Universidade de São Paulo
Instituto de Física

O Operador de Tempo na Mecânica Quântica e uma Extensão para o Caso Relativístico

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Dissertação de mestrado apresentada ao Instituto de Física da Universidade de São Paulo, como requisito parcial para a obtenção do título de Mestre(a) em Ciências.

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São Paulo
2023

University of São Paulo
Physics Institute

The Operator of Time in Quantum Mechanics and an Extension to the Relativistic Case

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Dissertation submitted to the Physics Institute of the University of São Paulo in partial fulfillment of the requirements for the degree of Master of Science.

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São Paulo
2023

Agradecimentos

Esse trabalho foi o fruto do esforço de muitas pessoas.

Gostaria, em primeiro lugar, de dedicar este trabalho aos meus pais. Ao meu pai, Amauri Garcia Pereira, gostaria de agradecer por, mesmo após muitos anos de seu falecimento, ainda continuar me ensinando o significado de dedicação, de amor pelo trabalho, e principalmente, do amor e respeito àqueles que te acompanham em sua jornada. À minha mãe, Célia Cavalheri Pereira, gostaria de agradecer pelo apoio incansável e interminável, para o qual palavras não fazem jus, dado durante o que foi não somente um mestrado árduo, mas também uma pandemia sem precedentes, além das inúmeras conversas e conselhos dados durante esse período.

Gostaria também de dedicar este trabalho aos meus irmãos, Ana Luiza Cavalheri Pereira e Mateus Cavalheri Pereira, e também à minha cunhada Flávia de Andrade Pereira, que apesar de estarem longe se mostraram confiantes de que eu conseguiria terminar este trabalho até mesmo nos momentos em que eu não tinha certeza disto.

Este trabalho também não poderia ter sido realizado sem meus amigos e colegas nesta jornada acadêmica, Eduardo Higa, César Pinardi, Leonardo Barreto de Oliveira Campos, Felipe Machado Salvador, Giancarlo Ammirati Rodrigues Tosto, Rafael Funicelli Spaziani e Daniel Sampaio Beltrão, que me ajudaram a descontrair e a seguir em frente, compartilhando suas experiências em suas respectivas áreas e mostrando que os percalços estão presente para todos, e que no fim das contas não são tão diferentes para todos nós.

Gostaria de agradecer aos meus colegas de trabalho e amigos Ivan Romualdo de Oliveira e Felipe Dilho Alves, que com inúmeras discussões sobre assuntos extremamente ricos e intrincados, me possibilitaram explorar este trabalho de formas inovadoras.

Por fim, gostaria de agradecer ao meu orientador, João Carlos Alves Barata, que durante todos estes anos foi uma fonte inesgotável de conhecimento, clareza, inspiração e paciência frente a um trabalho que é impossível de ser feito sem estas características.

I shall not be alive half a decade hence, [...] and yet it is of overpowering concern to me. Call it idealism. Call it an identification of myself with that mystical generalization to which we refer by the term, 'humanity'.

Isaac Asimov, *Foundation*

Abstract

The question of time in quantum mechanics has been a topic of intense discussion since its inception. Dating back to the works of the founders of the subject, the notion of time and how it fit in the theory was already a point with a lack of a formal explanation, especially in aspects such as the uncertainty principle and canonical commutation relations. A theorem by Pauli set in stone that any attempt at trying to understand time with the formalism in use at the time for quantum mechanics was impossible. Through studies on the logical basis of the subject and establishing different sets of axioms, more modern works were able to relax sufficiently the impositions on the concept of observables so that time could then be understood as a 'generalized' observable. Not only that, but it was shown that the correct way to interpret eventual time measurements was through the formalism of Positive Operator Valued Measures. Latest works then were focused in the direction of constructing either direct measures and time operators, or, in the case that was followed throughout this work, to establish a procedure to construct measures. This work was then an in-depth study of the work done by Brunetti and Fredenhagen, with the intent to do the first steps in generalizing their ideas by establishing their method to be covariant with respect to Poincaré transformations.

Keywords: Quantum Mechanics, Positive Operator Valued Measures, Haar Integrals, Group Theory

Resumo

A questão do tempo em mecânica quântica tem sido um tópico de intensa discussão desde o nascimento da mesma. Até mesmo nos trabalhos mais antigos dos fundadores do assunto, o tempo já apresentava dificuldades em como ser tratado formalmente dentro da teoria, especificamente levando em conta assuntos como o princípio de incerteza e as relações de comutação canônicas. Um teorema por Pauli mostrou de forma concreta que o tempo não era compatível com o formalismo usual da mecânica quântica. Por estudos nas bases lógicas do assunto, e utilizando um sistema axiomático modificado, trabalhos mais recentes relaxaram as condições impostas sobre um observável o suficiente, de forma que o tempo possa ser entendido como uma forma de observável 'generalizado'. Não somente isto, mas também foi mostrado que o modo correto de se tratar medidas temporais era através do formalismo de Medidas com Valores em Operadores Positivos. Os trabalhos mais recentes foram então direcionados para a construção de tais medidas diretamente e de seus operadores temporais associados, ou então, como no caso deste trabalho, no sentido de se estabelecer um método geral para a construção de tais medidas. Este trabalho foi uma análise a fundo do trabalho de Brunetti e Fredenhagen, com a intenção de se realizar os primeiros passos na generalização do método proposto pelos autores citados, estendendo o trabalho deles para o caso de medidas covariantes por transformações de Poincaré.

Palavras-chave: Mecânica Quântica, Medidas com Valores em Operadores Positivos, Integrais de Haar, Teoria de Grupos

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Chapter 1

What are Observables? And how is Time seen in Quantum Mechanics

Quantum Mechanics is a theory that has begun its development approximately one hundred years ago. Its study has been approached in many different ways, but of particular note are two fronts in which the theory has been scrutinized. The chronological first front, and the one that originated the subject in itself, is the heuristic approach. As was known before Planck's seminal work, classical physics could not predict finite energies for photons of frequency starting in the ultraviolet range [1]. It was then that Planck came with the rather practical solution of the quantisation of the energy of photons, which was shown to be remarkably effective at predicting the values of the energy of ultraviolet and higher frequency photons [1], [2].

This was the beginning of what is now known as quantum mechanics. Through the discussions of many authors such as [3], [4], [5] and [6], the more practical view of quantum mechanics was established, with means to calculate and predict a wide array of different phenomena. What was still in question though, was the more formal aspect of quantum mechanics, in that its mathematical foundations were found to be lacking. As is known, it was thought at the time that the wave mechanics of quantum mechanics was also described by a theory of infinite dimensional matrices [7], [5]. But as more authors started to study these foundations it was found that such theory came upon many foundational and computational problems, and as such, the second front of study of quantum mechanics began, and an effort to find a mathematically rigorous description of quantum mechanics started to be organized. To see a very good technical exposition on how far one can take the matrix formalism in quantum mechanics, see [8].

The author who could wrap up many of the concepts of quantum mechanics in a

rigorous form was John von Neumann, in his seminal work [9]. An important aspect of his work was exactly how to understand the physical observables that are possible to be measured in an experiment in a laboratory. To formalize these fundamental concepts, he utilized the theory of operators and of operator algebras.

1.1 A Shallow Dive in the Logic of Quantum Mechanics and how its Mathematical Structure arises naturally

The main difference between a classical theory of physics and a quantum theory of physics comes from the fact that, while a classical theory of physics is deterministic, that is, given a set of dynamical variables, it is possible to determine every aspect of the system in question, a quantum theory is given strictly by a **probabilistic** theory. This gives rise to a lower bound for the accuracy with which we can measure a quantum system, the celebrated Heisenberg's Uncertainty Relations.

To formalize these concepts, and to serve as a base for what will come next, it is useful to go through a brief exposition on logic theory.

A great part of the exposition in this section was inspired by [10], which has a more modern approach to the topic, which implies in a few results being translated into a more up to date language.

Given a classical system, it is possible to describe in the phase space at the time t, $\mathscr{P}_t$, whether any of the propositions about the system are true, false, or their probability to be true or false. It is rather natural then to define a relation between propositions about the system and points in the phase space, by identifying propositions with the sets of points in the phase space where they are true.

Given this relation, it is interesting to see if there is a way to relate the structure of propositions with the structure of points in the phase space. Using the same identification for propositions and the set of points in the phase space where they are true, then for any two propositions A and B, there are three basic logical connectives between them that can be identified with set operations:

1. $A \mathscr{O} B \rightarrow A \cup B$;

2. $A \mathscr{E} B \rightarrow A \cap B$;

3. $\dashv A \to \mathscr{P}_t \setminus A$.

Here, $\mathscr{O}$ is the logical connective corresponding to or, $\mathscr{E}$ is the connective correspond-ing to and, and $\dashv$ is the one corresponding to negation.

From this point, it is rather intuitive to define physical quantities as functions over the phase space taking real values, constrained to having at least some sort of regularity. Given that there are physical quantities that can only take discrete values, it is reasonable to restrain, at least in classical physics, to functions that are measurable. It is natural then to identify propositions about the physical quantity as the inverse image of an open borelian set in $\mathbb{R}$ under the function f which defines the quantity.

With these identifications, it is rather straightforward to see as in [10], that it is possible to identify the logical structure of propositions about a system with a σ-algebra, a concept that will be expanded upon in chapter 2.

It is now a good point to also do a rather brief excursion on lattice theory, to wrap up the logical aspects of classical theory and to identify its differences with quantum mechanics.

Definition 1.1: A partially ordered set $(X, \prec)$ is a **lattice** when, for any $a, b \in X$, it holds:

1. $\sup a, b$ exists and is denoted as $a \vee b$;

2. $\inf a, b$ exists and is denoted as $a \wedge b$.

It is straightforward to see that lattices obey:

Associativity: For any $a, b, c \in X$, $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$;

Commutativity: For any $a, b \in X$, $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$;

Absorption: For any $a, b \in X$ and $a \prec b$, $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$;

Idempotency: For any $a \in X$, $a \vee a = a$ and $a \wedge a = a$.

Besides that, that are other properties that a lattice may have, which are not direct consequences of its definition, such as being:

Distributive: given $a, b, c \in X$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

Limited: there are two elements, denoted 0 and 1, called respectively the contradiction and the tautology, such that $0 \prec a$ and $a \prec 1$ for any $a \in X$;

Orthocomplemented: if the lattice is limited and there is an endomorphism in X, $a \mapsto \neg a$, with $\neg a$ called the orthogonal complement of a, such that, for any $a, b \in X$:

1. $a \vee \neg a = 1$;

2. $a \wedge \neg a = 0$;

3. $\neg(\neg a) = a$;

4. $a \prec b \rightarrow \neg b \prec \neg a$.

Modular: given $a, b, c \in X$, $a \prec b \rightarrow (a \vee b) \wedge c = a \vee (b \wedge c)$;

Orthomodular: if the lattice is orthocomplemented and it holds $a \prec b \rightarrow a = b \vee ((\neg b) \wedge a)$ for every $a, b \in X$;

σ-complete: given a countable set $\{a_n\}_{n \in \mathbb{N}}$ of elements of X, then it has a least upper bound $\vee_{n \in \mathbb{N}} a_n$.

With all of these definitions at hand, it is finally possible to define a **boolean σ-algebra** as being a lattice that also has all of the previously listed properties.

Having finished this slight cruise through lattice theory, and taking these definitions to the context of physical systems and of σ-algebras, it is rather natural to associate the partial ordering of a lattice to the partial ordering of set inclusion, with $\cup \rightarrow \vee$, $\cap \rightarrow \wedge$, the tautology being associated with X, the contradiction with $\emptyset$ and complementation on sets being associated with orthocomplementation on lattices. Through this identification, a σ-algebra of sets is also a boolean σ-algebra. Thus, the following final characterization can be made:

Proposition 1.1 [adapted from [10], Pg. 263]: Propositions relative to classical physical systems form a boolean σ-algebra, where:

1. the ordering is that of set inclusion, conjunction is intersection, and disjunction is union;

2. the tautology and the contradiction are respectively, the whole space $\mathscr{P}_t$ and $\emptyset$;

3. orthocomplementation is identified with negation.

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If a measurable function $f : \mathscr{P} \to \mathbb{R}$ represents a physical quantity, then:

1. As $E \in \mathscr{B}(\mathbb{R})$ varies between all the borelian sets of $\mathbb{R}$, the propositions identified with $f^{-1}(E)$ form a boolean σ-algebra;

2. The map that takes a borelian set E to its respective proposition is a homomorphism between boolean σ-algebras.

This then summarizes the logical foundations of classical physics. It can now be more concretely understood in which ways quantum physics differs from classical physics.

It is best to begin with the axioms that are at the base of quantum physics. The first one is the fact that measures done on quantum systems are intrinsically probabilistic. There is no possible way in which one can determine previously with certainty the measurement of a system, being only possible to assign probabilities to each possible outcome. The second axiom is that there are *observables* which are **incompatible**. Using for instance measurements of two incompatible quantities A and B in sequence, taking care not to let eventual differences be due to the time evolution of the system, this will affect a subsequent measure of the other quantity. For instance, measuring first the quantity related to A, then the quantity related to B, and afterwards doing another measurement of A will result in an independent measurement for A.

These axioms are translated into differences in the logical structure of quantum physics when compared to the one of classical physics. The second axiom then is to be understood as the existence of **incompatible** propositions, that is, there is no way to attribute physical meaning to the simultaneous realization of the truth values of two incompatible propositions, since there is no measuring instrument capable of measuring the physical quantities related to both propositions at once.

Due to this aspect of the theory, it is not possible to do the identifications that were done previously to the classical case, relating the logical structure to a σ-algebra of sets. It was then that von Neumann came up with the idea [9] that the way to properly describe quantum systems would be through relating propositions to orthogonal projections defined on a complex Hilbert Space $\mathscr{H}$. This set of orthogonal projections on $\mathscr{H}$ form a lattice structure, and even though it will not be a boolean σ-algebra, it allows the differentiation between compatible and incompatible propositions, and the association of $\mathscr{E} \to \wedge$ and $\mathscr{O} \to \vee$.

Consider then some properties about orthogonal projections that commute:

- if P and Q commute, then PQ is an orthogonal projection in $P(\mathscr{H}) \cap Q(\mathscr{H})$;

- if P and Q commute, the $P + Q - PQ$ is an orthogonal projection in the closed subspace $\overline{P(\mathscr{H}) \oplus Q(\mathscr{H})}$

These properties lead naturally to the following association:

$$P \mathscr{E} Q \rightarrow PQ;$$

$$P \mathscr{O} Q \rightarrow P + Q - PQ;$$

$$\neg P \rightarrow I - P.$$

Here, I is the identity on the space of bounded operators acting on $\mathscr{H}$, $\mathscr{B}(\mathscr{H})$.

To clarify on the reason why this allows for the differentiation between compatible and incompatible propositions, it is worthy of note that, in the case where P and Q do not commute, then PQ and $P + Q - PQ$ may not even be orthogonal projections, which means that the previous correspondence between logical operators and orthogonal projections does not hold. In summary, the propositions about quantum systems are in a one-to-one correspondence with the set of orthogonal projections defined on a Hilbert space $\mathscr{H}$. This correspondence is such that:

1. the logical implication between two propositions corresponds to the partial ordering between the associated orthogonal projections, that is, if $P \Rightarrow Q$, then $P \leq Q$;

2. two propositions P and Q are compatible if and only if their corresponding projections satisfy $PQ = QP$.

Introducing another concept from lattice theory, in an orthocomplemented lattice $(X, \prec)$, two elements a and b are said to be **orthogonal**, or $a \perp b$, if $\neg b \prec a$ (or symmetrically, if $\neg a \prec b$). Also, they are said to be **commuting** if $a = c_1 \vee c_3$, and $b = c_2 \vee c_3$, with $c_i \perp c_j$ when $i \neq j$.

When one considers the structure made up by all possible orthogonal projections in a Hilbert space $\mathscr{H}$, it is seen that it is a limited, orthocomplemented, σ-complete lattice, though it will not be distributive (in the case where the dimension of $\mathscr{H}$ is greater than one). Besides the previously listed properties, the lattice also satisfy:

separability: in case the Hilbert space $\mathscr{H}$ is separable, that is, if there is a dense subset in $\mathscr{H}$, then if $\{P_\lambda\}_{\lambda \in \Lambda}$ satisfies $P_{\lambda_i} \perp P_{\lambda_j}$ for every $i \neq j$, then Λ is at most enumerable;

atomicity: there are elements Q in the lattice $\mathfrak{P}(\mathscr{H}) \setminus \{0\}$ such that, for every element P of $\mathfrak{P}(\mathscr{H})$, if $0 \prec P \prec Q$, then either $P = 0$, or $P = Q$. For any $P \in \mathfrak{P}(\mathscr{H})$ there exists a Q such that $Q \prec P$ and Q has the previous property. These elements Q are called the **atoms** of the lattice;

orthomodularity: given two elements P and Q in $\mathfrak{P}(\mathscr{H})$, it holds $P \prec Q \Rightarrow P \vee ((\neg P) \wedge Q)$;

covering property: given an atom Q, and P an orthogonal projection such that $P \wedge Q = 0$, then $P \prec Q \vee P$, with $P \neq Q \vee P$, and $P \prec R \prec Q \vee P \Rightarrow R = P$ or $R = Q \vee P$;

irreducibility: only the contradiction, which is the null projection, and the tautology, which is the identity, commute with every element of $\mathfrak{P}(\mathscr{H})$;

the only orthogonal projections in one dimensional subspaces are the atoms of $\mathfrak{P}(\mathscr{H})$.

