$$T' = T/\sqrt{1 - v^2}$$

- While Landsberg (1966) raised the possibility that,

$$T' = T$$

- In the eighties Israel and Stewart formulate a consistent Relativistic Thermodynamics.

Related Questions

The answer to the question of "how temperature transforms under boosts" is intrinsically related with the answer to other questions:

- What a Relativistic Thermodynamical Theory should look like?
- What is the physical meaning of the relative temperature?
- How is this relative temperature measured?

What the Relativistic Thermodynamical Theory should look like?

In Relativity observables associated with some particular measurement will depend in general on:

- The state of the system and
- the state of movement and orientation of the observer,

but not on

- the particular coordinate system chosen to describe the problem.

Tensors (an spinors too) comply with the last condition so they are ideals for describing these states.

The state of movement of an observer endowed with gyroscopes is described by a future-directed timelike 4-vector (its 4-velocity) tangent to its trajectory in space time at every event of its history. The orientation of its gyroscopes are described by tree spacelike 4-vectors orthogonal to its 4-velocity.

Parallel with Poisson Equation

Non Relativistic	$\nabla^2\phi = 4\pi\rho$	ϕ: Newtonian gravitational potential
	$\downarrow$	ρ: mass density
Relativistic	$G_{\mu\nu} = 8\pi T_{\mu\nu}$	$G_{\mu\nu}$: Encodes the geometry of the spacetime
		$T_{\mu\nu}$: stress-energy momentum tensor

The relativistic energy density as measured by some observer with four-velocity v^μ, $v^\mu v_\mu = -1$, will be given by $T_{\mu\nu}v^\mu v^\nu$.

$$\exists_{v_0^\mu} \Big| \rho \approx \lim_{v^\mu \to v_0^\mu} T_{\mu\nu}v^\mu v^\nu$$

Question: How the Newtonian potential transforms under boosts?

Answer: Pointless!

The problem of making Thermodynamics compatible with Relativity is not simply a matter of determining "how the 'usual' thermodynamical variables transform under boosts", but of reformulating Thermodynamics as such, in which context the very query may not be well posed.

What is the physical meaning of the relative temperature and how it can be measured?

Let us consider a plain thermal bath in Minkowski spacetime with temperature T, as defined by static inertial observers in the bath's rest frame S. How some particular thermometer moving with velocity v with respect to S behaves?

If a spectral analysis is performed by some apparatus at rest in s, It will indicates that the photon number density distribution is:

$$n(\omega, T) d^3\mathbf{k} = \frac{\omega^2}{2\pi^2(e^{\omega/T} - 1)} d^3\mathbf{k}, \tag{1}$$

where $\mathbf{k} \equiv (k_x, k_y, k_z)$ is the photon three-momentum and $\omega = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

Note that Eq. (1) is characterized by a single parameter T.

"Honest" thermometers: Devices designed to measure T when lying at rest and in equilibrium with the system.

Matsas and Costa (1995)

Photon number distribution, in the moving reference frame S'

In the twenties Pauli showed that

$$n'_{\theta'}(\omega', T'_{\theta'})d\omega' d\Omega' = \frac{{\omega'}^2}{2\pi^2(e^{\omega'/T'_{\theta'}} - 1)} d\omega' d\Omega' \; . \tag{2}$$

This result involves a "directional" temperature:

$$T'_{\theta'}(T, v, \theta') = \frac{T\sqrt{1-v^2}}{1 - v\cos\theta'} . \tag{3}$$

For the moving observer the "thermal bath" is not isotropic.

As shown by Matsas and Costa (1995) deferent "honest" thermometers measure (by construction) the same magnitude when they are at rest in the thermal bath but they will in general disagree when moving with respect to it.

