Lecture 5

Quantum Field Theory and Many-body Systems

So far we have considred aplications of quantum field theory (QFT) for either relativistic systems or the long range behavior of many-body systems such as a ferromagnet. Here we will start studying systems of a large number of particles and develop statistical methods to deal with them. We will also introduce temperature in the treatment. The tools of QFT we have developed so far will be also applicable here, with some minor adjustments. But we also need to incorporate a few new elements. The aim is to describe many-body systems at finite temperature. Applications will be mostly in condensed matter physics although there are some also in cold atoms and nuclear physics. We describe these systems by writing their partition functions as functional integrals. This will give us an interesting way to introduce the collective phenomena associated with symmetry breaking in condensed matter physics. We will first review quickly some basics of thermodynamics and statistical mechanics that we will be using next. We will then introduce coherent states so as to have all the elements that will allow us to write the partition function as a functional integral.

5.1 Thermodynamics and Statistical Mechanics

The internal energy of a closed systems (i.e. with a fixed number of particles N) is

$$dE = dQ - dW = T\,dS - P\,dV \ , \tag{5.1}$$

where Q stands for heat, T is the temperature of the system, S its entropy, P the pressure and V the volume. For N variable we introduce the chemical potential contribution, so now we have

$$dE = T\,dS - P\,dV + \mu\,dN \ . \tag{5.2}$$

The thermodynamic potentials of interest are the Helmholtz free energy F and the Gibbs free energy G. F is defined by

$$F \equiv E - T\,S \ , \tag{5.3}$$

which translates into

$$dF = -S\,dT - P\,dV + \mu\,dN \ , \tag{5.4}$$

showing that F is a function of T, V and N. On the other hand, the Gibbs free energy is defined by

$$G \equiv E - T\,S - P\,V = F - P\,V \ , \tag{5.5}$$

resulting in

$$dG = -S\,dT + \mu\,dN - V\,dP \ , \tag{5.6}$$

which means that G is a function of T, P and N instead. We will be mainly concerned with systems at fixed P, so we will make use of the Helmholtz free energy F. In particular, we notice that in minimizing F there is a competition between the internal energy minimization and disorder, the latter represented by the second term, which is minimized for larger values of S.

For fixed number of particles N, the probability of the so-called canonical ensamble is given by

$$P_c = \frac{1}{Z_N}\, e^{-\beta E} \ , \tag{5.7}$$

where $\beta = 1/T$, E is the eigenvalue of the energy and Z_N is the partition function of the canonical ensamble

$$Z_N = \mathrm{Tr}\left[\mathrm{e}^{-\beta \mathrm{H}}\right] \ . \tag{5.8}$$

The trace above is taken over all the states of the system, and it can include continuous spatial and/or momentum indexes. The grand canonical ensamble allows for the variation of N. In this case the probability is

$$P_{gc} = \frac{1}{Z}\, e^{-\beta(E-\mu N)} \ , \tag{5.9}$$

where the partition function Z is now a function of T μ and V and is given by

$$Z = \mathrm{Tr}\left[\mathrm{e}^{-\beta(\mathrm{H}-\mu\mathrm{N})}\right] \ . \tag{5.10}$$

Another way pf writing Z is in terms of the canonical partition function Z_N. This is

$$Z(T,\mu,V) = \sum_N e^{\beta\mu N}\, Z_N(T,V) \ . \tag{5.11}$$

Then the Helmholtz free energy F can be written in terms of Z_N. For this purpose, we first notice that

$$Z_N = \int dE\, \omega(E) e^{-\beta E} \ , \tag{5.12}$$

where $\omega(E)$ is the density of states with eigenvalue of the hamiltonian E. The expression (5.12) is just an integral form of (5.8). The density of states $\omega(E)$ enters in the expression of the entropy S since

$$S = \ln\left[\omega(E)\, \Delta E\right] \ , \tag{5.13}$$

where the argument of the logarithm is just the number of states with energies between E and $E + dE$. But then we can write