With all of the previous properties and notions established, Jauch and Piron [11] and Jauch in [12] were able to use methods on projective geometry to prove that the lattice of quantum propositions can be canonically identified with the closed subspaces of a Hilbert space defined by the system being analyzed.

Having discussed how the second axiom of quantum physics is translated into a logical structure, it is now time to turn our attention to the first axiom. As said before, it asserts that states, and resulting measurements, are inherently probabilistic. It is then advisable, in a first approach, to define what is a state in quantum systems. States are then defined as:

Definition 1.2: A state ω in a time t in a quantum system is a map from the lattice of orthogonal projections $\mathfrak{P}(\mathscr{H})$ to the interval $[0, 1]$, that is $\omega : \mathfrak{P}(\mathscr{H}) \to [0, 1]$, such that:

1. $\omega(I) = 1$;

2. if $\{P_i\}_{i \in \mathbb{N}} \subset \mathfrak{P}(\mathscr{H})$ satisfies $P_i P_j = 0$ for $i \neq j$, then $\omega \left(s - \sum_{i=0}^{\infty} P_i \right) = \sum_{i=0}^{\infty} \omega(P_i)$.

where in the second property $s - \sum$ is the strong limit on the operator topology on $\mathfrak{P}(\mathscr{H})$.

This definition of state will be generalized eventually to include states defined on C^*-algebras, another concept expanded upon in chapter 2.

Taking these previous concepts into account, along with the fact proven in [11] that $\mathfrak{P}(\mathscr{H})$ is identified with the closed subspaces of the Hilbert space defined by the system, Gleason then proved a fundamental theorem in [13], fully characterizing the probability measures able to satisfy the proposed conditions:

Theorem 1.1: Let μ be a measure defined on the closed subspaces of a separable Hilbert space $\mathscr{H}$ with dimension greater than 3. Then, there exists a positive semi-definite, self-adjoint, trace class operator T, such that for every closed subspace of $\mathscr{H}$ it holds:

$$\mu(A) = \text{Tr}\{TP_A\}$$

where P_A is the orthogonal projection onto the subspace $A \subset \mathscr{H}$.

This theorem, along with the well known Kochen-Specker theorem [14]:

Theorem 1.2 [[10], Pg. 281]: Let $\mathscr{H}$ be a separable Hilbert space of **finite** dimension greater than three. Then, there exists no function $f : \mathfrak{P}(\mathscr{H}) \to [0,1]$ that satisfies the properties of a state, and take values only on $\{0,1\}$.

determine the basis of the foundations under which quantum physics can be constructed.

This being established, the last point to be discussed is exactly how to define one of the main points of this work: **observables**. Let us define:

Definition 1.3: Let $\mathscr{H}$ be a Hilbert space, and A a function taking borelian sets $E \in \mathscr{B}(\mathbb{R})$ into orthogonal projections $P_E^{(A)} \in \mathfrak{P}(\mathscr{H})$. Then A is called an **observable** provided it satisfies:

1. $P_E^{(A)} P_{E'}^{(A)} = P_{E'}^{(A)} P_E^{(A)}$ for any E and E' in $\mathscr{B}(\mathbb{R})$;

2. $P_E^{(A)} \wedge P_{E'}^{(A)} = P_{E \cap E'}^{(A)}$ for any E and E' in $\mathscr{B}(\mathbb{R})$;

3. $P_{\mathbb{R}}^{(A)} = I$;

4. given any enumerable family $\{E_n\}_{n \in \mathbb{N}}$ of borelian subsets E_n, it holds $\vee_{n \in \mathbb{N}} P_{E_n}^{(A)} = P_{\bigcup_{n \in \mathbb{N}} E_n}^{(A)}$.

With this definition, we have that a function $P : \mathscr{B}(\mathbb{R}) \to \mathcal{B}(\mathscr{H})$ is an observable if and only if it satisfies:

1. $P(E) \geq 0$ for every $E \in \mathscr{B}(\mathbb{R}) \setminus \emptyset$;

2. $P(E)P(E') = P(E \cap E')$ for any $E, E' \in \mathscr{B}(\mathbb{R})$;

3. $P(\mathbb{R}) = I$;

4. for any disjoint enumerable family of borelian subsets $\{E_n\}_{n \in \mathbb{N}}$, $E_n \cap E_m = \emptyset$ when $n \neq m$, it holds $s - \sum_{n=0}^{\infty} P(E_n) = P(\bigcup_{n \in \mathbb{N}} E_n)$.

But these characteristics are exactly the ones that define the objects known as Projective Valued Measures, also called PVMs. That is, there is a bijection between PVMs in $\mathbb{R}$ and observables.

It is now better to introduce some definitions from the theory of operators in Hilbert spaces (actually for normed spaces in general, but the focus in this text will be on Hilbert spaces) in order to do the final connection of what was shown in this section with the celebrated spectral theorem, in order to wrap up the current exposition.

In a space in which its elements are operators, it is possible to define the operatorial norm:

Definition 1.4: Given an operator $A : \mathscr{H}_1 \to \mathscr{H}_2$, its **norm** is given as:

$$\|A\| = \sup_{x \in \mathscr{H}_1} \frac{\|Ax\|_{\mathscr{H}_2}}{\|x\|_{\mathscr{H}_1}}$$

An **unbounded** operator then is one for which this previously defined norm is not limited, that is, there is no finite M for which $\frac{\|Ax\|}{\|x\|} < M$ for every $x \in \mathscr{H}_1$.

Definition 1.5: Given an operator A, it being either bounded or unbounded, such an operator is called **self-adjoint** if it satisfies:

1. the dense domain of the operator A coincides with the dense domain of its adjoint: $\mathscr{D}(A) = \mathscr{D}(A^*)$;

2. $A\phi = A^*\phi$ for $\phi \in \mathscr{D}(A) = \mathscr{D}(A^*)$.

It is worth to note that this notion coincides with the usual notion of self-adjoint operator in the bounded case, since there it holds $\mathscr{D}(A) = \mathscr{H}$.

Definition 1.6: Given a function f defined on some space X taking on values on some field $\mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$), its **support** is defined as

$$supp(f) = \overline{\{x \in X | f(x) \neq 0\}}$$

Definition 1.7: Given an operator A, its **spectrum** is defined as the set of values λ in $\mathbb{C}$ such that $A - \lambda I$ is not invertible. That being said, the spectrum of an operator can be separated into three disjoint entities:

- the **point spectrum**, or the discrete spectrum, is the part of the spectrum composed by the points of $\mathbb{C}$ such that the equation $Ax = \lambda x$ for some x in $\mathscr{H}$. That is:

$$\sigma_p(A) = \{\lambda \in \mathbb{C} | \exists x \in \mathscr{H}, x \neq 0, s.t. Ax = \lambda x\}$$

  These λ are usually called the eigenvalues of the operator A, with the x that satisfy the previous equation called the eigenvectors related to the eigenvalue λ;

- the **continuous spectrum** is defined by the $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is injective, and its range is dense in $\mathscr{H}$. That is, the inverse of $A - \lambda I$ exists in the range of the operator, but it cannot be bounded (in case it were bounded the range of $A - \lambda I$ would be a closed subspace and as such the range would be the whole space). That is:

$$\sigma_c(A) = \{\lambda \in \mathbb{C} | \ker(A - \lambda I) = \{0\}, \overline{\operatorname{Ran}(A - \lambda I)} = \mathscr{H}\};$$

- the **residual spectrum** is defined in a much similar fashion to the continuous spectrum, but the range of $A - \lambda\mathbb{1}$ need not be dense in $\mathscr{H}$. Thus:

$$\sigma_r(A) = \{\lambda \in \mathbb{C} | \ker(A - \lambda I) = \{0\}, \overline{\operatorname{Ran}(A - \lambda I)} \subset \mathscr{H}\}.$$

It is now possible to enunciate the spectral theorem for both bounded and unbounded operators:

Theorem 1.3 [[10], Pg. 393]: Let T be a self-adjoint operator, defined on a Hilbert space $\mathscr{H}$. Then:

- there exists a unique PVM $P^{(T)} : \mathscr{B}(\mathbb{R}) \to \mathscr{B}(\mathscr{H})$ such that:

$$T = \int_{\mathbb{R}} \lambda dP^{(T)}(\lambda)$$

- it holds:

$$supp(P^{(T)}) = \sigma(T)$$

and also:

1. $\lambda \in \sigma_p(T) \iff P^{(T)}(\{\lambda\}) \neq 0$;

2. $\lambda \in \sigma_c(T) \iff P^{(T)}(\{\lambda\}) = 0$, and for any open set $\{\lambda\} \subset U_\lambda \subset \mathbb{R}$, $P^{(T)}(U_\lambda) \neq 0$;

3. if $\lambda \in \sigma(T)$ is isolated, that is, there is an open set $U_\lambda \supset \{\lambda\}$ such that $(U_\lambda \setminus \{\lambda\}) \cap \sigma(T) = \emptyset$, then $\lambda \in \sigma_p(T)$;

4. if $\lambda \in \sigma_c(T)$, then for any $\epsilon \geq 0$ there exists a unit vector $\phi(\epsilon)$ in the domain of T such that:
$0 \leq \|T\phi(\epsilon) - \lambda\phi(\epsilon)\| \leq \epsilon$.

This theorem asserts that there is a bijection between PVMs and self-adjoint operators. This bijection, along with the previous one between PVMs and observables, show that there is a bijection between observables and self-adjoint operators. That is the reason, coming from first principles, on why every observable in quantum physics is described by self-adjoint operators, and at this point it is now easier to understand why time stands out as much when compared to other physical quantities.

1.2   A Canonical Problem and a Timely Solution

As stated before, even before the development of the logical basis of quantum mechanics, a heuristical approach to the subject had already established that observables were given as self-adjoint operators. One such observable, and one of the first studied quantities in quantum mechanics was the one associated with position, given that it was also present in one of the foundational works of Heisenberg, pertaining to the uncertainty relation of position and momentum [15]. It was also in the founding years of quantum mechanics that through works like [16], [17], [18], a heuristic argument was constructed to arrive at a uncertainty relation between the energy of a system and the time of a process

or measurement made on the system, this argument based on the uncertainty relation between position and momentum. Also, it was at about the same time that it was known that time and energy were "conjugate" quantities, in the sense that the Hamiltonian of a system was the generator of time translations on a system.

It was thus natural to construct a "time observable", that would act as a clock for processes on the quantum system, and for such an observable to be self-adjoint and conjugate to the Hamiltonian, that is:

$$T = T^*$$

$$[T, H] = -i\hbar I$$

It was Pauli that showed that the construction of an observable with such characteristics is incompatible with most physical systems in [7]. In fact, it is rather straightforward to see the reason why that is so. Suppose that there exists an operator satisfying the previous properties. Then, since the Hamiltonian is the generator of time translations, there is an unitary operator $U = e^{i\hbar T a}$, where a is some energy value of the system.

Consider now a normalized state $\psi \in \mathscr{H}$. Then:

$$E_\psi = \langle \psi, H\psi \rangle$$

is the expected value of the Hamiltonian on the state ψ. Using the unitarity property of U:

$$E_\psi = \langle \psi, U^* U H U^* U \psi \rangle = \langle U\psi, U H U^* U \psi \rangle$$

Analyzing the operator $U H U^*$ we see:

$$UHU^* = H + \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} [T, [T, [T, ..., [T, [T, H]]...]]] =$$

$$= H - i\hbar(ia)I + \sum_{n=2}^{\infty} \frac{(ia)^n}{n!} (-i\hbar)[T, [T, ..., [T, [T, I]]...]] = H + a\hbar I$$

Thus, we can conclude that:

$$E_\psi = \langle U\psi, (H + \hbar a)U\psi \rangle = \langle U\psi, HU\psi \rangle + a\langle U\psi, U\psi \rangle = E_{U\psi} + a \rightarrow E_{U\psi} = E_\psi - a$$

We see from this last result that if there was an operator T satisfying the properties of an observable and such that it is canonically conjugate to the Hamiltonian, any system

would be such that there would be no lower bound to its energy. Besides that, since a can be any real number, the spectrum of every Hamiltonian would be continuous, that is, there would be no system that had eigenvalues, which is clearly not the case, considering the many systems that have been studied, such as the hydrogen atom, that show an energy spectrum with eigenvalues. As such, for a long time this has been interpreted as time not being able to be treated as a measurable quantity in quantum mechanics, and as such, it was treated simply as a parameter in quantum systems. It would be rather natural for time to be treated this way in most cases, but there are experiments in which events can occur at randomly distributed times in relation to a laboratory clock, and as such time can be understood as a random variable, and thus there can be the need to construct an associated "observable" to time.

Though interest on the lifting of time to the level of "observable" has been recent, it was already suggested by Ludwig [19] that the concept of observable should be generalized. This generalization should be taken in the direction of defining "generalized observables" as associated to Positive Operator Valued Measures, as observed by Naimark in [20], which will also be called POVMs. From then, it has been quite quickly developed POVMs that were covariant with relation to time translations, such as in [21], [22], [23], [24] and [25]. One such work was that of Brunetti and Fredenhagen [26]. In this work, they lay out a general method of construction of such POVMs, starting from bounded observables, which is interpreted as the effect, and the resulting POVM and operator associated with it as its first moment is understood as the time of occurrence of the event.

Due to the generality that the POVM approach has, it is interesting to see whether this could be extended in order to complement studies made on subjects such as localization observables, or other concepts that involve relativistic phenomena. This then lead naturally to the question of whether the construction method laid out by Brunetti and Fredenhagen could be extended to other groups. This would mean that a POVM covariant with respect to Poincaré transformations could be constructed, with its first moment in the time coordinate being the associated time operator to the event that occurred. The object of this work was then exactly to build such an extension, and do a bit more in depth description of the original method of Brunetti and Fredenhagen.

Chapter 2

A Systematic Procedure and a First Step

2.1 The Creation and an Overview of the Method

This work was done mainly by making an in-depth analysis of the work of Brunetti and Fredenhagen [26]. By exploring the mechanisms through which the authors came up with the construction method for positive operator valued measures, the means to extend it, and how to create measures covariant with respect to more symmetries become clear.

As with the authors in [26] then, the starting point of this work will be by first setting out the basic objects of the theory. In most quantum theories, the first object to be defined is the Hilbert space of the theory.

Definition 2.1: A Hilbert space is a complete space with an inner product $\langle .,. \rangle$ defined on it.

Even though the Hilbert space of the theory will be detailed, the explicit nature of this Hilbert space won't be needed as will be shown in the next sections, and this will be particularly useful when it comes to generalizing these considerations. In the case of this work, since its goal will be to create a method of constructing positive operator valued measures that are covariant with respect to Poincaré transformations, then the systems considered will be relativistic ones. It is then well know that for these systems, their Hilbert space is usually given by:

$$\mathcal{H} = \int^{\oplus} \mathcal{H}_k d\mu(k)$$

This previous construction is known as a **direct integral**. According to [27]:

Definition 2.2: Let there be a continuous indexing set Λ. Then, let there be a collection of Hilbert spaces $\{\mathcal{H}_\lambda\}_{\lambda \in \Lambda}$. Let there be continuous functions (in fact, vector fields) such that $x : \Lambda \to \mathcal{H}_\lambda$. If they are such that $\langle x, x \rangle := \int \langle x(\lambda), x(\lambda) \rangle d\mu(\lambda) = \int \|x(\lambda)\|^2_{\mathcal{H}_\lambda} d\mu(\lambda) < \infty$ for a measure $\mu(\lambda)$ defined on Λ, then we may define the space of the vector fields x as the **direct integral** over the Hilbert spaces $\{\mathcal{H}\}_{\lambda \in \Lambda}$. This will be denoted as $\int^{\oplus} \mathcal{H}_{\lambda \in \Lambda} d\mu(\lambda)$.

The Hilbert space of relativistic theories is given by this direct integral because the space is understood as being decomposed into the possible mass hyperboloids parametrized by the 4-momenta k.

Considering that we now have established the Hilbert space $\mathcal{H}$ of interest, we can then say that we have an associated C^*-algebra of bounded operators acting on $\mathcal{H}$, denoted $\mathcal{B}(\mathcal{H})$. Then, on $\mathcal{B}(\mathcal{H})$ we have states defined as:

Definition 2.3: A **state** ω on a C^*-algebra is a positive linear functional on the algebra, such that it is normalized as to satisfy $\|\omega\| = 1$.

With the Hilbert space $\mathcal{H}$ and its associated C^*-algebra established, as well as states acting on $\mathcal{B}(\mathcal{H})$, we can start building up Positive Operator Valued Measures covariant with respect to translations in $\mathbb{R}^n$. This will be done since the Poincaré group is defined as $\mathcal{P} = \mathbb{R}^4 \rtimes \mathcal{L}$. Here, $\rtimes$ is the semidirect product between the groups $\mathbb{R}^4$ and $\mathcal{L}$, with $\mathcal{L}$ being the Lorentz Group, defined as $\mathcal{L} := O_{3,1}(\mathbb{R})$. Here, $O_{1,3}(\mathbb{R}) \cong O(\mathbb{R}^4, \beta)$, with $O(\mathbb{R}^4, \beta)$ being the group of orthogonal transformations that maintain the quadratic form $\beta(x, y) := x^1 y^1 + x^2 y^2 + x^3 y^3 - x^4 y^4$ invariant.

The first step, as suggested in [26] is then, for a given state ω to analyse $\omega(\alpha_x(A))$ for a given observable A, with characteristics to be determined soon. In these next few paragraphs the reason for this will be detailed.