An Example

Two "honest" thermometers (both give T when at rest in S):

$$T'_{\rm aver1}(T,v) \equiv \langle T'_{\theta'}\rangle = \frac{1}{4\pi}\int T'_{\theta'} d\Omega' \tag{4}$$

$$T'_{\rm aver2}(T,v) \equiv \langle T'^4_{\theta'}\rangle^{1/4} = \left(\frac{1}{4\pi}\int T'^4_{\theta'} d\Omega'\right)^{1/4} , \tag{5}$$

When they are set lying at rest in S', they measure quite distinct values:

$$\begin{aligned} T'_{\rm aver1}(T,v) &= \frac{T}{2v\gamma}\ln\frac{1+v}{1-v} \\ T'_{\rm aver2}(T,v) &= T\left(\frac{1+v^2/3}{1-v^2}\right)^{1/4} . \end{aligned} \tag{6}$$

So, in this simple case it is not clear how a possible relative temperature could be unambiguously defined and measured.

Parallel with the mass of a classical point particle

Rest mass m is the only proper parameter necessary to characterize a classic point particle (as well as fermionic and bosonic quantum particles described by Dirac and Klein-Gordon equations, respectively).

The particle four-momentum is $p^\mu = mu^\mu$, where u^μ is the particle four-velocity satisfying $u^\mu u_\mu = -1$. An observer with four-velocity v^μ will measure the following value for the energy of the particle:

$$E' = -p_\mu v^\mu = m\gamma$$

with $\gamma = 1/\sqrt{1 - v^2}$, where $v = |\mathbf{v}|$ is the absolute value of the relative three-velocity $\mathbf{v}$ between the observer and the particle.

Notice that in the particular case where the particle is at rest with the observer, we obtain $E' = m$. Analogously, the only proper parameter necessary to characterize a black body is its (proper) temperature T.

Temperature in the classical theory of thermodynamics

In the classical theory of thermodynamics the extensive parameters: internal energy U, particle numbers N and volume V are sufficient to characterize the macroscopic equilibrium states of a simple system with only one chemical component. Then, the entropy, $S = S(U, N, V)$, is defined as a *continuous*, *differentiable* and *monotonically increasing function of* U and contains all the thermodynamical information about the system. $S = S(U, N, V)$ is called the *fundamental relation of thermodynamics.* The thermodynamical intensive parameters *temperature* T , *pressure* P and *chemical potential* μ are defined as

$$T \equiv \left(\frac{\partial S}{\partial U}\right)_{V,N}^{-1} , \tag{7}$$

$$P \equiv T\left(\frac{\partial S}{\partial V}\right)_{U,N} , \tag{8}$$

$$\mu \equiv -T\left(\frac{\partial S}{\partial N}\right)_{U,V} . \tag{9}$$

The entropy of any system is assumed to vanish at zero temperature.

In the so called *Euler form*, the fundamental relation has the form

$$S = \left(\frac{1}{T}\right) U + \left(\frac{P}{T}\right) V - \left(\frac{\mu}{T}\right) N . \tag{10}$$

One can show that the entropy density $s \equiv S/V$ can be cast in the form $s = s(\rho, n)$ where $\rho \equiv U/V$ and $n \equiv N/V$, and will carry the same thermodynamical information as Eq, (10).

Temperature in a Relativistic Thermodynamical Theory

This scheme can be covariantly generalized [Israel and Stewart (1980-86)] for a simple continuous medium with only one chemical component in an open region $\mathcal{O}$ of a spacetime with metric $g^{\mu\nu}$. In this case, instead of U, N and V, the *primary variables* that will characterize completely the macroscopic *equilibrium states* of the system are the stress-energy tensor $T^{\mu\nu}$, the particle four-current N^{μ} associated with the only chemical component and a four-velocity field u^{μ} which will be assumed to coincide with the four-velocity of the fluid elements. The stress-energy tensor $T^{\mu\nu}$ is symmetric, $T^{[\mu\nu]} = 0$, conserved, $T^{\mu\nu}{}_{;\nu}$ and satisfy the condition[1], $T^{\mu\nu}\omega^{\mu}\omega^{\nu} > 0$ for any non spacelike ω^{μ}. The particle current N^{μ} is also conserved ($N^{\nu}_{;\nu} = 0$) and u^{μ} is a smooth four-vector field satisfying $u^{\mu}u_{\mu} = -1$. Note that u^{μ} contains the information about the rates of rotation, deformation, expansion and shear of the fluid

Here the *entropy* will be represented by a four-vector S^{μ} and the fundamental relation, $S^{\mu} = S^{\mu}(T^{\mu\nu}, N^{\mu}, u^{\mu})$ will contain again the complete thermodynamical information about the system.