$$\omega(E) = \frac{e^S}{\delta E} \ . \tag{5.14}$$

Replacing this in (5.12) we have

$$Z_N = \int dE\, \frac{1}{\Delta E} e^{-\beta(E-TS)} \ , \tag{5.15}$$

which can be readily approximated to be

$$Z_N \simeq e^{-\beta(\langle E\rangle - TS)} \ , \tag{5.16}$$

where we replaced E by its thermal average. This is the case since

$$\omega(E)\, e^{-\beta E} \ , \tag{5.17}$$

peaks at $\langle E\rangle$ and effectively acts as a delta function. We then can write

$$F = -T\, \ln\left[Z_N\right] \ . \tag{5.18}$$

5.2 Second Quantization

We will review here some aspects of second quantization. We are mainly interested in showing how to write operators in terms of creation and annihilation operators. We start with the action of creation and annihilation operators on the eigenstates of the hamiltonian.

$$\begin{aligned} a\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned} \tag{5.19}$$

We are interested in the occupation number representation, which allows us to write states as

$$|n_1, n_2, \ldots, n_j, \ldots\rangle \ . \tag{5.20}$$

In the state above, the subscripts i are labels defining the state, such as momentum, position or spin, among others. The n_i's are the number of particles in each of the states. These are states belonging to Fock space. In them, the total number of particles is not fixed. We could imagine that these states belong to the direct sum of spaces with fixed total number of particles N. In this way, the states with fixed N written as

$$|n_1, n_2, \ldots, n_N\rangle \ , \tag{5.21}$$

satisfy $\sum_j n_j = N$. The states in (5.21) span a Hilbert space we call $\mathcal{F}^N$. Then the states in (5.20) defined the Fock space as

$$\mathcal{F} = \oplus_{N=0}^{\infty}\mathcal{F}^N \tag{5.22}$$

Coming back to the variable N states from (5.20), the action of creation and annihilation operators is

$$a_j\,|n_1,\ldots,n_j,\ldots,\rangle \;=\; \sqrt{n_j}\,|n_1,\ldots,n_j-1,\ldots\rangle \tag{5.23}$$

$$a_j^\dagger\,|n_1,\ldots,n_j,\ldots,\rangle \;=\; \sqrt{n_j+1}\,|n_1,\ldots,n_j+1,\ldots\rangle\ . \tag{5.24}$$

We can then rewrite the states (5.20) as

$$|n_1,n_2,\ldots,n_j,\ldots\rangle = \prod_j \frac{1}{\sqrt{n_j!}}\left(a_j^\dagger\right)^{n_j}\,|0\rangle\ . \tag{5.25}$$

Of course, the annihilation and creation operators above satisfy the commutation relations [1]

$$[a_i,a_j]=0=[a_i^\dagger,a_j^\dagger]\ ,\qquad [a_i,a_j^\dagger]=\delta_{ij}\ . \tag{5.26}$$

5.2.1 One-body Operators

These are single-particle operators that act on the states (5.20). We would like to write them interms of annihilation and creation operators. A first step is to consider a generic operator as expanded in terms of the states $\{|\alpha\rangle\}$ as

$$\hat{\mathcal{A}} = \sum_{\alpha,\beta}\mathcal{A}_{\alpha\beta}\,|\alpha\rangle\langle\beta|\ , \tag{5.27}$$

where $\mathcal{A}_{\alpha\beta}=\langle\alpha|\mathcal{A}|\beta\rangle$ is the matrix element of the operator. The action of (5.27) on a generic multi-particle state is

$$\hat{\mathcal{A}}\,|\psi_1,\ldots,\psi_j,\ldots\rangle = \sum_{\alpha\beta}\mathcal{A}_{\alpha\beta}\sum_j\langle\beta|\psi_j\rangle\,|\psi_1,\ldots,\alpha,\ldots\rangle\ . \tag{5.28}$$

But this is the same we would obtain if we replaced

$$\hat{\mathcal{A}} = \sum_{\alpha\beta}\mathcal{A}_{\alpha\beta}\,a_\alpha^\dagger a_\beta\ , \tag{5.29}$$

[1] Here we are considering the example of bosons. The case of fermions will carry the appropriate signs resulting from using anti-commutation relations instead.

which is valid in general, even for fermions. The detailed proof of (5.29) is a bit more involved when done generally, but is enough to see here that it works.