As detailed in the first chapter, according to Theorem 1.3, every observable can be associated to a self-adjoint operator in a given Hilbert space. Thus, some of the observables that carry information about the system $\mathcal{H}$ will be represented by self-adjoint operators

in $\mathscr{B}(\mathscr{H})$. As is known, the following result for self-adjoint operators hold:

Proposition 2.1 [[28], Prop. 39.17]: If A is a bounded, self-adjoint operator acting on a Hilbert space $\mathscr{H}$, then its eigenvalues will be real valued, in case they exist. Its associated eigenvectors respective to different eigenvalues will be orthogonal.

It is then possible to define positive operators as:

Definition 2.4: A **Positive Operator** is a self-adjoint operator acting in a Hilbert space $\mathscr{H}$ such that its spectrum is contained in $[0, \infty]$. In other words, positive operators are operators T acting in $\mathscr{H}$ such that $\langle \phi, T\phi \rangle \geq 0$ for every $\phi \in \mathscr{H}$.

Thus, giving more details on the properties the observable must have in order for it to be possible to construct a Positive Operator Valued Measure, they have to be **bounded** and **positive**. It is also necessary to understand what α_x means in $\omega(\alpha_x(A))$. α_x is the action of the translation group on observables in $\mathscr{H}$. Now, there are two fundamental results that characterize symmetry transformations such as the one of translations:

Theorem 2.1 (Wigner's Theorem) [[29], Pg. 45]: Given a Hilbert space $\mathscr{H}$ such that its dimension is equal to or greater than three, any bijection s that conserves the probability transition of the unitary rays of $\mathscr{H}$ can be represented either by a unitary operator or by an antiunitary operator as $s\Phi = U\Phi$, with Φ being an element of an unitary ray in $\mathscr{H}$.

Theorem 2.2 (Dye's Theorem) [[29], Pg. 47]: Given a Hilbert space $\mathscr{H}$ such that its dimension is equal to or greater than three, then any orthoautomorphism in $\mathfrak{P}(\mathscr{H})$ can be uniquely extended to a $*$-automorphism or to a $*$-antiautomorphism in $\mathscr{B}(\mathscr{H})$.

These results show that any symmetry α, which by definition are either $*$-automorphisms or $*$-antiautomorphisms, can be represented by $\alpha(A) = UAU^*$. Since symmetry transformations are given by unitary or antiunitary operators, then they are isometric, which means that positive operators transformed by symmetries will remain positive operators.

This means then that $\omega(\alpha(A))$ assumes a positive value for every state ω, positive observable A, and symmetry α. For the case of translations symmetries then,

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$\alpha_x(A) = U_x A U_x^*$. By the following theorem:

Theorem 2.3 (Stone) [[10], Pg. 419]: Let $\mathscr{H}$ be a Hilbert space. Then:

- If A is an operator $A : D(A) \to \mathscr{H}$, with $D(A)$ dense in $\mathscr{H}$, that is self-adjoint with associated spectral measure $P^{(A)}$, then the operators:

$$U_t = e^{itA} := \int_{\sigma(A)} e^{i\lambda t} dP^{(A)}(\lambda)$$

with $t \in \mathbb{R}$ form a strongly continuous one-parameter unitary group. Moreover:

1. the limit:

$$s - \left.\frac{dU_t}{dt}\right|_{t=0} \Psi := \lim_{t \to 0} \frac{U_t \Psi - \Psi}{t}$$

exists in $\mathscr{H} \iff \Psi \in D(A)$;

2. if $\Psi \in D(A)$:

$$s - \left.\frac{dU_t}{dt}\right|_{t=0} \Psi = iA\Psi.$$

- If $\{U_t\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\mathscr{H}$, there exists a unique self-adjoint operator $A : D(A) \to \mathscr{H}$, with dense domain $D(A)$, such that:

$$e^{itA} = U_t, \forall t \in \mathbb{R}.$$

The unitary operator describing translations is then: $U_x = e^{-iPx}$. Since this work is dealing with relativistic systems, then x is a point in the Minkowski space $\mathcal{M}$, and P is a notation to represent $P = (P^0, P^1, P^2, P^3)$, with P^0 being the Hamiltonian of the system, and the P^i are commuting, unbounded, self-adjoint operators corresponding to momentum observables in the three spatial directions.

As said previously in this work, most of it was based on a paper written by Brunetti and Fredenhagen [26]. In the next section, what was done by the authors will be explained, and an in depth analysis of their ideas will be made.

2.2 A Covariant Measure for Translations

To better understand the construction method delineated by Brunetti and Fredenhagen, some fundamentals on some topics need to be covered, and from them, the path to an extension to Poincaré covariance is rather straightforward.

Results relating to C^*-algebras will be taken note of, and their application to the case at hand will then be made. Some definitions of objects from Measure Theory are in order before the first steps are taken:

Definition 2.5: A σ-**algebra** over a set X is a collection of subsets of X, denoted $\Sigma(X)$, such that:

1. $X \in \Sigma(X)$;

2. $E \in \Sigma(X) \Rightarrow X \setminus E \in \Sigma(X)$;

3. given a family $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma(X)$, then $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma(X)$.

A **measurable space** is thus given by the pair $(X, \Sigma(X))$.

If X is a topological space with topology τ, then the σ-algebra generated by the sets in τ is called the **Borel σ-algebra** on X, with the elements of σ-algebra being the Borel sets.

Definition 2.6: Let $(X, \Sigma(X))$ be a measurable space. A **measure** on X with respect to $\Sigma(X)$ is a function $\mu : \Sigma(X) \to [0, \infty]$, satisfying:

1. $\mu(\emptyset) = 0$;

2. Given a pairwise disjoint family of sets of $\Sigma(X)$, it holds $\mu(\bigcup_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} \mu(E_n)$;

The triple $(X, \Sigma(X), \mu)$ is said to form a **measure space**. Given a set $E \in \Sigma(X)$, E is said to have **finite measure** if $\mu(E) < \infty$.

Definition 2.7: Given a measurable space $(X, \Sigma(X))$, a **complex measure** is a function $\mu : \Sigma(X) \to \mathbb{C}$, such that the following properties hold:

1. $\mu(\emptyset) = 0$;

2. Given a pairwise disjoint family of sets of $\Sigma(X)$, it holds $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \mu(E_n)$, *independently of the summing order*. In case the values of μ fall only on the real line $\mathbb{R}$, μ is said to be a **signed measure**.

Now, as per the previous section, in [26], the authors started by noting that $\omega(\alpha_t(A))$ can be seen as a function of t, and if A is a **bounded** and **positive** operator, then $\omega(\alpha_t(A))$ will be a continuous, positive, bounded function. From this, they then started by considering the following operators:

$$B(I) = \int_I \alpha_t(A)dt \qquad (2.1)$$

Here, I is a Borel set from $\mathbb{R}$ of **finite** measure. This sort of operator is to be understood as coming from:

$$\langle \Psi, B(I)\Phi \rangle := \int_I \langle \Psi, \alpha_t(A)\Phi \rangle dt$$

These operators are well defined through this procedure due to the following result:

Proposition 2.2 [[28], Prop. 39.11]: Let $\mathscr{S} : \mathscr{H}_2 \times \mathscr{H}_1 \to \mathbb{C}$ be a bicontinuous sesquilinear form. Then, there exists a unique bounded linear operator $S : \mathscr{H}_2 \to \mathscr{H}_1$ such that:

$$\mathscr{S}(\psi, \phi) = \langle S\psi, \phi \rangle_{\mathscr{H}_1} \qquad (2.2)$$

for all $\psi \in \mathscr{H}_2$ and $\phi \in \mathscr{H}_1$.

In this case, $\int_I \langle \Psi, \alpha_t(A)\Phi \rangle dt$ is a bicontinuous sesquilinear form for any $\Psi, \Phi \in \mathscr{H}$, and thus, there exists a unique bounded operator associated with it, namely, $B(I)$.

The same can be done for the case of translations on any n-dimensional space. For a given Borel set I of finite measure in $\mathbb{R}^n$ then, one can define:

$$B(I) := \int_I \alpha_x(A)dx \qquad (2.3)$$

for a given positive bounded observable A.

Proposition 2.3: Operators $B(I)$ as defined in (2.3) are, themselves, positive, bounded and self-adjoint for a positive and bounded observable A.

Proof. Let $\Psi \in \mathscr{H}$. Then:

$$\langle \Psi, B(I)\Psi \rangle = \int_I \langle \Psi, \alpha_x(A)\Psi \rangle dx = \int_I \langle \Psi, U_x^* A U_x \Psi \rangle dx = \int_I \langle U_x \Psi, A U_x \Psi \rangle dx$$

The integrand then can be understood as the expectation value of the observable A in the state $U_x\Psi$: $\langle A \rangle_{U_x\Psi}$. Since A is a positive and bounded observable, then $\langle A \rangle_{U_x\Psi}$ is positive and bounded for any $U_x\Psi$. Thus, the integration over a Borel set of finite measure also results in a finite value for the integral. As such, the operator $B(I)$, defined through the process of proposition 2.2 is a bounded and positive operator.

Let us consider then:

$$\langle B(I)\Psi, \Phi \rangle = \int_I \langle \alpha_x(A)\Psi, \Phi \rangle dx = \int_I \langle U_x^* A U_x \Psi, \Phi \rangle dx = \int_I \langle \Psi, (U_x^* A U_x)^* \Phi \rangle dx =$$

$$= \int_I \langle \Psi, U_x^* A^* (U_x^*)^* \Phi \rangle dx = \int_I \langle \Psi, U_x^* A U_x \Phi \rangle dx = \langle \Psi, B(I)\Phi \rangle$$

where on this calculation were used the facts that $U_x^* A U_x$ is a bounded operator, since A is bounded and U_x is unitary, and thus its adjoint is defined for any $\Phi \in \mathscr{H}$, and that A is an observable, and thus, self-adjoint. $\qquad\qquad\square$

Two basic definitions will be needed for what follows.

Definition 2.8: A **directed set** is a non empty, partially ordered set X, such that, for any two pair of elements $x, y \in X$, there is an element $z \in X$ such that it holds $x \prec z$ and $y \prec z$.

Definition 2.9: A **net** in a space X is a function from a directed set Λ into X.

Consider now a net of Borel sets of finite measure $\{I_\lambda\}_{\lambda \in \Lambda}$, Λ being an indexing set. This net is such that all possible translated sets of $\mathbb{R}^n$ of the form $I + x$ for any $I \in \mathcal{B}(\mathbb{R}^n)$ of finite measure and any $x \in \mathbb{R}^n$ are in the net, and also such that, if $\lambda \prec \lambda'$ for some $\lambda, \lambda' \in \Lambda$, then $I_\lambda \subset I_{\lambda'}$.

Now, a core concept for what follows will be defined:

Definition 2.10: An **algebra** over $\mathbb{C}$ is a vector space $\mathcal{A}$, along with a binary

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operation, denoted $\cdot$, which is called the **product** of the algebra, that satisfy the following properties:

1. The distributive property:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

where $a, b, c \in \mathcal{A}$;

2. for any $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$, it holds:

$$\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$$

The algebra is said to be **commutative** if it holds:

$$a \cdot b = b \cdot a$$

for any $a, b \in \mathcal{A}$, and is said to be **associative** if:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

also for any $a, b, c \in \mathcal{A}$.

From here on out, the $\cdot$ will be omitted, unless needed for further clarification.

Definition 2.11: An **involution** over an algebra $\mathcal{A}$ is an operation $* : \mathcal{A} \to \mathcal{A}$, that to every $a \in \mathcal{A}$ associates an element denoted a^*. The operation also must have the following properties:

1. $(a^*)^* = a$ for any $a \in \mathcal{A}$;

2. $(ab)^* = b^*a^*$ for any $a, b \in \mathcal{A}$;

3. $(\alpha a + \beta b)^* = \overline{\alpha}a^* + \overline{\beta}b^*$ for any $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$;

4. In case the algebra has a **neutral element** $\mathbb{1}$ such that $\mathbb{1}a = a\mathbb{1} = a$ for any $a \in \mathcal{A}$, then $\mathbb{1}^* = \mathbb{1}$.

An algebra is said to be a $*$-**algebra** in case it has an involution $*$ defined on it.

Definition 2.12: An associative algebra $\mathcal{A}$ is said to be **normed** if it has a norm $\|\cdot\|$ that satisfies $\|ab\| \leq \|a\|\|b\|$ for any $a, b \in \mathcal{A}$.

A **Banach** algebra is an associative normed algebra that is complete in relation to the norm $\|\cdot\|$.

A **Banach** $*$-**algebra** is a Banach algebra with an involution $*$ defined on it, and such that it holds $\|a\| = \|a^*\|$ for any $a \in \mathcal{A}$.

Definition 2.13: A C^*-**algebra** is a Banach $*$-algebra where the C^* property holds:

$$\|aa^*\| = \|a\|^2$$

Following the previous paragraph, one is able to see that the partial ordering of the sets in the net is translated into the ordering of the cone of positive, self-adjoint operators in the C^*-algebra $\mathscr{B}(\mathscr{H})$ of bounded operators in $\mathscr{H}$. Given $\Psi \in \mathscr{H}$, and $\lambda \prec \lambda'$ with $\lambda, \lambda' \in \Lambda$:

$$\langle \Psi, B(I_{\lambda'})\Psi \rangle = \int_{I_{\lambda'}} \langle \Psi, \alpha_x(A)\Psi \rangle dx = \int_{I_\lambda} \langle \Psi, \alpha_x(A)\Psi \rangle dx + \int_{I_{\lambda'} \setminus I_\lambda} \langle \Psi, \alpha_x(A)\Psi \rangle dx =$$

$$= \langle \Psi, B(I_\lambda)\Psi \rangle + \langle \Psi, B(I_{\lambda'} \setminus I_\lambda)\Psi \rangle \tag{2.4}$$

Since $B(I)$ is a positive operator for any Borel set I of finite measure, then the previous equation implies:

$$B(I_{\lambda'}) = B(I_\lambda) + B(I_{\lambda'} \setminus I_\lambda) \Rightarrow B(I_{\lambda'}) \geq B(I_\lambda)$$

Consider the following proposition:

Proposition 2.4 [[28], Prop. 39.50]: Given a C^*-algebra with unity $\mathbb{1}$, and let $\mathcal{A}_+$ denote the cone of positive, self-adjoint elements of the algebra. Then, if $a \in \mathcal{A}_+$, and $x \in [0, \infty)$, then $\mathbb{1} + xa$ is invertible, and $(\mathbb{1} + xa)^{-1} \in \mathcal{A}_+$.

One can see through it that the net of operators given by $\{\mathbb{1} + B(I_\lambda)\}_{\lambda \in \Lambda}$ is a net of invertible operators, themselves being positive, self-adjoint operators. The following

proposition makes it clear what kind of ordering is present in this net:

Proposition 2.5 [[28], Prop. 39.51]: Let $\mathcal{A}$ be a C^*-algebra with unit $\mathbb{1}$, and let $a, b \in \mathcal{A}$ be self-adjoint elements of the algebra, such that $a \geq b \geq 0$. Then, for every $x \in [0, \infty)$, it holds:

$$(x\mathbb{1} + b)^{-1} \geq (x\mathbb{1} + a)^{-1} \geq 0 \tag{2.5}$$

This means then that the net $\{[\mathbb{1} + B(I_\lambda)]^{-1}\}_{\lambda \in \Lambda}$ inherits the partial ordering from the net $\{I_\lambda\}_{\lambda \in \Lambda}$, but is a *decreasing* net, in the sense that, if $\lambda \prec \lambda'$, then $[\mathbb{1} + B(I_{\lambda'})]^{-1} \leq [\mathbb{1} + B(I_\lambda)]^{-1}$.

The ordering properties of this net $\{[\mathbb{1} + B(I_\lambda)]^{-1}\}_{\lambda \in \Lambda}$ being established, and knowing that it forms a decreasing net, then it is possible to show that this net converges strongly to a lower bound C. Stating some results:

Proposition 2.6 [[28], Prop. 39.70]: Let $A \in \mathcal{B}(\mathcal{H})$, $A \neq 0$. Then, the following are equivalent:

1. $\forall \Psi \in \mathcal{H}$, it holds $\langle \Psi, A\Psi \rangle \geq 0$;

2. A is self-adjoint and positive;

3. A is self-adjoint and $\left\| \mathbb{1} - \frac{1}{\|A\|} A \right\| \leq 1$;

4. A is self-adjoint and there exists a self-adjoint C such that $A = C^2$.

Proposition 2.7 [[28], Prop. 39.49]: Let $\mathcal{A}$ be a C^*-algebra with unity $\mathbb{1}$, and let $a, b \in \mathcal{A}$ be self-adjoint elements of the algebra. Then the following holds:

1. If $a \geq 0$, then $a\|\mathbb{1}\| \geq a \geq 0$;

2. If $a \geq b \geq 0$, then $\|a\| \geq \|b\|$;

3. If $a \geq 0$, then $a\|a\| \geq a^2 \geq 0$.

One is now in position to show the convergence of this net of operators:

Proposition 2.8: The net of operators $\{[\mathbb{1} + B(I_\lambda)]^{-1}\}_{\lambda \in \Lambda}$ converges strongly in $\mathcal{B}(\mathcal{H})$.

