Lecture 9

Spontaneous Breaking of Continuous Global Symmetries

In what follows, we will formalize some of the results we have encountered in the context of Bose-Einstein condensation and superfluidity. Namely, that the appearence of a non-trivial ground state, characterized there by the non-zero value of the density macroscopically populating the zero-momentum state at low temperatures, is associated with the appearence of a *massless* (or sometimes referred to as *gapless*) excitation in the spectrum. We will see bellow that this is always the case when the ground state of the system is not invariant under the symmetries of the Hamiltonian. This mismatch is what is called spontaneous symmetry breaking, and in the case of a continuous symmetry it leads to the appearance of gapless excitations, the so-called Nambu-Goldstone states. Below, we approach this subject through a simple quantum mechanical derivation of the Goldstone theorem. The simplest example, that of a complex scalar field with a non-trivial ground state, is reviewed after that, and formalizes our description of a superfluid from the previous lecture, in particular the appearance of a massless state.

9.1 Spontaneous Breaking and Gapless Excitations

Noether's theorem tells us that for each continuous symmetry in the Lagrangian $\mathcal{L}(\phi, \partial_\mu\phi)$ there is a conserved current J^μ, i.e.[1]

$$\partial_\mu J^\mu = 0 \ . \tag{9.1}$$

We can restate this by saying that the charge associated with this symmetry

[1] Here we go back to relativistic notation and Minkowski space.

$$Q = \int d^3x \, J^0 \ , \tag{9.2}$$

is conserved. This is easily checked by computing

$$\frac{dQ}{dt} = \int d^3x \, \partial_0 J^0 = \int d^3x \, \vec{\nabla} \cdot \mathbf{J} = \int_{S_\infty} d\mathbf{s} \cdot \mathbf{J} = 0 \ , \tag{9.3}$$

where in the last step we assume there are no sources at infinity.

Now, in the presence of a continuous symmetry, quantum states transform under the symmetry as

$$|\psi\rangle \to e^{i\alpha Q} \, |\psi\rangle \ , \tag{9.4}$$

where α is a real constant, i.e. a continuous parameter. In particular, if the ground state is invariant under the symmetry this means that

$$|0\rangle \to e^{i\alpha Q}|0\rangle = |0\rangle \ , \tag{9.5}$$

with the last equality implying

$$Q|0\rangle = 0 \ . \tag{9.6}$$

In other words, if the ground state is invariant under a continuous symmetry the associated charge Q annihilates it. This is the normal realization of a symmetry.

But if

$$Q|0\rangle \neq 0 \ , \tag{9.7}$$

then this means that

$$|0\rangle \to e^{i\alpha Q}|0\rangle \equiv |\alpha\rangle \neq |0\rangle \ , \tag{9.8}$$

where we defined the states $|\alpha\rangle$ by the continuous parameter of the transformation connecting it to the ground state. In general, this is the situation when a symmetry is broken. But it is possible to have (9.7) and still have a conserved charge. In other words to have

$$\frac{dQ}{dt} = 0 \ . \tag{9.9}$$

Having both (9.7) and (9.9) satisfied at the same time corresponds to what we call spontaneous symmetry breaking (SSB): the charge is still conserved, but the ground state is not invariant under a symmetry transformation.

$$\boxed{\left(Q|0\rangle \neq 0, \quad \frac{dQ}{dt} = 0\right) \Rightarrow \text{SSB}} \; . \tag{9.10}$$

For instance, this is what happens in a ferromagnet below a critical temperature. The free energy

$$F = E - TS \; , \tag{9.11}$$

can be minimized, at high temperature, by increasing the entropy S. So at high T disorder rules. However, below a critical temperature, the free energy would be minimized by minimizing E, which is achieved by aligning the interacting spins, resulting in a macroscopic magnetization. This is an ordered phase. But since the magnetization picks a direction in space it corresponds to the spontaneous breaking the symmetry of the system, i.e. $O(3)$. Since the charge is conserved we have that $[H, Q] = 0$. Then, given a Hamiltonian H acting on a state $|\alpha\rangle$ connected to the ground state, we can write

$$\begin{aligned} H|\alpha\rangle &= He^{i\alpha Q}|0\rangle = e^{i\alpha Q}H|0\rangle = E_0 e^{i\alpha Q}|0\rangle \\ &= E_0|\alpha\rangle \; . \end{aligned} \tag{9.12}$$

So we conclude that (9.10) results in a continuous family of degenerate states $|\alpha\rangle$ with the same energy of the ground state, E_0. Going from the ground state $|0\rangle$ to the $|\alpha\rangle$ states costs no energy. These are the gapless states characteristic of SSB. They are the Nambu-Goldstone modes. In a relativistic quantum field theory they correspond to massless particles, as we will see in the following example.