We recall that, $\rho \equiv T^{\mu\nu}u_{\mu}u_{\nu}$, $n \equiv -N^{\nu}u_{\nu}$ and $s \equiv -S^{\nu}u_{\nu}$ are the energy density, the particle density and the entropy density, as measured by observers with four-velocity u^{μ}.

[1]This guarantees the existence of a timelike eigenvector of $T^{\mu\nu}$.

The fundamental relation in the Euler form (10) can be covariantly generalized by imposing that $s = s(\rho, n)$ is continuous, differentiable, grows monotonically with ρ, has the same formal dependence with respect to ρ, and n at every space time point, and that

$$S^\mu = -T^{\mu\nu}\beta_\nu - \alpha N^\mu + P\beta^\mu \ , \tag{11}$$

where

$$\beta^\mu \equiv \beta u^\mu \ , \qquad \beta \equiv \frac{\partial s}{\partial \rho} \geq 0 \ , \tag{12}$$

$$\alpha \equiv -\frac{\partial s}{\partial n} \ , \tag{13}$$

$$P = \beta^{-1}\left[s(\rho, n) - \frac{\partial s}{\partial n} n\right] - \rho \ . \tag{14}$$

Here $\beta = 1/T$ is the inverse of temperature T, $\mu \equiv \alpha/\beta$ is the relativistic chemical potential and P is the thermodynamical pressure where we emphasize that T, μ and P are assumed to be measured by observers at rest with the fluid elements with four-velocity u^μ. The function $s(\rho, n)$ vanishes in the state for which $T = 0$.

Note that when Eq. (11) is contracted with u^μ and multiply by the scalar field V representing the volume of N particles as measured by observers with four-velocity u^μ it reproduces the Euler equation (10).

Comparing both theories

Comparing the fundamental relation in its classic and relativistic versions, we could say that the inverse of temperature (T^{-1}) is formally replaced, in the relativistic theory, by a four-vector β^μ whose modulus $\sqrt{\beta^\mu \beta_\mu} = T^{-1}$ gives information about the thermodynamical temperature of the system and whose direction $(\beta^\mu \beta_\mu)^{-1/2} \beta^\mu$ gives the four-velocity of the observers with respect to whom this temperature makes sense.

Although, in the classical thermodynamics it turns out that, the *thermal equilibrium* is associated with *equality and homogeneity of the temperature*; in the relativistic context it is found that the inverse temperature β, as defined in Eq. (12), is no more a homogeneous parameter when the system is in global thermal equilibrium. Instead the vector β^μ satisfies the Killing equation $\beta_{\mu;\nu} + \beta_{\nu;\mu} = 0$ which implies that observers with the four-velocity u^μ of the fluid see the gravitational field as stationary and that the temperature they measure varies with position according to Tolman's law $T\sqrt{-g_{00}} = \text{const}$. This means that temperature is higher in regions of lower gravitational potential so that the redshift of the radiation propagating from a point of intense gravitational field to a point of weak gravitational field will exactly compensate the temperature difference between these tow points and, this way, maintain the equilibrium.

Final Comments

Temperature should be seen as a parameter which characterizes a thermal bath. As such, it is a thermodynamical variable in the usual Thermodynamical Theory associated with observers at rest with the system.

In the context of a Covariant Thermodynamics, (the inverse of) temperature is formally replaced by a four-vector and the question "how temperature transforms under Lorentz boosts" becomes meaningless.

Despite the nonexistence of a transformation law for temperature, the legitimate observables defined by observers moving in the thermal bath can be calculated.