Proof. Per the results shown before, the elements of the net obey, for $\lambda \prec \lambda'$, $[\mathbb{1} + B(I_{\lambda'})]^{-1} \leq [\mathbb{1} + B(I_{\lambda})]^{-1}$. This in turn translates to, according to item 1. of proposition 2.6:

$$\langle \Psi, [\mathbb{1} + B(I_{\lambda'})]^{-1} \Psi \rangle \leq \langle \Psi, [\mathbb{1} + B(I_{\lambda})]^{-1} \Psi \rangle$$

Now, for fixed $\Psi \in \mathscr{H}$, $\langle \Psi, [\mathbb{1} + B(I_{\lambda})]^{-1} \Psi \rangle$ is a decreasing net of real numbers for $\lambda \in \Lambda$, and as such, must converge to a real number. It is thus also a Cauchy sequence, and for $\lambda \prec \lambda'$ appropriately chosen in Λ, one has:

$$\langle \Psi, [\mathbb{1} + B(I_{\lambda})]^{-1} \Psi \rangle - \langle \Psi, [\mathbb{1} + B(I_{\lambda'})]^{-1} \Psi \rangle < \epsilon_{\Psi}^2$$

Let us then define $D_{\lambda} := [\mathbb{1} + B(I_{\lambda})]^{-1}$ and $D_{\lambda'} := [\mathbb{1} + B(I_{\lambda'})]^{-1}$. Then:

$$\langle \Psi, D_{\lambda} \Psi \rangle - \langle \Psi, D_{\lambda'} \Psi \rangle = \langle \Psi, (D_{\lambda} - D_{\lambda'}) \Psi \rangle < \epsilon_{\Psi}^2$$

Now $D_{\lambda} - D_{\lambda'}$ is a self-adjoint operator, since it is the difference of two self-adjoint operators D_{λ} and $D_{\lambda'}$. Then:

$$\| (D_{\lambda} - D_{\lambda'}) \Psi \|^2 = \langle (D_{\lambda} - D_{\lambda'}) \Psi, (D_{\lambda} - D_{\lambda'}) \Psi \rangle = \langle \Psi, (D_{\lambda} - D_{\lambda})^2 \Psi \rangle$$

Now, it is worth noting that, since $B(I)$ is a positive operator for any I of finite measure, then $\mathbb{1} + B(I) \geq \mathbb{1}$ for any $I \in \mathcal{B}(\mathbb{R}^4)$. Due to proposition 2.5 then, $[\mathbb{1} + B(I)]^{-1} \leq \mathbb{1}^{-1} = \mathbb{1}$. This then means that $D_{\lambda} - D_{\lambda'} \leq \mathbb{1} \Rightarrow \| D_{\lambda} - D_{\lambda'} \| \leq 1$. As a consequence of item 3. of proposition 2.7, one can obtain:

$$1 \cdot (D_{\lambda} - D_{\lambda'}) \geq \| D_{\lambda} - D_{\lambda'} \| (D_{\lambda} - D_{\lambda'}) \geq (D_{\lambda} - D_{\lambda'})^2 \geq 0$$

It is thus possible to conclude that:

$$\langle \Psi, (D_{\lambda} - D_{\lambda'})^2 \Psi \rangle \leq \langle \Psi, (D_{\lambda} - D_{\lambda'}) \Psi \rangle = \langle \Psi, [\mathbb{1} + B(I_{\lambda})]^{-1} \Psi \rangle - \langle \Psi, [\mathbb{1} + B(I_{\lambda'})]^{-1} \Psi \rangle < \epsilon_{\Psi}^2$$

Since for any fixed Ψ, ϵ_{Ψ}^2 can be taken to be arbitrarily small for appropriate $\lambda \prec \lambda'$, $\lambda, \lambda' \in \Lambda$, one has:

$$\left\| [[\mathbb{1} + B(I_{\lambda})]^{-1} - [\mathbb{1} + B(I_{\lambda'})]^{-1}] \Psi \right\| = \| (D_{\lambda} - D_{\lambda'}) \Psi \| < \epsilon_{\Psi}^2$$

This last expression is exactly the strong convergence of the net of operators $\{[\mathbb{1} +$

$B(I_\lambda)]^{-1}\}_{\lambda \in \Lambda}$.

There is still the question of whether or not this limit operator C is unique, considering different choices of indices in a net could lead to different limits.

Consider then, that there are certain choices of indices in Λ, one converging to the operator C and one converging to the operator C'. Take from these choices of indices, two particular indices λ' and λ'', such that it holds $\|(C - [\mathbb{1} + B(I_{\lambda'})]^{-1})\Psi\| < \frac{\epsilon_\Psi}{4}$ and $\|(C' - [\mathbb{1} + B(I_{\lambda''})]^{-1})\Psi\| < \frac{\epsilon_\Psi}{4}$.

Since Λ is a directed set, there is an index $\lambda''' \in \Lambda$, such that $\lambda', \lambda'' \prec \lambda'''$. In the level of the net of operators, this means that $B(I_{\lambda'''}) \geq B(I_{\lambda'}), B(I_{\lambda''})$, which implies $[\mathbb{1} + B(I_{\lambda'''})]^{-1} \leq [\mathbb{1} + B(I_{\lambda'})]^{-1}, [\mathbb{1} + B(I_{\lambda''})]^{-1}$.

Since it holds that $C \leq [\mathbb{1}+B(I_{\lambda'''})]^{-1} \leq [\mathbb{1}+B(I_{\lambda'})]^{-1}$, and $\|(C - [\mathbb{1} + B(I_{\lambda'})]^{-1})\Psi\| < \frac{\epsilon_\Psi}{4}$, it holds in particular that $\|([\mathbb{1} + B(I_{\lambda'})]^{-1} - [\mathbb{1} + B(I_{\lambda'''})]^{-1})\Psi\| < \frac{\epsilon_\Psi}{4}$. The same holds for $\|([\mathbb{1} + B(I_{\lambda''})]^{-1} - [\mathbb{1} + B(I_{\lambda'''})]^{-1})\Psi\|$.

Denoting once again $D_\lambda := [\mathbb{1} + B(I_\lambda)]^{-1}$, we can see then:

$$\|(C - C')\Psi\| \leq \|(C - D_{\lambda'})\Psi\| + \|(D_{\lambda''} - C')\Psi\| + \|(D_{\lambda'} - D_{\lambda''})\Psi\| \leq$$

$$\leq \|(C - D_{\lambda'})\Psi\| + \|(D_{\lambda''} - C')\Psi\| + \|(D_{\lambda'} - D_{\lambda'''})\Psi\| + \|(D_{\lambda'''} - D_{\lambda''})\Psi\| < \frac{\epsilon_\Psi}{4} + \frac{\epsilon_\Psi}{4} + \frac{\epsilon_\Psi}{4} + \frac{\epsilon_\Psi}{4}$$

That is, it holds

$$\|(C - C')\Psi\| < \epsilon_\Psi$$

This means that the limit operator is unique. $\qquad\square$

At this point, one question needs to be addressed. As was well noted by Brunneti and Fredenhagen in [26], an important part of constructing a Positive Operator Valued Measure is to determine whether the operator understood as:

$$B := \int_{\mathbb{R}^4} \alpha_x(A) dx \tag{2.6}$$

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is well defined.

To do this, two spaces are considered. The first is the space of the kernel of the operator C. This space, as comes from definition, is the space corresponding to the vectors in $\mathscr{H}$ that annihilate C. The kernel of C will be suggestively denoted by $\mathscr{H}_\infty$. This is because the vectors in $\mathscr{H}_\infty$ will correspond to the vectors where the time spent in a detector measuring the quantity related to the observable A would be infinitely long. The other would be the space formed by the joint kernel of the operators $B(I_\lambda)$. This space will in turn be denoted as $\mathscr{H}_0$ since it will correspond to the vectors where the effect related to A would not happen at all. In other words, the time spent in a detector would be null.

If one then considers the orthogonal complement to the sum of these spaces $\mathscr{H}_{finite} = (\mathscr{H}_0 + \mathscr{H}_\infty)^\perp$, it would be a space where the following operator:

$$B := C^{-1} - \mathbb{1} \tag{2.7}$$

would be well defined. It can be understood as the total time duration of the event associated with the observable A, that is:

$$B = \lim_{I \to \mathbb{R}^4} \int_I \alpha_x(A) dx \tag{2.8}$$

This operator is, in fact, positive, self-adjoint, and dominates all other $B(I)$, as will be shown. Prior to this, a result is necessary:

Proposition 2.9 [[30], Pg. 512]: Given an injective and self-adjoint operator A on a Hilbert space $\mathscr{H}$, its inverse A^{-1}, defined on a dense domain $\mathrm{Ran}(A)$, is also self-adjoint.

Proposition 2.10: The operator B as defined per equation (2.7) is self-adjoint, positive, and dominates all operators $B(I)$.

Proof. To show that it dominates all $B(I)$ is rather simple, as seen:

$$C \leq [\mathbb{1} + B(I)]^{-1} \Rightarrow C^{-1} \geq \mathbb{1} + B(I) \Rightarrow B = C^{-1} - \mathbb{1} \geq B(I) \forall I$$

To show that it is positive, further comments on the domain of definition of B must be made. As established, B is to be defined in the space $\mathscr{H}_{finite} = (\mathscr{H}_0 + \mathscr{H}_\infty)^\perp$. As defined, $\mathscr{H}_\infty$ is the kernel of the operator C, and considering the orthogonal subspace to

it, there could be vectors $\psi, \phi \neq 0$, $\phi, \psi \in \mathscr{H}_{finite}$ such that $C\psi = C\phi$, which would mean that $\psi - \phi \in \mathscr{H}_\infty$. But, since we're considering the subspace $\mathscr{H}_{finite} = (\mathscr{H}_0 + \mathscr{H}_\infty)^\perp$, then any finite linear combination of elements of $\mathscr{H}_{finite}$ must be in $\mathscr{H}_{finite}$. Thus, $\psi - \phi \in \mathscr{H}_0 + \mathscr{H}_\infty$, and $\psi - \phi \in (\mathscr{H}_0 + \mathscr{H}_\infty)^\perp$, but, by definition of these spaces $(\mathscr{H}_0 + \mathscr{H}_\infty) \cap (\mathscr{H}_0 + \mathscr{H}_\infty)^\perp = \{0\}$, which means that $\psi - \phi = 0 \Rightarrow \psi = \phi$. And thus, in $\mathscr{H}_{finite}$, C is an injective operator. Since it is injective, we may apply proposition 2.9, and conclude that C^{-1} is self-adjoint. Let us then consider, for $\psi, \phi \in \mathscr{H}_{finite}$:

$$\langle \psi, B\phi \rangle = \langle \psi, (C^{-1} - \mathbb{1})\phi \rangle = \langle \psi, C^{-1}\phi \rangle - \langle \psi, \phi \rangle = \langle (C^{-1})^*\psi, \phi \rangle - \langle \psi, \phi \rangle =$$

$$= \langle C^{-1}\psi, \phi \rangle - \langle \psi, \phi \rangle = \langle (C^{-1} - \mathbb{1})\psi, \phi \rangle = \langle B\psi, \phi \rangle$$

Thus, B is hermitian. But since the domain of definition of both B and its adjoint are the same, then B is self-adjoint. To see that it is positive, it suffices to recall that, since it dominates every other $B(I)$, then necessarily $B \geq 0$. $\qquad\square$

With these properties of the operator B being established, one can make use of the Spectral Theorem 1.3 to define:

$$B^{-\frac{1}{2}} := \int_{\sigma(B)} \lambda^{-\frac{1}{2}} dE_\lambda \tag{2.9}$$

where $\sigma(B)$ is the spectrum of B and E_λ are its spectral projectors. At this point, it is finally possible to define:

$$P(I) := B^{-\frac{1}{2}} B(I) B^{-\frac{1}{2}} \tag{2.10}$$

Let us now consider the following result from:

Proposition 2.11 [[28], Prop. 39.48]: Let $\mathcal{A}$ be a unital C^*-algebra and let $a, b \in \mathcal{A}$ be self-adjoint elements such that $a \geq b$. Then, for every $c \in \mathcal{A}$, it holds:

$$c^*ac \geq c^*bc \tag{2.11}$$

With this previous result, it is straightforward to conclude that the operator $P(I)$ as defined per equation (2.10) is bounded by the identity. In fact:

$$\mathbb{1} - P(I) = B^{-\frac{1}{2}}BB^{-\frac{1}{2}} - B^{-\frac{1}{2}}B(I)B^{-\frac{1}{2}}$$

Now, since B dominates all other $B(I)$, then $B - B(I) \geq 0$, which means that $B^{-\frac{1}{2}}(B - B(I))B^{-\frac{1}{2}} \geq 0 \forall I$. Thus, $\mathbb{1} - P(I) \geq 0 \Rightarrow \mathbb{1} \geq P(I) \forall I$.

Another important property to verify about the operator $P(I)$ is its behaviour under disjoint unions. For $I, I' \subset \mathbb{R}^4$, with $I \cap I' = \emptyset$, one sees that:

$$P(I \cup I') = B^{-\frac{1}{2}}B(I \cup I')B^{-\frac{1}{2}} = B^{-\frac{1}{2}}\left[\int_{I \cup I'} \alpha_x(A)dx\right]B^{-\frac{1}{2}} =$$

$$= B^{-\frac{1}{2}}\left[\int_I \alpha_x(A)dx + \int_{I'} \alpha_x(A)dx\right]B^{-\frac{1}{2}} = B^{-\frac{1}{2}}(B(I) + B(I'))B^{-\frac{1}{2}} = P(I) + P(I')$$

At this point, the defining characteristic of this construction can be verified. That is, if $P(I)$ is covariant with respect to translations in $\mathbb{R}^n$:

Proposition 2.12: The operator $P(I)$ as defined per equation (2.10) is covariant with respect to translations in $\mathbb{R}^n$.

Proof. To show that $P(I)$ is covariant is to show that $\alpha_y(P(I)) = P(I + y)$ for any $I \in \mathcal{B}(\mathbb{R}^n)$ and any $y \in \mathbb{R}^4$. Thus:

$$\alpha_y(P(I)) = \alpha_y(B^{-\frac{1}{2}}B(I)B^{-\frac{1}{2}})$$

Now, since symmetry transformations are represented in the Hilbert space of physical systems as inner automorphisms as per Wigner's Theorem, then:

$$\alpha_y(B^{-\frac{1}{2}}B(I)B^{-\frac{1}{2}}) = U_y^*(B^{-\frac{1}{2}}B(I)B^{-\frac{1}{2}})U_y = U_y^*B^{-\frac{1}{2}}U_yU_y^*B(I)U_yU_y^*B^{-\frac{1}{2}}U_y =$$

$$= \alpha_y(B^{-\frac{1}{2}})\alpha_y(B(I))\alpha_y(B^{-\frac{1}{2}})$$

One can study $\alpha_y(B)$:

$$\alpha_y(B) = \alpha_y\left(\lim_{I \to \mathbb{R}^n}\int_I \alpha_x(A)dx\right) = \lim_{I \to \mathbb{R}^n}\int_I \alpha_y(\alpha_x(A))dx = \lim_{I \to \mathbb{R}^n}\int_I \alpha_{y+x}(A)dx =$$

$$= \lim_{I \to \mathbb{R}^n} \int_{I-y} \alpha_x(A)dx = \int_{\mathbb{R}^n} \alpha_x(A)dx = B$$

Thus B is invariant under translations α_y. Since B is invariant, this means that its spectral projectors will also be invariant to translations. Since:

$$B^{-\frac{1}{2}} = \int_{\sigma(B)} \lambda^{-\frac{1}{2}} dE_\lambda$$

then $B^{-\frac{1}{2}}$ will also be invariant to translations. The only task left is that of seeing the properties of $B(I)$ under translations. Consider:

$$\alpha_y(B(I)) = \alpha_y \left(\int_I \alpha_x(A)dx \right) = \int_I \alpha_y(\alpha_x(A))dx = \int_I \alpha_{y+x}(A)dx = \int_{I-y} \alpha_x(A)dx = B(I-y)$$

And thus:

$$\alpha_y(P(I)) = B^{-\frac{1}{2}} B(I-y) B^{-\frac{1}{2}} = P(I-y)$$

that is, $P(I)$ is covariant with respect to translations on $\mathbb{R}^4$. $\qquad\square$

Now, these are all the characteristics that define a Positive Operator Valued Measure, and P will be the wanted measure on the Hilbert space $\mathscr{H}$. Though, there is still a slight question on the domain of definition of the measure P. Every measure must be defined on a σ-algebra of some space X. This measure P was constructed to act upon sets of **finite** measure of $\mathbb{R}^n$, and as such, the domain of definition of P as of now is not comprised of a σ-algebra (particularly, we will want to define this measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$). The last step in constructing this Positive Operator Valued Measure is then to extend its definition to all Borel sets of $\mathbb{R}^n$.

According to definition 2.5, a σ-algebra must be such that it is defined in the whole space, which it is in this case, since $P(\mathbb{R}^n) = \mathbb{1}$, and also in the empty set, which is rather natural to define as $P(\emptyset) = 0$. For the complement of sets of finite measure, one has:

$$\mathbb{1} = \int_{\mathbb{R}^n} \alpha_x(A)dx = \int_I \alpha_x(A)dx + \int_{\mathbb{R}^n \setminus I} \alpha_x(A)dx = P(I) + P(\mathbb{R}^n \setminus I) \Rightarrow P(\mathbb{R}^n \setminus I) = \mathbb{1} - P(I)$$

There are now two questions which must be addressed. How to evaluate P on sets of infinite measure which can not be written as the complement of a set of finite measure,

and the σ-additivity of P. Addressing the σ-additivity of P solves the question of defining P in the sets that are left, due to the basic result that every open set in a second countable space, which $\mathbb{R}^n$ is, can be covered by a countable family of open sets. In this case, the family of open sets which form a basis for $\mathbb{R}^n$ are open balls, that are of finite measure. Let then $\{E_n\}_{n\in\mathbb{N}}$ be a family of disjoint sets that cover a set E of non-finite measure in $\mathbb{R}^n$ such that $E = \bigcup_{n\in\mathbb{N}} E_n$. Consider then:

$$P(E) = B^{-\frac{1}{2}} B(E) B^{-\frac{1}{2}} = B^{-\frac{1}{2}} \left[\int_E \alpha_x(A) dx \right] B^{-\frac{1}{2}} = B^{-\frac{1}{2}} \left[\int_{\bigcup_{n\in\mathbb{N}} E_n} \alpha_x(A) dx \right] B^{-\frac{1}{2}} =$$

$$= B^{-\frac{1}{2}} \left[\sum_{n\in\mathbb{N}} \int_{E_n} \alpha_x(A) dx \right] B^{-\frac{1}{2}} = \sum_{n\in\mathbb{N}} B^{-\frac{1}{2}} \left[\int_{E_n} \alpha_x(A) dx \right] B^{-\frac{1}{2}} = \sum_{n\in\mathbb{N}} B^{-\frac{1}{2}} B(E_n) B^{-\frac{1}{2}}$$

Thus:

$$P(E) = \sum_{n\in\mathbb{N}} P(E_n)$$

which is σ-additivity.