9.2 Spontaneous Breaking of a Global $U(1)$ Symmetry

We will consider a complex scalar field, the simplest systems to illustrate the spontaneous breaking of a global symmetry and the appearance of massless particles. This is the relativistic version of the superfluid. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{1}{2}\mu^2\phi^*\phi - \frac{\lambda}{4}\left(\phi^*\phi\right)^2 \; . \tag{9.13}$$

As we well know, $\mathcal{L}$ is invariant under the $U(1)$ symmetry transformations

$$\phi(x) \to e^{i\alpha}\phi(x) \ , \qquad \phi^*(x) \to e^{-i\alpha}\phi^*(x) \ , \tag{9.14}$$

where α is a real constant. Here the $U(1)$ symmetry is equivalent (isomorphic) to a rotation in the complex plane defined by

$$\phi(x) = \phi_1(x) + i\phi_2(x) \ , \qquad \phi^*(x) = \phi_1(x) - i\phi_2(x) \ , \tag{9.15}$$

where $\phi_{1,2}(x)$ are real scalar fields. Then we see that $U(1) \simeq O(2)$. For instance, had we started with a purely real field $\phi(x) = \phi_1(x)$, i.e. $\phi_2(x) = 0$, the $U(1)$ transformations (9.14) would result in

$$\phi(x) = \phi_1(x) \to \cos\alpha\phi_1(x) + i\sin\alpha\phi_1(x) \ , \tag{9.16}$$

as illustrated in Figure 9.1 below.

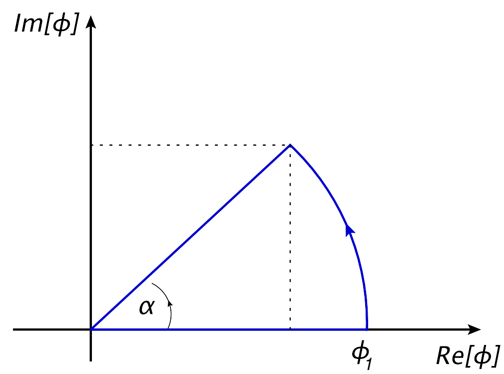

Figure 9.1: The $U(1)$ rotation $\phi \to e^{i\alpha}\phi$ for an initially real field.

We now consider the (classical) potential

$$V = \frac{1}{2}\mu^2\phi^*\phi + \frac{\lambda}{4}\left(\phi^*\phi\right)^2 \ . \tag{9.17}$$

For $\mu^2 > 0$ V has a minimum at $(\phi^*\phi)_0 = 0$. On the other hand, if $\mu^2 < 0$ there is a non trivial minimum for $\lambda > 0$ resulting from the competition of the first and second terms in (9.17). Redefining

$$\mu^2 \equiv -m^2 \ , \tag{9.18}$$

with $m^2 > 0$, the minimum of the potential now is

$$(\phi^*\phi)_0 = \frac{m^2}{\lambda} \equiv v^2 \ . \tag{9.19}$$

Here v^2 is the expectation value of the $\phi^*\phi$ operator in the ground state, i.e.

$$\langle 0|\phi^*\phi|0\rangle = v^2 \ . \tag{9.20}$$

The potential looks just as the one for the superfluid case in the previous lecture, shown in Figure 8.1. The projection onto the (ϕ_1, ϕ_2) plane is shown in Figure 9.2 below.

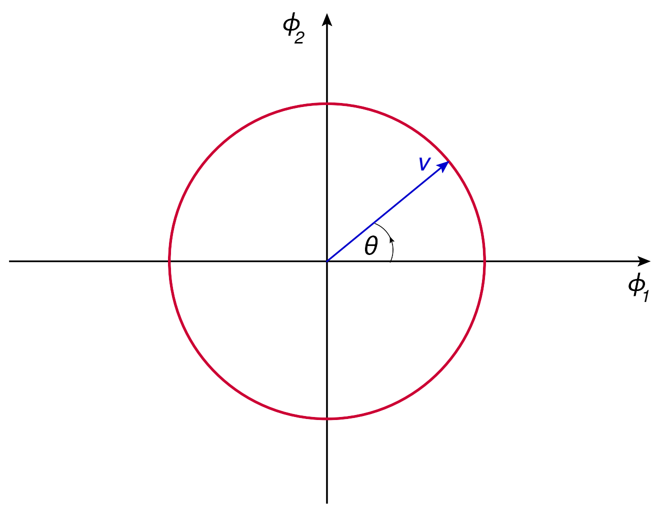

Figure 9.2: The red circle represents the locus points of the minimum of the potential (9.17) for $\mu^2 < 0$. The radius is v, a real number. The phase is not determined by the minimization.