More transparently, if the operator $\hat{\mathcal{A}}$ is diagonal in the basis $\{|\alpha\rangle\}$, then we have

$$\hat{\mathcal{A}} = \sum_{\alpha} \mathcal{A}_\alpha a^\dagger_\alpha a_\alpha = \sum_{\alpha} \mathcal{A}_\alpha \hat{n}_\alpha \ , \tag{5.30}$$

where $\hat{n}_\alpha$ is the number operator counting the number of particles in the state $|\alpha\rangle$, and therefore the sum is just adding up the number of particles with this eigenvalue times the eigenvalue. As a first simple example let us consider the momentum operator in one dimension

$$\hat{p} = \sum_{p_i} p_i \, |p_i\rangle\langle p_i| \ , \tag{5.31}$$

where it is obvious that the operator $\hat{p}$ is diagonal in the basis $\{|p_i\rangle\}$, with $\{p_i\}$ the eigenvalues. Then we can write the second quantization form as

$$\hat{p} = \sum_{p_i} p_i \, a^\dagger_{p_i} a_{p_i} = \sum_{p_i} p_i \, \hat{n}_{p_i} \ . \tag{5.32}$$

Then, when acting on a state

$$\hat{p}\,|n_1, \ldots, n_i, \ldots\rangle = \sum_{p_i} p_i \, n_i \, |n_1, \ldots, n_i, \ldots\rangle \ , \tag{5.33}$$

$\hat{p}$ will be summing up all the momenta of all particles by counting with $\hat{n}$ how many have each eigenvalue, resulting in the total momentum eigenvalue. We can generalize this for any operator that is a function of $\hat{p}$. For instance the kinetic energy in momentum representation is

$$\hat{T} = \sum_{p_i} \frac{p_i^2}{2m} \, a^\dagger_{p_i} \, a_{p_i} = \sum_{p_i} \frac{p_i^2}{2m} \, \hat{n}_{p_i} \ . \tag{5.34}$$

5.2.2 Field Operators

Here we will introduce the operators $a(\mathbf{x})$ and $a^\dagger(\mathbf{x})$ that annihilate and create particle at a given position $\mathbf{x}$. They are related to the ones creating and annihilation states of definite momentum by the Fourier transforms

$$a(\mathbf{x}) = \frac{1}{\sqrt{L^d}} \sum_{\mathbf{p}} a_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{x}}$$
$$a^\dagger(\mathbf{x}) = \frac{1}{\sqrt{L^d}} \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \,, \tag{5.35}$$

where we are considering a finite d dimensional volume $V = L^d$ and, as a consequence, the momentum is quantized as in

$$\mathbf{p} = \frac{2\pi\mathbf{m}}{L} \,, \tag{5.36}$$

where $\mathbf{m} = (m_1, m_2, \ldots, m_d)$ with the m_i integers. We call this position representation of the annihilation and creation operators field operators since they do resemble the fields as we defined them before in terms of the momentum ones. The missing terms corresponding to the anti-particles are not present here since we are dealing in principle with non-relativistic systems. This is why we use a and $a^\dagger$ in the notation for the left hand sides of (5.35). As an example, let us consider the kinetic energy operator in the position representation. As we will see below, the correct form must be

$$\hat{T} = \int d^d x\, a^\dagger(\mathbf{x})\, \frac{\hat{\mathbf{p}}^2}{2m}\, a(\mathbf{x}) \,, \tag{5.37}$$

where $\hat{\mathbf{p}} = -i\vec{\partial}$, the momentum operator in position space. We can easily verify that this way of writing the kinetic term in (5.37) results in (5.34), once we use the Fourier transforms (5.35) and the fact that

$$\frac{1}{L^d} \int d^d x\, e^{-i(\mathbf{p}_1 - \mathbf{p}_2)\cdot\mathbf{x}} = \delta_{\mathbf{p}_1 - \mathbf{p}_2} \,. \tag{5.38}$$