Thus, the Positive Operator Valued Measure P has been finally properly established, it being associated to the event measured by the observable A. One may then turn the attention to:

$$T_A = \int t P(dx) \tag{2.12}$$

This is the first moment of the first coordinate of the points in $\mathbb{R}^n$, and it can be interpreted as the operator of time of occurrence of the event associated to the observable A.

At this point, half of the job of constructing a time operator for relativistic systems is done. As is known, given a representation U of the Poincaré group (actually of the proper orthochronous Poincaré group, if one is to deal with non-discrete symmetries that preserve causality), it can be written as $U(\Lambda, a)$, where Λ is a Lorentz transformation and a a point of the translation group in $\mathbb{R}^4$. Now, due to the fact that the Poincaré group is a semidirect product of the Lorentz group with the translation group, its properties of transformation translates to its representations as:

$$U(a, \Lambda) = U(a, \mathbb{1})U(0, \Lambda) \tag{2.13}$$

If the goal is to construct a Positive Operator Valued Measure covariant with respect to Poincaré transformations of relativistic systems, the previous formalism must be generalized so that a POVM that is covariant with respect to Lorentz transformations can be constructed, which will be done in the next section.

Chapter 3

Extending the Method to Another Symmetry

3.1 A Covariant Measure for Lorentz Transformations

As was seen in section 2.2, the starting point of the construction of these POVMs is to consider a given state ω, and its evaluation on the action of the inner automorphism corresponding to a symmetry transformation α_g on a positive, bounded observable A: $\omega(\alpha_g(A))$. The point is then to consider operators in the spirit of equation (2.3):

$$B(V) = \int_V \alpha_g(A) dg. \tag{3.1}$$

This then begs the question on how one may define such integrals on groups. This is a matter well studied and defined by the means of Haar integrals. To establish these concepts, an excursion on topological groups will be made. Beginning with a few basic concepts [31]:

Definition 3.1: A **topological group** is a pair (G, τ), where G is a group and τ is a topology defined on the group, such that the map $f : G \times G \to G$ given by $f(x, y) = xy^{-1}$ is continuous in this topology. That is, given an an open set $V \in \tau$, then $f^{-1}(V) \in \tau \times \tau$, that is $f^{-1}(V)$ is an open set of $G \times G$, where $\tau \times \tau$ is the product topology in $G \times G$.

Because of the topological structure of topological groups, many of the usual concepts apply.

Definition 3.2: A **covering** of a set $Y \subset X$ is a collection of sets from $\mathbb{P}(X)$ such that the union of all the sets of the collection contains Y. That is, given a collection of sets $\{V_\lambda\}_{\lambda \in \Lambda}$, with Λ an indexing set, then this collection is a covering of Y if $Y \subset \bigcup_{\lambda \in \Lambda} V_\lambda$. A covering is said to be **finite** if it contains a finite amount of sets.

Definition 3.3: A **compact space** is a pair (X, τ) such that **any** covering of X by open sets has a finite subcovering, that is, a finite sub collection of sets from the original covering that covers X.

Definition 3.4: A **compact group** is a topological group that is also a compact space considering its topology.

Definition 3.5: A **locally compact group** is a topological group that for every element $g \in G$, there is an open set V, and a compact set C such that $g \in V \subset C$.

Now, as is known from measure theory [32], a Lebesgue measure satisfies, between many other properties, translation invariance. That is, for a given open set $I \subset \mathbb{R}^n$, a measure μ and $x \in \mathbb{R}^n$, it holds $\mu(I) = \mu(I + x)$. But it is possible to understand $\mathbb{R}^n$ as the locally compact additive group $(\mathbb{R}^n, +)$. Thus, it is rather natural to ask whether one may construct measures with such a property in an arbitrary locally compact group.

Among some other properties one loses when abstracting to a general locally compact group from the additive group is the one of commutativity. That is $I + x$ and $x + I$ represent exactly the same sets in $\mathbb{R}^n$. This leads then to the separation between two kinds of invariant measures:

Definition 3.6: Consider a group G, and a measure μ defined on the group. If, for any given $g \in G$ and $V \subset G$ it holds $\mu(gV) = \mu(V)$, then the measure μ is said to be **left invariant**. In case it holds $\mu(Vg) = \mu(V)$, then the measure μ is said to be **right invariant**.

The question then of whether, given an arbitrary locally compact group G, there exists defined on this group a measure μ, either left or right invariant, was proved by Haar in 1933, and became the celebrated:

Theorem 3.1 (Haar) [[31], Pg. 65]: On every locally compact group G, there

exists at least one left invariant positive measure μ. This measure is unique up to a positive constant of proportionality, that is, given another left invariant measure ν on G, it is of the form $\nu = c\mu$ for a positive $c \in \mathbb{R}$.

Due to this theorem, the measures μ and the integrals associated with them are known as **Haar integrals**.

Though this theorem is asserting the existence only of **left** invariant measures, it is a rather straightforward consequence that it also guarantees the existence of **right** invariant measures, by means of a one to one correspondence. First, let us define the following concept that will relate left and right invariant measures:

Definition 3.7: Let ν be a measure in a locally compact group G, such that $\int f(x)d\nu(x) = \int (tf)(x)d\mu(x)$ for an arbitrary element $t \in G$, where $(tf)(x) := f(xt^{-1})$ and a left invariant measure μ in G. Then, defined in this way, the measure ν will also be left invariant. Now, due to Haar's theorem, μ and ν differ by at most a positive real number. This number will depend on the element t, and will be denoted as $\Delta_r(t)$. It will be called the **right hand modulus** of G. One may equivalently define a **left hand modulus** starting from a right invariant measure μ.

Proposition 3.1 [[31], Pg. 78]: Given a left invariant Haar integral μ on a locally compact group G guaranteed by Theorem 3.1, then it holds

$$\int f(x^{-1})d\mu(x) = \int \frac{f(x)}{\Delta_r(x)}d\mu(x) \tag{3.2}$$

for every function of compact support in G. Here $\Delta_r(x)$ is the right hand modulus of G. Now, the linear functional ν which associates to every function of compact support f the value in the previous expression is a right invariant measure on G. That is:

$$\int f(x)d\nu(x) = \int f(x^{-1})d\mu(x) = \int \frac{f(x)}{\Delta_r(x)}d\mu(x)$$

is a right invariant Haar integral.

This begs the question then of what kind of measure is the Lebesgue measure on $\mathbb{R}^n$ when seen as the additive group, since in it the following holds: $l(I + y) = l(y + I) = l(I)$

where $I \subset \mathbb{R}^n$, $y \in \mathbb{R}^n$ and l is the Lebesgue measure of the set I. This means that the Lebesgue measure is both left and right invariant. This may be translated to the level of moduli as per the following proposition:

Proposition 3.2 [[31], Pg. 80]: In a locally compact group G, it holds $\Delta_l(g)\Delta_r(g) = 1$, where $\Delta_{l,r}$ are, respectively, the left and the right hand modulus.

This shows that, when either the left hand or the right hand modulus of a locally compact group G is identically equal to 1, then the other will be, and this means that the left invariant measures will be equal to their right invariant counterparts. This leads to the following concept:

Definition 3.8: A locally compact group G is said to be **unimodular** if every left invariant measure is also right invariant and vice-versa.

Thus, in unimodular groups, Haar measures are both left and right invariant. These measures will be denoted simply as **invariant** measures. Unimodular groups are slightly easier to work with in this construction method, due to the modulus functions being identically equal to one, which means that one does not need to take them into account when normalizing the resulting Positive Operator Valued Measure.

The next step in constructing a covariant Positive Operator Valued Measure with respect to Lorentz transformations is then to find the explicit Haar measure of the Lorentz group. A first property to verify is then whether the Lorentz group is unimodular. For this, some conceptual aspects must first be exposed.

The Lorentz group, which will be the main interest in this section, has more structure than merely being a locally compact topological group. By being a closed subgroup of the Lie Group $GL(4,\mathbb{R})$, it is itself a Lie Group, which has many benefits, in particular, analytical behaviour in its product and inversion operations due to the structure itself forming a manifold. To define it more formally:

Definition 3.9: A **second countable space** is a topological space with a countable base.

Definition 3.10: A **locally Euclidian space** of dimension n is a topological space that has at least one covering by open sets that are homeomorphic to open balls in $\mathbb{R}^n$.

The definition of a manifold in this work will then be:

Definition 3.11: A **second countable manifold** is a topological space that is second countable and locally Euclidian.

Definition 3.12: A **local coordinate chart** is an open set V, along with a homeomorphism f to an open ball in $\mathbb{R}^n$.

Definition 3.13: Given two local coordinate charts of two sets U and V, with $U \cap V \neq \emptyset$, and its two corresponding homeomorphisms f and g, then a **transition function** is a homeomorphism from $f(U \cap V)$ to $g(U \cap V)$ (or vice versa), defined as $t := g \circ f^{-1}$ (respectively $t := f \circ g^{-1}$).

To introduce the concepts of analiticity and differentiability, the notion of diffeomorphism must first be defined:

Definition 3.14: A **diffeomorphism** is a homeomorphism between spaces, that is, a bijective function with continuous inverse, such that the function and its inverse are also differentiable.

Definition 3.15: A manifold is said to be C^k-differentiable if all of its transition functions are k times differentiable. A manifold is said be smooth if all of its transition functions are C^∞-differentiable.

A Lie Group can thus be defined as:

Definition 3.16: A **Lie Group** is a topological group which is also a C^∞-differentiable manifold.

Some properties from Lie Groups are essential in a characterization of whether there are invariant measures on it that will be enunciated soon. Firstly:

Definition 3.17: Let H be a topological subgroup of a group G. H is said to be an invariant subgroup if for each $h \in H$ and $g \in G$ we have $g^{-1}hg \in H$, that is, if $g^{-1}Hg \subset H$.

And with the concept of invariant subgroups, one may define:

Definition 3.18: A Lie group is said to be a **semisimple group** if it contains no proper invariant connected abelian Lie subgroup.

With these concepts established, the groundwork for a fundamental result is complete, which characterizes which Lie Groups are unimodular:

Theorem 3.2 [[33], Pg. 114]: The following groups are unimodular:

1. Lie Groups for which the set of modular functions $\{\Delta_r(g), g \in G\}$ is compact;

2. Semisimple Lie Groups;

3. Connected Nilpotent Lie Groups.

Since the Lorentz Group is a semisimple Lie group, then the Lorentz group is unimodular. Thus, there exists an invariant Haar measure on the Lorentz group, which will be denoted $\mu(\Lambda)$.

One can then define on the Lorentz group:

$$B(V) := \int_V \alpha_\Lambda(A) d\mu(\Lambda) \tag{3.3}$$

where V is a Borel set of finite μ measure in the topology of the Lorentz group, and α_Λ is the inner automorphism corresponding to the action of a Lorentz transformation on a positive, bounded observable A.

Proposition 3.3: Operators $B(V)$ as defined in (3.3) are, themselves, positive, bounded and self-adjoint for a positive and bounded observable A.

Proof. Let $\Psi \in \mathscr{H}$. Then:

$$\langle \Psi, B(V)\Psi \rangle = \int_V \langle \Psi, \alpha_\Lambda(A)\Psi \rangle d\mu(\Lambda) = \int_V \langle \Psi, U_\Lambda^* A U_\Lambda \Psi \rangle d\mu(\Lambda) = \int_V \langle U_\Lambda \Psi, A U_\Lambda \Psi \rangle d\mu(\Lambda)$$

Here, U_Λ is a representation of the Lorentz group in $\mathscr{H} = \int^\oplus \mathscr{H}_k$. The explicit form of these representations are well known, and will be shown in Appendix A. The integrand then can be understood as the expectation value of the observable A in the state $U_\Lambda\Psi$: $\langle A\rangle_{U_\Lambda\Psi}$. Since A is a positive and bounded observable, then $\langle A\rangle_{U_\Lambda\Psi}$ is positive and bounded for any $U_\Lambda\Psi$. Thus, the integration over a Borel set of finite measure also results in a finite value for the integral. As such, the operator $B(V)$ is a bounded and positive operator.

Let us consider then:

$$\langle B(V)\Psi, \Phi\rangle = \int_V \langle \alpha_\Lambda(A)\Psi, \Phi\rangle d\mu(\Lambda) = \int_V \langle U_\Lambda^* A U_\Lambda \Psi, \Phi\rangle d\mu(\Lambda) = \int_V \langle \Psi, (U_\Lambda^* A U_\Lambda)^* \Phi\rangle d\mu(\Lambda) =$$

$$= \int_V \langle \Psi, U_\Lambda^* A^* (U_\Lambda^*)^* \Phi\rangle d\mu(\Lambda) = \int_V \langle \Psi, U_\Lambda^* A U_\Lambda \Phi\rangle d\mu(\Lambda) = \langle \Psi, B(V)\Phi\rangle$$

where on these calculations were used the facts that $U_\Lambda^* A U_\Lambda$ is a bounded operator, since A is bounded and U_Λ is unitary, and thus its adjoint is defined for any $\Phi \in \mathscr{H}$, and that A is an observable, and thus, self-adjoint. $\qquad\square$

Consider now a net of Borel sets of finite measure $\{V_\gamma\}_{\gamma\in\Gamma}$, Γ being an indexing set. This net is such that all sets of finite measure in the Lorentz group, including also the pure Lorentz transformed sets of any set of finite measure are in the net, and also such that, if $\gamma \prec \gamma'$ for some $\gamma, \gamma' \in \Gamma$, then $V_\gamma \subset V_{\gamma'}$.

Given $\Psi \in \mathscr{H}$, and $\gamma \prec \gamma'$ with $\gamma, \gamma' \in \Gamma$:

$$\langle\Psi, B(V_{\gamma'})\Psi\rangle = \int_{V_{\gamma'}} \langle\Psi, \alpha_\Lambda(A)\Psi\rangle d\mu(\Lambda) = \int_{V_\gamma} \langle\Psi, \alpha_\Lambda(A)\Psi\rangle d\mu(\Lambda) + \int_{V_{\gamma'}\backslash V_\gamma} \langle\Psi, \alpha_\Lambda(A)\Psi\rangle d\mu(\Lambda) =$$

$$= \langle\Psi, B(V_\gamma)\Psi\rangle + \langle\Psi, B(V_{\gamma'} \backslash V_\gamma)\Psi\rangle \tag{3.4}$$

Since $B(V)$ is a positive operator for any Borel set of finite measure V, then the previous equation implies:

$$B(V_{\gamma'}) = B(V_\gamma) + B(V_{\gamma'} \backslash V_\gamma) \Rightarrow B(V_{\gamma'}) \geq B(V_\gamma)$$

Considering once again proposition 2.4 one can conclude that the net of operators

given by $\{\mathbb{1} + B(V_\gamma)\}_{\gamma \in \Gamma}$ is a net of invertible operators, themselves being positive, self-adjoint operators. Proposition 2.5 once again establishes the ordering in this net.

This means then that the net $\{[\mathbb{1} + B(V_\gamma)]^{-1}\}_{\gamma \in \Gamma}$ inherits the partial ordering from the net $\{V_\gamma\}_{\gamma \in \Gamma}$, but is a *decreasing* net, in the sense that, if $\gamma \prec \gamma'$, then $[\mathbb{1} + B(V_{\gamma'})]^{-1} \leq [\mathbb{1} + B(V_\gamma)]^{-1}$.

Following then the same steps that were done in the construction of the translation covariant POVM, one can use this ordering to establish the strong convergence to an operator C:

Proposition 3.4: The net of operators $\{[\mathbb{1} + B(V_\gamma)]^{-1}\}_{\gamma \in \Gamma}$ converges strongly in $\mathscr{B}(\mathscr{H})$.