The radius is fixed through

$$(\phi^*\phi)_0 = v^2 = \phi_1^2 + \phi_2^2 \ , \tag{9.21}$$

but the phase is undetermined. We need to fix it in order to choose a ground state to expand around. Any choice should be equivalent

$$
\begin{aligned}
\langle\phi_1\rangle &= v & \langle\phi_2\rangle &= 0 \\
\langle\phi_1\rangle &= \frac{v}{\sqrt{2}} & \langle\phi_2\rangle &= \frac{v}{\sqrt{2}} \\
&\vdots & &\vdots \\
\langle\phi_1\rangle &= 0 & \langle\phi_2\rangle &= v\ .
\end{aligned}
$$

This particular choice is what constitutes spontaneous symmetry breaking. We need to fix the phase $\theta = \theta_0$ arbitrarily in order to expand around *this* ground state. For instance, let us choose the first line above, i.e. $\langle\phi_1\rangle = v$, and $\langle\phi_2\rangle = 0$. This allows us to expand the field $\phi(x)$ around this ground state as

$$\phi(x) = v + \eta(x) + i\xi(x)\ , \tag{9.22}$$

where $\eta(x)$ and $\xi(x)$ are *real* scalar fields satisfying

$$\langle 0|\eta(x)|0\rangle = 0, \qquad \langle 0|\xi(x)|0\rangle = 0\ . \tag{9.23}$$

This obviously corresponds to $\phi_1(x) = v + i\eta(x)$ and $\phi_2(x) = \xi(x)$. We can now rewrite the Lagrangian (9.13) in terms of $\eta(x)$ and $\xi(x)$. This is

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}m^2\left(v+\eta-i\xi\right)\left(v+\eta+i\xi\right) \\
&\quad -\frac{\lambda}{4}\left[\left(v+\eta-i\xi\right)\left(v+\eta+i\xi\right)\right]^2\ ,
\end{aligned} \tag{9.24}
$$

where we used (9.18). Using (9.19) and focusing on the terms quadratic in the fields, we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - m^2\eta^2 + \ \text{interactions}\ . \tag{9.25}$$

So we see that when we expand around the ground state defined by (9.22) we end up with a theory of a real scalar field with mass (η) and a massless state ξ. That is

$$m_\eta = \sqrt{2}m, \qquad m_\xi = 0\ . \tag{9.26}$$

This result is a reflection of Goldstone's theorem: a spontaneously broken continuous symmetry, here a $U(1)$, results in massless states. Notice that the result would be exactly the same had we chosen any other angle in Figure 9.2 instead of $\theta = 0$. One simple way to check this is to use a different parametrization of $\phi(x)$. We write

$$\phi(x) \equiv [v + h(x)]\, e^{i\pi(x)} \ , \tag{9.27}$$

where $h(x)$ and $\pi(x)$ are real scalar fields, also satisfying

$$\langle 0|h(x)|0\rangle = 0, \qquad\qquad \langle 0|\pi(x)|0\rangle = 0 \ . \tag{9.28}$$

Then from (9.27) it is pretty obvious that $\pi(x)$ does not enter in the potential, and therefore will not have a mass term. It is very simple to obtain the Lagrangian (9.13) in terms of $h(x)$ and $\pi(x)$ using (9.27). This is

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu \pi \partial^\mu \pi - m^2 h^2 + \quad \text{interactions} \ , \tag{9.29}$$

which is exactly the same theory as the one in (9.25), i.e. a massive state with $m_h = \sqrt{2}m$ and a massless particle, here the $\pi(x)$.

We will later see a derivation of Goldsone's theorem that is more geared towards quantum field theory. We will see that there will be a NGB for each *broken* symmetry generator, i.e. for each spontaneously broken symmetry.

Additional suggested readings

- *Condensed Matter Field Theory*, Altland and Simons, Section 6.3.
- *Dynamics of the Standard Model*, J. F. Donoghue, E. Golowich and B. Holstein, Chapter I-5.