Proof. By the results shown before, the elements of the net obey, for $\gamma \prec \gamma'$, $[\mathbb{1} + B(V_{\gamma'})]^{-1} \leq [\mathbb{1} + B(V_\gamma)]^{-1}$. This in turn translates to, according to item 1. of proposition 2.6:

$$\langle \Psi, [\mathbb{1} + B(V_{\gamma'})]^{-1}\Psi \rangle \leq \langle \Psi, [\mathbb{1} + B(V_\gamma)]^{-1}\Psi \rangle$$

Now, for fixed $\Psi \in \mathscr{H}$, $\langle \Psi, [\mathbb{1} + B(V_\gamma)]^{-1}\Psi \rangle$ is a decreasing net of real numbers for $\gamma \in \Gamma$, and as such, must converge to a real number. It is thus also a Cauchy sequence, and for $\gamma \prec \gamma'$ appropriately chosen in Γ, one has:

$$|\langle \Psi, [\mathbb{1} + B(V_\gamma)]^{-1}\Psi \rangle - \langle \Psi, [\mathbb{1} + B(V_{\gamma'})]^{-1}\Psi \rangle| < \epsilon_\Psi^2$$

Let us then define $D_\gamma := [\mathbb{1} + B(V_\gamma)]^{-1}$ and $D_{\gamma'} := [\mathbb{1} + B(V_{\gamma'})]^{-1}$. Then:

$$|\langle \Psi, D_\gamma \Psi \rangle - \langle \Psi, D_{\gamma'} \Psi \rangle| = |\langle \Psi, (D_\gamma - D_{\gamma'})\Psi \rangle| < \epsilon_\Psi^2$$

Once again, $D_\gamma - D_{\gamma'}$ is a self-adjoint operator, since it is the difference of two self-adjoint operators D_γ and $D_{\gamma'}$. Then:

$$\|(D_\gamma - D_{\gamma'})\Psi\|^2 = \langle (D_\gamma - D_{\gamma'})\Psi, (D_\gamma - D_{\gamma'})\Psi \rangle = \langle \Psi, (D_\gamma - D_{\gamma'})^2\Psi \rangle$$

As before, since $B(V)$ is a positive operator for any V of finite measure, then $\mathbb{1} + B(V) \geq \mathbb{1}$ for any $V \in \mathcal{B}(L_+^\uparrow)$. Due to proposition 2.5 then, $[\mathbb{1} + B(V)]^{-1} \leq \mathbb{1}^{-1} = \mathbb{1}$. This then means that $D_\gamma - D_{\gamma'} \leq \mathbb{1} \Rightarrow \|D_\gamma - D_{\gamma'}\| \leq 1$. As a consequence of item 3. of proposition 2.7, we have:

$$1 \cdot (D_\gamma - D_{\gamma'}) \geq \|D_\gamma - D_{\gamma'}\|(D_\gamma - D_{\gamma'}) \geq (D_\gamma - D_{\gamma'})^2 \geq 0e$$

One can thus conclude that:

$$|\langle \Psi, (D_\gamma - D_{\gamma'})^2 \Psi \rangle| \leq |\langle \Psi, (D_\gamma - D_{\gamma'}) \Psi \rangle| =$$

$$= |\langle \Psi, [\mathbb{1} + B(V_\gamma)]^{-1} \Psi \rangle - \langle \Psi, [\mathbb{1} + B(V_{\gamma'})]^{-1} \Psi \rangle| < \epsilon_\Psi^2$$

Since for any fixed Ψ, ϵ_Ψ^2 can be taken to be arbitrarily small for appropriate $\gamma \prec \gamma'$, $\gamma, \gamma' \in \Gamma$, the following holds:

$$\left\| [[\mathbb{1} + B(V_\gamma)]^{-1} - [\mathbb{1} + B(V_{\gamma'})]^{-1}]\Psi \right\| = \|(D_\gamma - D_{\gamma'})\Psi\| < \epsilon_\Psi$$

This last expression is exactly the strong convergence of the net of operators $\{[\mathbb{1} + B(V_\gamma)]^{-1}\}_{\gamma \in \Gamma}$.

The uniqueness of this limit operator comes once again from the fact that the set of indices Γ is a directed set. $\qquad\square$

It is now the moment to turn to the question of whether the following operator:

$$B := \int_{L_+^\uparrow} \alpha_\Lambda(A) d\mu(\Lambda) \tag{3.5}$$

is well defined for the case of the restricted Lorentz group.

Following in the same steps as in the translation case, the space is decomposed into two parts. The first is the space of the kernel of the operator C to which the net $\{[\mathbb{1} + B(V_\gamma)]^{-1}\}_{\gamma \in \Gamma}$ converges. This space corresponds to the vectors in $\mathscr{H}$ that annihilate C. The kernel of C will be denoted again by $\mathscr{H}_\infty$. The second space would be the one formed by the joint kernel of the operators $B(V_\gamma)$. This space will also be denoted as $\mathscr{H}_0$.

If we then consider the orthogonal complement to the sum of these spaces $\mathscr{H}_{finite} = (\mathscr{H}_0 + \mathscr{H}_\infty)^\perp$, it would be a space where the following operator:

$$B := C^{-1} - \mathbb{1} \tag{3.6}$$

would be well defined. It can be understood as the total time duration of the event associated with the observable A, that is:

$$B = \lim_{V \to L_+^\uparrow} \int_V \alpha_\Lambda(A) d\mu(\Lambda) \tag{3.7}$$

This operator is, as in the translation case, positive, self-adjoint, and dominates all other $B(V)$, as will be now shown:

Proposition 3.5: The operator B as defined per equation (3.7) is self-adjoint, positive, and dominates all operators $B(V)$.

Proof. To show that it dominates all $B(V)$ is again seen by:

$$C \leq [\mathbb{1} + B(V)]^{-1} \Rightarrow C^{-1} \geq \mathbb{1} + B(V) \Rightarrow B = C^{-1} - \mathbb{1} \geq B(V), \forall V \in \mathcal{B}(L_+^\uparrow)$$

The same considerations as those done in proposition 2.10 may be applied to this case in order to see that B is well defined in the previously defined Hilbert space and that, in $\mathscr{H}_{finite}$, C is an injective operator. Since it is injective, we may apply proposition 2.9, and conclude that C^{-1} is self-adjoint. Let us then consider, for $\psi, \phi \in \mathscr{H}_{finite}$:

$$\langle \psi, B\phi \rangle = \langle \psi, (C^{-1} - \mathbb{1})\phi \rangle = \langle \psi, C^{-1}\phi \rangle - \langle \psi, \phi \rangle = \langle (C^{-1})^*\psi, \phi \rangle - \langle \psi, \phi \rangle =$$

$$= \langle C^{-1}\psi, \phi \rangle - \langle \psi, \phi \rangle = \langle (C^{-1} - \mathbb{1})\psi, \phi \rangle = \langle B\psi, \phi \rangle$$

Thus, B is hermitian. But since the domain of definition of both B and its adjoint are the same, then B is self-adjoint. To see that it is positive, it suffices to recall that, since it dominates every other $B(V)$, then necessarily $B \geq 0$. $\qquad\square$

With these properties of the operator B being established, one can make use of the Spectral Theorem 1.3 to define:

$$B^{-\frac{1}{2}} := \int_{\sigma(B)} \lambda^{-\frac{1}{2}} dE_\lambda \tag{3.8}$$

where $\sigma(B)$ is the spectrum of B and E_λ are its spectral projectors. At this point, one can define:

$$P(V) := B^{-\frac{1}{2}} B(V) B^{-\frac{1}{2}} \tag{3.9}$$

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Again, through Proposition 2.11 it is straightforward to conclude that the operator $P(V)$ as defined per equation (3.9) is bounded by the identity. In fact:

$$\mathbb{1} - P(V) = B^{-\frac{1}{2}}BB^{-\frac{1}{2}} - B^{-\frac{1}{2}}B(V)B^{-\frac{1}{2}}$$

Now, since B dominates all other $B(V)$, then $B - B(V) \geq 0$, which means that $B^{-\frac{1}{2}}(B - B(V))B^{-\frac{1}{2}} \geq 0, \forall V \in \mathcal{B}(L_+^\uparrow)$. Thus, $\mathbb{1} - P(V) \geq 0 \Rightarrow \mathbb{1} \geq P(V) \forall V \in \mathcal{B}(L_+^\uparrow)$.

Another important property to verify about the operator $P(V)$ is its behaviour under disjoint unions. For $V, V' \subset \mathcal{B}(L_+^\uparrow)$, with $V \cap V' = \emptyset$, we see that:

$$P(V \cup V') = B^{-\frac{1}{2}}B(V \cup V')B^{-\frac{1}{2}} = B^{-\frac{1}{2}}\left[\int_{V \cup V'} \alpha_\Lambda(A)d\mu(\Lambda)\right]B^{-\frac{1}{2}} =$$

$$= B^{-\frac{1}{2}}\left[\int_V \alpha_\Lambda(A)d\mu(\Lambda) + \int_{V'} \alpha_\Lambda(A)d\mu(\Lambda)\right]B^{-\frac{1}{2}} = B^{-\frac{1}{2}}(B(V)+B(V'))B^{-\frac{1}{2}} = P(V)+P(V')$$

At this point, it is possible to verify that $P(V)$ is indeed covariant with respect to restricted Lorentz transformations:

Proposition 3.6: The operator $P(V)$ as defined per equation (3.9) is covariant with respect to restricted Lorentz transformations in $\mathcal{B}(L_+^\uparrow)$.

Proof. To show that $P(V)$ is covariant is to show that $\alpha_\Lambda(P(V)) = P(\Lambda V)$ for any $V \in \mathcal{B}(L_+^\uparrow)$ and any $\Lambda \in L_+^\uparrow$. Thus:

$$\alpha_\Lambda(P(V)) = \alpha_\Lambda(B^{-\frac{1}{2}}B(V)B^{-\frac{1}{2}})$$

Now, since symmetry transformations are represented in the Hilbert space of physical systems as inner automorphisms as per Wigner's Theorem, then:

$$\alpha_\Lambda(B^{-\frac{1}{2}}B(V)B^{-\frac{1}{2}}) = U_\Lambda^*(B^{-\frac{1}{2}}B(V)B^{-\frac{1}{2}})U_\Lambda = U_\Lambda^*B^{-\frac{1}{2}}U_\Lambda U_\Lambda^*B(I)U_\Lambda U_\Lambda^*B^{-\frac{1}{2}}U_\Lambda =$$

$$= \alpha_\Lambda(B^{-\frac{1}{2}})\alpha_\Lambda(B(V))\alpha_\Lambda(B^{-\frac{1}{2}})$$

Let us study $\alpha_\Lambda(B)$:

$$\alpha_\Lambda(B) = \alpha_\Lambda\left(\lim_{V\to L_+^\uparrow}\int_V \alpha_{\Lambda'}(A)d\mu(\Lambda')\right) = \lim_{V\to L_+^\uparrow}\int_V \alpha_\Lambda(\alpha_{\Lambda'}(A))d\mu(\Lambda') = \lim_{V\to L_+^\uparrow}\int_V \alpha_{\Lambda\Lambda'}(A)d\mu(\Lambda') =$$

$$= \lim_{V\to L_+^\uparrow}\int_{\Lambda^{-1}V} \alpha_{\Lambda'}(A)d\mu(\Lambda') = \int_{L_+^\uparrow} \alpha_{\Lambda'}(A)d\mu(\Lambda') = B$$

It is important to note that the step in which the measure is translated, so as to let $d\mu(\Lambda') \to d\mu(\Lambda^{-1}\Lambda') = d\mu(\Lambda')$, is justified in the basis of the measure $\mu(\Lambda')$ being an invariant measure, and thus, in particular, a left-invariant measure.

Thus B is invariant under restricted Lorentz transformations α_Λ. Since B is invariant, this means that its spectral projectors will also be invariant to the same transformations. The spectral theorem then yields:

$$B^{-\frac{1}{2}} = \int_{\sigma(B)} \lambda^{-\frac{1}{2}}dE_\lambda$$

then $B^{-\frac{1}{2}}$ will also be invariant to restricted Lorentz transformations. The only task left is that of seeing the properties of $B(V)$ under such transformations. It holds:

$$\alpha_\Lambda(B(V)) = \alpha_\Lambda\left(\int_V \alpha_{\Lambda'}(A)d\mu(\Lambda')\right) = \int_V \alpha_\Lambda(\alpha_{\Lambda'}(A))d\mu(\Lambda) = \int_V \alpha_{\Lambda\Lambda'}(A)d\mu(\Lambda') =$$

$$= \int_{\Lambda^{-1}V} \alpha_{\Lambda'}(A)d\mu(\Lambda') = B(\Lambda^{-1}V)$$

Thus:

$$\alpha_\Lambda(P(V)) = B^{-\frac{1}{2}}B(\Lambda^{-1}V)B^{-\frac{1}{2}} = P(\Lambda^{-1}V)$$

that is, $P(V)$ is covariant with respect to restricted Lorentz transformations. $\qquad\square$

Now, these are all the characteristics that define a Positive Operator Valued Measure, and P will be the wanted measure on the Hilbert space $\mathcal{H}$. In the same spirit as happened in the case of translation covariant measures, there is still the question on the domain of definition of the measure P, in which its domain of definition should be a σ-algebra. The measure P was constructed to act upon sets of **finite** measure of $L_+^\uparrow$, and as such,

the domain of definition of P as of now is not comprised of a σ-algebra. The last step in constructing this Positive Operator Valued Measure is then to extend its definition to all Borel sets of $L_+^\uparrow$.

According to definition 2.5, a σ-algebra must be such that it is defined in the whole space, which it is in this case, since $P(L_+^\uparrow) = \mathbb{1}$, and also in the empty set, which is rather natural to define as $P(\emptyset) = 0$. For the complement of sets of finite measure, one has:

$$\mathbb{1} = \int_{L_+^\uparrow} \alpha_\Lambda(A)d\mu(\Lambda) = \int_V \alpha_\Lambda(A)d\mu(\Lambda) + \int_{L_+^\uparrow \setminus V} \alpha_\Lambda(A)d\mu(\Lambda) = P(V) + P(L_+^\uparrow \setminus V)$$

$$P(L_+^\uparrow \setminus V) = \mathbb{1} - P(V)$$

There are now two questions which must be addressed. How to evaluate P on sets of infinite measure which cannot be written as the complement of a set of finite measure, and the σ-additivity of P. Addressing the σ-additivity of P solves the question of defining P in the sets that are left, due to the basic result that every open set in a second countable space can be covered by a countable family of open sets. In this case, $L_+^\uparrow$ is indeed a second countable space, due to it being a Lie group, and thus, a manifold. Let then $\{F_n\}_{n \in \mathbb{N}}$ be a family of disjoint sets that cover a set F of non finite measure in $L_+^\uparrow$ such that $F = \bigcup_{n \in \mathbb{N}} F_n$. We have:

$$P(F) = B^{-\frac{1}{2}} B(F) B^{-\frac{1}{2}} = B^{-\frac{1}{2}} \left[\int_F \alpha_\Lambda(A)d\mu(\Lambda) \right] B^{-\frac{1}{2}} = B^{-\frac{1}{2}} \left[\int_{\bigcup_{n \in \mathbb{N}} F_n} \alpha_\Lambda(A)d\mu(\Lambda) \right] B^{-\frac{1}{2}} =$$

$$= B^{-\frac{1}{2}} \left[\sum_{n \in \mathbb{N}} \int_{F_n} \alpha_\Lambda(A)d\mu(\Lambda) \right] B^{-\frac{1}{2}} = \sum_{n \in \mathbb{N}} B^{-\frac{1}{2}} \left[\int_{F_n} \alpha_\Lambda(A)d\mu(\Lambda) \right] B^{-\frac{1}{2}} = \sum_{n \in \mathbb{N}} B^{-\frac{1}{2}} B(F_n) B^{-\frac{1}{2}}$$

Thus:

$$P(F) = \sum_{n \in \mathbb{N}} P(F_n)$$

which is σ-additivity.

Thus, the Positive Operator Valued Measure P has been finally properly established, it being associated to the event measured by the observable A and covariant with respect

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to Lorentz transformations.

At this point then, two separate POVMs have been constructed, each covariant with respect to the group in relation to which the operators $B(I)$ or $B(V)$ are constructed. The next step is clearly piecing both POVMs together to form a fully Poincaré covariant measure, having then attained the goal of this work.

3.2   A Restricted Poincaré Covariant Measure

As was stated at the end of section 2.2, one interesting property of a representation of an arbitrary Poincaré transformation is its separation into two parts, a representation corresponding to a pure translation and a representation corresponding to a pure Lorentz transformation, that is:

$$U(a, \Lambda) = U(a, \mathbb{1})U(0, \Lambda) \tag{3.10}$$

That is so due to the fact that the restricted Poincaré group is a semidirect product of the translation group with the restricted Lorentz group, which means that its transformation properties translates to the level of representations in this way. Now, the restricted Poincaré group being such a semidirect product implies in some properties that may be used.

To approach this next part, another definition will be needed. Firstly:

Definition 3.19 A **topological automorphism** defined on a group G is a homemorphic bijective map of G into itself.

Then, the following proposition holds:

Proposition 3.7 [adapted from [31], Pg. 83]: Let α be a topological automorphism of a locally compact group G. Then, there exists a positive real number $\delta(\alpha)$ such that, for an integrable function f under the Haar measure $d\mu(g)$ of the group G, it holds, for both left and for right invariant Haar measures:

$$\int f(\alpha^{-1}g)d\mu(g) = \delta(\alpha) \int f(g)d\mu(g) \tag{3.11}$$

The number $\delta(\alpha)$ is denominated the **modulus** of the automorphism α.

From this result, it follows:

Proposition 3.8 [[31], Pg. 99]: Let a group G be the semidirect product of two locally compact groups H and N, with an action α_n of N on H. Let $d\mu(h)$ and $d\nu(n)$ be the right invariant Haar measures defined on H and N, respectively. Then, the right invariant Haar measure on G, denoted $d\xi(g)$ is given by:

$$\int f(x)d\xi(g) = \int \int f(h, n)d\mu(h)d\nu(n) \tag{3.12}$$

where f is an integrable function defined on G. It also holds:

$$\Delta_l^G(u, v) = \Delta_l^H(u)\Delta_l^N(v)\delta^H(\alpha_v) \tag{3.13}$$

where the superscripts indicate which group the object is referring to.

This has a corresponding result for left invariant Haar measures:

Proposition 3.9 [[31], Pg. 100]: Let a group G be the semidirect product of two locally compact groups H and N, with an action α_n of N on H. Let $d\mu(h)$ and $d\nu(n)$ be the left invariant Haar measures defined on H and N, respectively. Then, the left invariant Haar measure on G, denoted $d\xi(g)$ is given by:

$$\int f(x)d\xi(g) = \int \int \frac{f(h, n)}{\delta^H(\alpha_n)}d\mu(h)d\nu(n) \tag{3.14}$$

where f is an integrable function defined on G. It also holds:

$$\Delta_r^G(u, v) = \frac{\Delta_r^H(u)\Delta_r^N(v)}{\delta^H(\alpha_v)} \tag{3.15}$$

where, once again, the superscripts indicate which group the object is referring to.

In this case then, since the restricted Poincaré group is exactly the semidirect product of the restricted Lorentz group with the translation group, both being locally compact groups, with an action of the Lorentz group on the translation group, we may use the

previous propositions to see at once that the invariant Haar measure will be given exactly by the product of the measures at each of the component groups.

Though the previous paragraph holds true, there is still the question of whether the Poincaré group is unimodular itself. To see that, we first need a definition, extracted from [33]:

Definition 3.20: Given an abstract group G, and two elements $x, y \in G$, the element formed by $xyx^{-1}y^{-1}$ is called the **commutator** of x and y. The set of all elements formed by finite products of commutators q_i of respective elements $x_i, y_i \in G$ is called the **commutant** of the group G, and is denoted Q.

Definition 3.21: Let K be the set of all elements formed by finite products of elements of the form: $xyx^{-1}y^{-1}$, with $x \in Q$ and $y \in G$. Consider then the sequence $K_0 \supset K_1 \supset \cdots \supset K_{n-1} \supset K_n \supset \cdots$, where $K_0 = G$, $K_1 = K$ and K_n is formed by finite products of elements of the form $xyx^{-1}y^{-1}$, with $x \in Q$ and $y \in K_{n-1}$. If there is a finite $m \in \mathbb{N}$ such that $K_m = \{e\}$, then the group is said to be **nilpotent**.

From this definition we can see that the proper ortochronous Poincaré group is a nilpotent and connected group. Thus, it falls in case 3. of Theorem 3.2, and thus it is a unimodular group. With the previous decomposition of its measure, we see that it can be written as:

$$d\nu(\{a, \Lambda\}) = da\,d\mu(\Lambda) \tag{3.16}$$

At this point, one can write a covariant POVM with respect to Poincaré transformations directly, first by constructing the operator $B(M)$:

$$B(M) := \int_M \alpha_{\{a,\Lambda\}}(A)d\nu(\{a, \Lambda\}) \tag{3.17}$$

where M is an open Borel set in the Poincaré group. If we write $M = I \times V$, with $I \in \mathcal{B}(\mathbb{R}^4)$ and $V \in \mathcal{B}(L_+^\uparrow)$ we have:

$$B(M) = \int_I \int_V \alpha_{\{a,\Lambda\}}(A)d\mu(\Lambda)da \tag{3.18}$$

Now, given a representation $U(a, \Lambda)$ of the Poincaré group, it holds:

$$B(M) = \int_I \int_V U(\{a, \Lambda\})^* A U(\{a, \Lambda\}) d\mu(\Lambda) da =$$

$$= \int_I \int_V [U(a, \mathbb{1}) U(0, \Lambda)]^* A U(a, \mathbb{1}) U(0, \Lambda) da d\mu(\Lambda) =$$

$$= \int_I \int_V U(0, \Lambda)^* U(a, \mathbb{1})^* A U(a, \mathbb{1}) U(0, \Lambda) d\mu(\Lambda) da =$$

$$= \int_V U(0, \Lambda)^* \left[\int_I U(a, \mathbb{1})^* A U(a, \mathbb{1}) da \right] U(0, \Lambda) d\mu(\Lambda)$$

Due to the nature of the Poincaré group, restricting the representation U to $U(a, \mathbb{1})$ and to $U(0, \Lambda)$ results exactly in the representations of the restricted Lorentz group and of the translation group in the same Hilbert space. As such, the results of sections 2.2 and 3.1 apply directly here, and the main result of this work then holds:

Proposition 3.10: The Positive Operator Valued Measure covariant with respect to causal Poincaré transformations, that generates a time operator associated to an positive, bounded observable A is given by:

$$P(M) := B^{-\frac{1}{2}} B(M) B^{-\frac{1}{2}} = B^{-\frac{1}{2}} \left[\int_M \alpha_{\{a, \Lambda\}}(A) d\nu(g) \right] B^{-\frac{1}{2}} =$$

$$= B^{-\frac{1}{2}} \left[\int_V U(0, \Lambda)^* \left(\int_I U(a, \mathbb{1})^* A U(a, \mathbb{1}) da \right) U(0, \Lambda) d\mu(\Lambda) \right] B^{-\frac{1}{2}} \tag{3.19}$$

with the associated time operator given by:

$$T_A = \int_{\mathbb{R}^4} \int_{L_+^\uparrow} t P(d\{a, \Lambda\}) \tag{3.20}$$

where the operator $B^{-\frac{1}{2}}$ is constructed in the same way as done in sections 2.2 and 3.1, and $d\{a, \Lambda\} = da d\mu(\Lambda)$.

3.3 A Covariant Measure with respect to the Universal Covering of the Lorentz Group

There is a detail of technical character that was omitted in section 3.1, but that is of importance in applications of this formalism. As is well known [33], the restricted Lorentz group is plagued in the level of its representations by the fact that some of them are not true representations.

Any value measured in an experiment is determined by a state up to a complex number of modulus 1. That is, given a state ψ and a state $e^{i\theta}\psi$ with $\theta \in \mathbb{R}$, then, for an observable A, $\langle \psi, A\psi \rangle = \langle e^{i\theta}\psi, Ae^{i\theta}\psi \rangle$. Now, if we consider a representation of a group G acting on the system, the previous fact has a consequence on these representations. Given two states ψ and $\phi = U(g)\psi$, such that they are in the same ray, that is, such that they differ by only a factor of modulus 1, then the product of two representations $U(g)$ and $U(g')$ is given by:

$$U(gg') = \omega(g, g')U(g)U(g') \tag{3.21}$$

where $\omega(g, g')$ is a complex number of modulus 1. In case this factor is constant and equal to 1, then the representations are called **true representations**. Otherwise they are called **projective representations**.

A seminal paper by Bargmann [34] has characterized the question of projective representations, and has given a method to circumvent the phase factor problem by lifting the system to one where the representations become true representations. Since some representations of the restricted Lorentz group are plagued by such a problem, most of the time the treatment of a physical problem, specially one that will need the explicit form of the representations of the symmetry groups of the system, is lifted to the space in which the representations become true representations. To cite one of the main results of Bargmann's work:

Theorem 3.3 (Lifting Criterion) [[33], Pg. 402]: Let G be a simply connected Lie group with Lie algebra L. Assume that for each skew-symmetric real valued bilinear form $\theta(x, y)$ on L satisfying

$$\theta([x, y], z) + \theta([y, z], x) + \theta([z, x], y) = 0$$

there exists a linear form f on L such that:

$$\theta(x, y) = f([x, y])$$

for all $x, y \in L$. Then each strongly continuous projective representation of G is induced by a strongly continuous unitary representation on the corresponding Hilbert space.

In the case of the restricted Lorentz group, its representations are induced from its universal covering group, which is $SL(2, \mathbb{C})$. Since $SL(2, \mathbb{C})$ is a simply connected, connected Lie group, then it has many properties adequate to the formalism previously studied in this work. Most important, it is a unimodular group, and thus has an invariant measure. For an arbitrary matrix of the group:

$$S = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

the measure is explicitly given in [33], Pg. 68:

$$d\mu(S) = \frac{1}{|\delta|^2} d\beta d\bar{\beta} d\gamma d\bar{\gamma} d\delta d\bar{\delta} \tag{3.22}$$

One may then define, for an arbitrary Borel set $I \times N = X \subset \mathbb{R}^4 \times SL(2, \mathbb{C})$ and a given bounded, positive observable A:

$$B(X) := \int_X \alpha_{\{a,S\}}(A) da d\mu(S) = \int_I \int_N \alpha_{\{a,S\}}(A) \frac{1}{|\delta|^2} d\beta d\bar{\beta} d\gamma d\bar{\gamma} d\delta d\bar{\delta} da \tag{3.23}$$

And through that, the steps delineated throughout sections 2.2 and 3.1 follow, with the resulting Positive Operator Valued Measure being:

$$P(X) := B^{-\frac{1}{2}} B(X) B^{-\frac{1}{2}} = B^{-\frac{1}{2}} \left[\int_X \alpha_{\{a,S\}}(A) d\nu(\{a, S\}) \right] B^{-\frac{1}{2}} =$$

$$= B^{-\frac{1}{2}} \left[\int_N U(0, S)^* \left(\int_I U(a, \mathbb{1})^* A U(a, \mathbb{1}) da \right) U(0, S) \frac{1}{|\delta|^2} d\beta d\bar{\beta} d\gamma d\bar{\gamma} d\delta d\bar{\delta} \right] B^{-\frac{1}{2}} \tag{3.24}$$

with the associated time operator given by:

$$T_A = \int_{\mathbb{R}^4} \int_{SL(2,\mathbb{C})} tP(d\{a, S\}) \tag{3.25}$$

where $d\{a, S\} = \frac{1}{|\delta|^2} d\beta d\overline{\beta} d\gamma d\overline{\gamma} d\delta d\overline{\delta} da$.

Chapter 4

Conclusion

This work has been taken in the direction of making a first step towards generalizing the initial work done by Brunetti and Fredenhagen in [26]. The formalism of Positive Operator Valued Measures has been used extensively in the last decades, through the works of [21], [22], [35], [36], [37] and [23], especially in the sense of determining formally what could be understood as a time of arrival operator for some particle at a detector. [26] differ from some of the most recent works [36], [35] by starting from a different point. While the latter works deal with the formalism of quantum clocks, and associate to a possible time operator the eigenvalues of these quantum clocks, [26] start from a more base point, requiring simply a positive, bounded observable to be considered, for example a detector located at a region of space M.

The work by Toller, [35], seems to have some advantages in relation to the method presented in this work in the sense that the time operator that the author constructs is canonically conjugate to the Hamiltonian of the system. This is in some form remedied by another paper by Brunetti and Fredenhagen [38]. In it, the authors show how their formalism can be used to deduce rigorously an uncertainty relation between the Hamiltonian of the system and the time operator constructed, and thus the canonical commutation relation is not explicitly needed to have the relation between time and energy.

Another point that should be noted is exactly the restrictions on the possible observables that can be considered to construct such time operators. As stated in [26], the observables to which this formalism applies must be **positive** and **bounded**. This is so because the Positive Operator Valued Measure to be constructed must be a probabilistic measure. This clearly shows the reason these restrictions were imposed. A question that remains and that shows the line of work of future works is: can this sort of formalism be extended to include:

1. non-positive operators;

2. unbounded operators.

The first one of these items is rather straightforward, and this author looks forward to expand on it in a coming work. The second item is a critical one in order for this method to be applicable to a wide range of problems of great physical interest, one of them being the case of Newton-Wigner localization operators [39], [40].

This work then could be summarized as being the first step in generalizing this method to a wide array of situations. To construct Poincaré covariant Positive Operator Valued Measures is of fundamental importance if this line of work is to be extended to relativistic cases, and eventually even to the level of Quantum Field Theories, being rather natural its usage in Algebraic Quantum Field Theories, where the building blocks are exactly C^*-algebras of local operators [41].

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Appendix A

Representations of the Universal Covering of the Restricted Poincaré Group

This appendix will be mainly interested in giving an overview on the representation theory applied to the universal covering of the Poincaré group, so that a explicit form of the representations of the restricted Poincaré group are given, which can be eventually used to calculate POVMs and their associated time operators explicitly. This section was greatly based on the exposition presented on [33].

As was said at the end of section 2.2, the Poincaré group is the semidirect product of the translation group in $\mathbb{R}^4$ with the Lorentz group also in 4 dimensions. In this work, the focus will be restricted to *causal* transformations, since the non-continuity of parity and time reversal transformations imply in severe modifications to the analysis realized in this work and also in the methods necessary to construct their representations.

The task of characterizing representations of the many known groups have been worked on for over a century, and has been tangled with many other aspects of group theory. Of special interest were the so called Lie groups, which abound in physical studies, and which are interesting by themselves to the great level of structure they present, by being themselves smooth manifolds. Some fundamental theorems were already presented in sections 3.1 and 3.2, so other aspects will be treated here.

One may start by noting that the Poincaré group is not only a semidirect product, but one of the groups which it is composed of is an abelian group, which is of great use in the task of finding its representations. Let us then consider a unitary representation U of the Poincaré group. As was already noted at the end of section 2.2, it holds:

$$U(a, \Lambda) = U(a, \mathbb{1})U(0, \Lambda) \tag{A.1}$$

where $a \in \mathbb{R}^4$ and $\Lambda \in L_+^\uparrow$. One can then understand $U(a, \mathbb{1})$ and $U(0, \Lambda)$ as the restriction of the representation U to the respective component subgroups of the semidirect product of the Poincaré group. One may denote $U(a, \mathbb{1}) \equiv W_a$ and $U(0, \Lambda) \equiv V_\Lambda$.

Consider then the transformation property of the Poincaré group:

$$(a, \Lambda)(b, \Lambda') = (a + \Lambda b, \Lambda\Lambda') \tag{A.2}$$

This translates to the level of representations as:

$$W_a V_\Lambda W_b V_{\Lambda'} = W_{a+\Lambda b} V_{\Lambda\Lambda'} = W_a W_{\Lambda b} V_\Lambda V_{\Lambda'}$$

which may be rewritten, by doing some basic manipulations, as:

$$V_\Lambda W_b V_{\Lambda^{-1}} = W_{\Lambda b} \tag{A.3}$$

In this case, the dual space to the group of translations is the object of interest. Define:

Definition A.1: A **character** of an abelian locally compact group G is a continuous function $c : G \to \mathbb{C}$ satisfying:

1. $|c(g)| = 1$ for any $g \in G$;

2. $c(gg') = c(g)c(g')$

A character is then a continuous unitary irreducible one dimensional representation. The definition of dual space to a group is then:

Definition A.2: The **dual space** $\hat{G}$ of a group G is the set of equivalence classes of all continuous irreducible unitary representations of the group G.

Now, due to the following proposition:

Proposition A.1 [[33], Pg. 159]: Any irreducible unitary representation of an abelian group G in $\mathbb{C}$ is one-dimensional.

one sees that the dual space to the translation group is exactly the space of its charac-
ters. Considering an automorphism α of the translation group in itself, given a character
c of the group, the composition $c \circ \alpha$ is also a character. Considering then the map
$\hat{\alpha} : \hat{\mathbb{R}}^4 \to \hat{\mathbb{R}}^4$, $\hat{\alpha}(c) = c \circ \alpha$, it can be noted that $\hat{\alpha}$ is an automorphism of the dual space
$\hat{\mathbb{R}}^4$. One sees then that this also applies to the case of the action of the Lorentz group on
the translation group, which is given through the means of automorphisms.

Before proceeding, note that the translation group, by being a non-compact vector
group, has the property that its dual group can be identified with itself, that is $\hat{\mathbb{R}}^4 \equiv \mathbb{R}^4$.

A more general form of Theorem 2.3 will now be presented:

Theorem A.1 (Stone, Naimark, Ambrose, Godement) [[33], Pg. 160]: Let
T be an unitary continuous representation of an abelian locally compact group G in a
Hilbert space $\mathscr{H}$. Then, there exists on the character group $\hat{G}$ a spectral measure $E(\cdot)$
such that:

$$T_g = \int_{\hat{G}} c(g) dE(c) \tag{A.4}$$

With this result, W_a can be written as:

$$W_a = \int_{\mathbb{R}^4} c(a) dE(c)$$

Considering equation (A.3), it holds:

$$V_\Lambda W_b V_\Lambda^{-1} = W_{\Lambda b} = W_{\alpha_\Lambda b} = \int_{\mathbb{R}^4} c(\alpha(b)) dE(c) = \int_{\mathbb{R}^4} c(b) dE(c\alpha_\Lambda^{-1})$$

Now, one may also write the left hand side of the previous equation as:

$$\int_{\mathbb{R}^4} c(b) d(V_\Lambda E(c) V_\Lambda^{-1})$$

Since characters separate points of the group, then:

$$V_\Lambda E(Z) V_\Lambda^{-1} = E(Z\Lambda^{-1}) \tag{A.5}$$

By the very definition of the spectral measure $E(c)$, we also haveit also holds:

$$W_a E(Z) W_a^{-1} = E(z) \tag{A.6}$$

This leads to the introduction of the concept of system of imprimitivity, done for the first time by Mackey in [42], [43]:

Definition A.3: Let X be a space in which the action of a group G is defined on it, and let U be a representation of said group in a Hilbert space $\mathscr{H}$. Let also $E(Z)$ be a spectral measure in X with values in $\mathscr{L}(\mathscr{H})$. Then, if:

$$U(g) E(Z) U(g)^{-1} = E(Zg^{-1}) \tag{A.7}$$

the system (E, U) is said to be a **system of imprimitivity** for the space X.

It can be noted that the system formed by the representation U and the spectral measure E induced in the character group of $\mathbb{R}^4$ is a system of imprimitivity for the Poincaré group.

Definition A.4: Given a semidirect product group $G = H \rtimes N$, and a fixed element c of the character group $\hat{G}$, the set:

$$\hat{O}_c = \{cn \mid n \in N\}$$

is said to be the **orbit** of the character c.

In order to avoid some pathologies from happening, one regularity condition will be imposed in the form of:

Definition A.5: A group G, which is the semidirect product $H \rtimes N$ is said to be a **regular semidirect product** of H and N if $\hat{H}$ contains a countable family $\{L_i\}_{i \in \mathbb{N}}$ of Borel subsets, each of the L_i being a union of orbits in G, such that every orbit in $\hat{H}$ is the intersection of the members of a subfamily containing that orbit.

To be more clear, let $\{L_i\}_{i \in \mathbb{N}}$ be the same family of Borel subsets as before. The regularity condition then is the assumption that, for a given orbit O_c in $\hat{H}$, it can be written as the intersection of all the L_i that contain O_c as a subset.

As said in section 3.3, the representations of the Poincaré group are not true repre-

sentations, so, in order to not have to deal with phase factors, the problem will be lifted to the universal covering of the Poincaré group, which is $SL(2, \mathbb{C})$.

What we mean by universal covering is:

Definition A.6: Let X, Y be two topological groups, and let $p : X \rightarrow Y$ be continuous and surjective. If every point of Y has a neighborhood V such that $p^{-1}(V)$ can be written as the union of disjoint sets $\{U_\lambda\}_{\lambda \in \Lambda}$, with the condition that the restriction of p to each U_λ is a homeomorphism of U_λ onto V, then p is said to be a **covering map**. In case X is a simply connected space, then X is said to be the **universal covering group** of Y.

Now, $SL(2, \mathbb{C})$ being the universal covering of the Poincaré group, there is a 'two-to-one' homomorphism from $SL(2, \mathbb{C})$ to $L_+^\uparrow$, which will be denoted as Λ_S. This being the case, and considering that, as said previously, only causal Poincaré transformations transformations, the next step is classifying the orbits of these transformations. The explicit form of the characters of $\mathbb{R}^4$ will be:

$$c(a) = e^{i(a_0 c_0 - a_1 c_1 - a_2 c_2 - a_3 c_3)} = e^{ia \cdot c}$$

the $\cdot$ product being the one understood by the Lorentz metric. The action of $SL(2, \mathbb{C})$ in the character group of $\mathbb{R}^4$ is a direct consequence of the properties of transformation of both the Poincaré group and of the Lorentz group itself. It holds:

$$c(\Lambda_S a) = \Lambda_S^{-1} c(a)$$

This means that $SL(2, \mathbb{C})$ acts in the character group of $\mathbb{R}^4$ in the same way as it acts in $\mathbb{R}^4$. Thus, fixing an element c of the character group, the set of all $\Lambda_S c$ with $\Lambda_S \in L_+^\uparrow$ is an orbit of the Poincaré group. Now, as is well known [28], the Lorentz transformations are characterized exactly by the fact that they maintain the interval between points of the Minkowski space invariant. This in turn means that every orbit will be contained in the hyperboloids:

$$c_0^2 - c_1^2 - c_2^2 - c_3^2 = m^2$$

with m^2 being a real number. There are three characteristic cases to be considered: $m^2 > 0$, $m^2 = 0$, and $m^2 < 0$. Their corresponding orbits can be illustrated as per figure A.1.
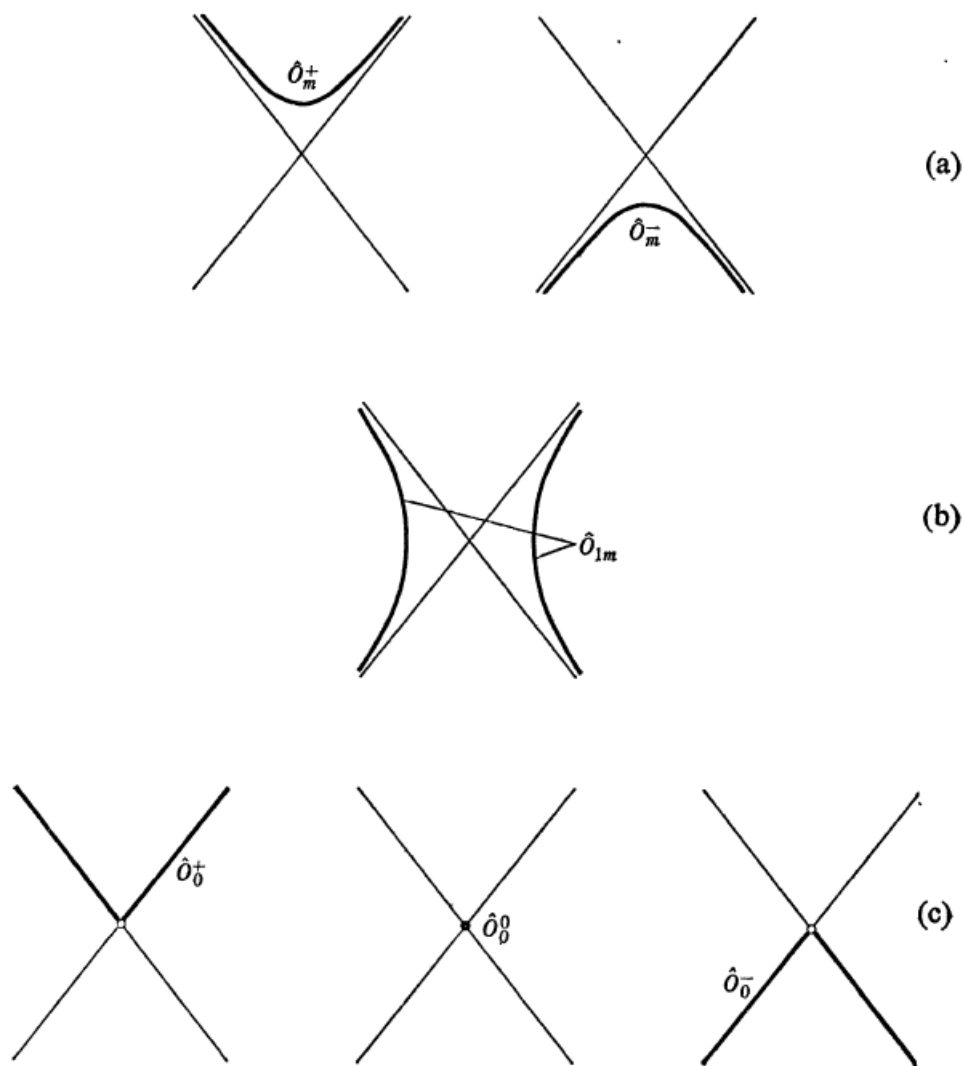
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Figure A.1: Orbits of the universal covering of the Poincaré group. Extracted from [33], Pg. 517

As is seen, there are six distinct orbits in the Poincaré group. The orbits of positive m and either positive or negative c_0, the orbits of $m = 0$ and again either positive or negative c_0, the orbit at the fixed point of the origin, and the orbit of negative m.

One can quickly see that the Poincaré group is indeed a regular semidirect product, if we consider in the character space of $\mathbb{R}^4$ the orbits of the $m = 0$ hyperboloids, and the following sets:

$$Z_m^+(r_1, r_2) = \bigcup_{r_1 < m < r_2} \hat{O}_m^+$$

$$Z_m^-(r_1, r_2) = \bigcup_{r_1 < m < r_2} \hat{O}_m^-$$

$$Z_{im}(r_1, r_2) = \bigcup_{r_1 < m < r_2} \hat{O}_{im}$$

where $r_1, r_2 \in \mathbb{Q}$, and $\hat{O}_m^+$, $\hat{O}_m^-$ and $\hat{O}_{im}$ are the hyperboloids of positive m, with either positive or negative first coordinate, and the hyperboloids of negative m, respectively.

It is now time to introduce a concept intensively studied by Mackey in [42] and [43], and which is the basis for his work in systems of imprimitivity.

Let H be a closed subgroup of a group G, and consider the right cosets of H by G, $H \backslash G = \{Hg, g \in G\}$. Let us then consider the set of all functions with domain in G and range in a Hilbert space $\mathscr{H}$, which have the following properties:

1. $\langle f(g), s \rangle$ is measurable with respect to the Haar measure on G for all $s \in \mathscr{H}$;

2. $f(hg) = U(h)f(g)$, for all $h \in H$ and $g \in G$, where $U(h)$ is a unitary representation of H in the Hilbert space $\mathscr{H}$;

3. $\int_{H \backslash G} \|f(g)\|^2 d\mu(Hg) < \infty$, this measure μ being the one defined in the homogeneous space that is $H \backslash G$.

The following lemma characterizes the space just considered:

Lemma A.1 [[33], Pg. 473]: The space previously introduced is isomorphic to the space of square integrable functions on $H \backslash G$ with range in $\mathscr{H}$. This isomorphism is seen uniquely determined by:

$$f(g) = U_{h_g} \hat{f}(Kg) \tag{A.8}$$

where h_g is a factor coming from the *Mackey decomposition* of g, $g = h_g b_g$.

Theorem A.2 (Mackey Decomposition Theorem) [[33], Pg. 70]: Let G be a locally compact, separable group, and let K be a closed subgroup of G. Then, there exists a Borel set $B \subset G$ such that every element $g \in G$ can be uniquely represented in the form:

$$g = kb$$

The following lemma introduces one of the central concepts in Mackey's theory:

Lemma A.2 [[33], Pg. 475]: The map $g_0 \to W^U(g_0)$ given by:

$$W^U(g_0)f(g) := \sqrt{\xi_{g_0}(g)} f(gg_0) \tag{A.9}$$

defines a unitary representation of G in $L^2(H\backslash G, \mu, \mathscr{H})$. Here, $\xi_{g_0}(g)$ is the Radon-Nykodym derivative of the measure μ in $H\backslash G$, given by:

$$\xi_{g_0}(g) = \frac{d\mu(Hgg_0)}{d\mu(Hg)}$$

Definition A.7: The representation defined in the previous lemma is said to be the **representation of G induced by U**.

Definition A.8: The **stability group** of an element $g \in G$, where G is a group with an action of a group H in it, is the set of elements $h \in H$ such that $hg = g$. That is $S_g = \{h \in H \mid hg = g\}$.

At this point, it is possible to enunciate the following two theorems, which completely characterize induced representations in regular semidirect products of groups, where the one in which the action of the transformation is defined on is an abelian group.

Theorem A.3 [[33], Pg. 508]: Let G be a regular semidirect product $G = H \rtimes N$ of separable, locally compact groups H and N, and let H be abelian. Let T be an irreducible

unitary representation of G. It holds:

1. One can associate with T an orbit $\hat{O}$ in $\hat{H}$;

2. The representation T is unitarily equivalent to an induced representation $W^{\hat{h}U}$, where U is an irreducible unitary representation in a Hilbert space $\mathscr{H}$ of the stability group $N_{\hat{O}}$ of a point $\hat{h}_0$ of the orbit $\hat{O}$. The representation $W^{\hat{h}U}$ is realized in the space $L^2(\hat{O}, \mu, \mathscr{H})$.

Theorem A.4 [[33], Pg. 509]: Let the group G be as in the previous theorem. It holds:

1. With each orbit $\hat{O}$ in $\hat{H}$ and with each irreducible unitary representation $\hat{h}U$ of the stability subgroup $H \rtimes N_{\hat{O}}$ one can associate the induced representation $W^{\hat{h}U}$ which is irreducible;

2. The spectral measure $E(\cdot)$, which is defined by the restriction of $W^{\hat{h}U}$ to H is concentrated on the orbit $\hat{O}$;

3. The representation $W^{\hat{h}U}$ is realized in the Hilbert space $L^2(\hat{O}, \mu, \mathscr{H})$, where $\mathscr{H}$ is the carrier space of the representation U and μ is a quasi-invariant measure in $\hat{O}$. We have:

$$W^{\hat{h}U}_{(h,n)}f(\hat{h}) = \hat{h}(h)Q^L(n)f(\hat{h}) \tag{A.10}$$

where Q^L is a representation of N induced by the representation U of the stability subgroup $N_{\hat{O}} \subset N$.

Now that most of the basic theoretical foundations for finding representations has been laid out, with many more results that went without mentioning, it is possible to finally apply these to the case of the Poincaré group and find explicit representations for its orbits. The focus in this section will be on the orbit $\hat{O}_m^+$ because it is one of the orbits with the most meaningful physical interpretations (the other being $\hat{O}_0^+$, which is related to the representations of massless particles, in particular, photons).

A good point to choose from the orbit $\hat{O}_m^+$ would then be $(m, 0, 0, 0)$, $m > 0$. The stability subgroup of this point is then the entirety of $SU(2)$, which, as is known [33], has irreducible representations labeled by a parameter $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. The representations of this orbit will thus be parametrized by m, j, which will be interpreted as the mass and spin of the particle, respectively.

Consider now the following result:

Proposition A.2 [pieces of excerpts extracted from [33]]: The explicit realization of the induced representation $W^U(g_0)$ in the space $L^2(G\backslash K, \mu, \mathcal{H})$, where U is a representation of a closed subgroup K of the group G, with $g_0 \in G$, is given by:

$$W^U(g_0)f(gK) = \sqrt{\frac{d\mu(g_0^{-1}gK)}{d\mu(gK)}} U^{-1}_{k_{g_0^{-1}s_g}} f(g_0^{-1}gK) \tag{A.11}$$

where $\sqrt{\frac{d\mu(g_0 x)}{d\mu(x)}}$ is the Radon-Nykodym derivative of the measure μ, $k_{g_0^{-1}}$ and s_g are the Mackey decomposition factors coming from g_0^{-1} and g, respectively.

As a first point, it can be noticed that, for this specific case, since the stabilizer subgroup is $\mathbb{R}^4 \rtimes SU(2)$ and the whole group is $\mathbb{R}^4 \rtimes SL(2, \mathbb{C})$, then the space of mass hyperboloids is associated with $\mathbb{R}^4 \rtimes SL(2, \mathbb{C}) \backslash \mathbb{R}^4 \rtimes SU(2)$. The mass on these mass hyperboloids is given by:

$$d\mu(p) = \frac{d^3\vec{p}}{p_0}$$

Now, this measure is invariant under Lorentz transformations, which is isomorphic with the coset previously formed. Thus, the Radon-Nykodym derivative for this case will be identically equal to 1.

This reduces the previous equation to:

$$W^U(g_0)f(gK) = U^{-1}_{k_{g_0^{-1}s_g}} f(g_0^{-1}gK)$$

and the only task left is that of finding $U^{-1}_{k_{g_0^{-1}s_g}}$.

Explicitly, since the stability subgroup of the point $(m, 0, 0, 0)$ is $\mathbb{R}^4 \rtimes SU(2)$, the irreducible representations of the stability subgroup of this orbit will be:

$$L^j_{(a,r)} = e^{i\underline{m}a} D^j(r) \tag{A.12}$$

where $\underline{m} = (m, 0, 0, 0)$ and:

$$D_j(r) = \sum_{M,M'=-j}^{j} e^{-iM\phi_r} d_{MM'}^{j}(\theta_r) e^{-iM\psi_r} \tag{A.13}$$

here $\phi_r \in [0, 2\pi)$, $\theta_r \in [0, \pi)$ and $\psi_r \in [0, 2\pi)$ being the unique angles associated to the matrix r through the isomorphism between $SO(3)$ and $SU(2)$, and

$$d_{MM'}^{j}(\theta_r) = \left(\frac{1 + \cos\theta_r}{2} \right)^{M} P_{J-M}^{0,2M}(\cos\theta_r)$$

$P_c^{a,b}$ being the Jacobi polynomials.

Now, for an element $S \in SL(2, \mathbb{C})$, the following decomposition holds:

$$S = S_p r$$

where $r \in SU(2)$ and

$$S_p = \begin{bmatrix} \lambda & z \\ 0 & \frac{1}{\lambda} \end{bmatrix}$$

with $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$. Thus, $S_p \equiv s_g$. This results in:

$$g_0^{-1} s_g = (a_0, S_0)^{-1}(0, S_p) = (0, S_{\Lambda_{S_0}^{-1}p})(-\Lambda_{\Lambda_{S_0p}^{-1}}^{-1} \Lambda_{S_0}^{-1} a_0, S_{\Lambda_{S_0}^{-1}p}^{-1} S_0^{-1} S_p)$$

the action of S_0 in a point p of the Minkowski space being understood as the point $p = (p_0, p_1, p_2, p_3)$ being associated to a hermitian matrix by the following one-to-one map:

$$\begin{bmatrix} p_0 - p_3 & p_2 + ip_1 \\ p_2 - ip_1 & p_0 + p_3 \end{bmatrix} = \sum_{\mu=0}^{3} p^{\mu} \sigma_{\mu}$$

where σ_{μ} are the Pauli matrices, including the identity.

From this point, one can then see that:

$$k_{g_0^{-1} s_g} = (-\Lambda_{\Lambda_{S_0p}^{-1}}^{-1} \Lambda_{S_0}^{-1} a_0, S_{\Lambda_{S_0p}^{-1}}^{-1} S_0^{-1} S_p)$$

This means then that:

$$U^{-1}_{k_{g_0^{-1}sg}} = e^{ipa} D^j(S_p^{-1}S_0 S_{\Lambda_{S_0 p}^{-1}})$$

Thus, for a representation of the orbit $\hat{O}_m^+$, one has:

$$U_m^{+,j}(a,S)f(p) = e^{ipa} D^j(r_S)f(\Lambda_S^{-1}p) \tag{A.14}$$

where $r_{S_0} = S_p^{-1}S_0 S_{\Lambda_S^{-1}p}$.