Then the one-body hamiltonian in position space would be

$$\hat{H} = \int d^d x\, a^\dagger(\mathbf{x}) \left(\frac{\hat{\mathbf{p}}^2}{2m} + U(\mathbf{x}) \right) a(\mathbf{x}) \,, \tag{5.39}$$

where $U(\mathbf{x})$ is the one-particle potential. Using (5.35) and the definition

$$U_{\mathbf{p}_1 - \mathbf{p}_2} \equiv \frac{1}{L^d} \int d^d x\, e^{-i(\mathbf{p}_1 - \mathbf{p}_2)\cdot\mathbf{x}}\, U(\mathbf{x}) \,, \tag{5.40}$$

we arrive at

$$\hat{H} = \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \sum_{\mathbf{p}_1,\mathbf{p}_2} U_{\mathbf{p}_1-\mathbf{p}_2}\, a^\dagger_{\mathbf{p}_1} a_{\mathbf{p}_2} \ . \tag{5.41}$$

We see in the second term above that we can think of the one-particle potential in momentum space representation as taking a particle of momentum $\mathbf{p}_2$, making it interact with the potential, and then creating a particle of momentum $\mathbf{p}_1$.

5.2.3 Two-body Operators

Operators acting on two particles in a given state can be expanded in terms of annihilation and creation operators as

$$\hat{\mathcal{A}} = \sum_{\alpha,\beta,\gamma,\delta} \mathcal{A}_{\alpha\beta\gamma\delta}\, a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta \ . \tag{5.42}$$

We are specifically interested in operators resulting in interactions between two particles. For instance, let us consider a symmetric two-body interaction potential

$$V(\mathbf{x},\mathbf{y}) = V(\mathbf{y},\mathbf{x}) \ . \tag{5.43}$$

We want to find the form of the operator $\hat{V}$ in the second quantized language. In particular, when applied to an N-particle state it should satisfy

$$\hat{V}\,|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N\rangle = \frac{1}{2}\sum_{i\neq j} V(\mathbf{x}_i,\mathbf{x}_j)\,|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N\rangle \ . \tag{5.44}$$

It is possible to prove that the correct form of $\hat{V}$ is

$$\hat{V} = \frac{1}{2}\int d^dx \int d^dy\, V(\mathbf{x},\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \ . \tag{5.45}$$

Before we prove that (5.45) is the correct expression for $\hat{V}$, we can see that it satisfies

$$\langle 0|\hat{V}|0\rangle = 0 \ . \tag{5.46}$$

Although we know that we need $a^\dagger(\mathbf{x})$, $a^\dagger(\mathbf{y})$, $a(\mathbf{x})$ and $a(\mathbf{y})$, their ordering could be different that the one given in (5.45). The chosen ordering, which guarantees (5.46), is called normal ordering. Had we chosen a different ordering of the annihilation and creation operators in (5.45), we would have ended with an undesirable and ill-defined self-interaction term. Normal ordering avoids this. In the formulation of QFT we went through in the first part of the course, normal ordering is closely related with Wick theorem.

To prove (5.45), we just apply $\hat{V}$ as defined by it on a multi-particle state $|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\rangle$ and check that we obtain the right hand side of (5.44). We first notice that

$$\begin{aligned} & a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})|\mathbf{x}_1, \ldots, \mathbf{x}_N\rangle = a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_N)\,|0\rangle \\ = & \sum_{j=1}^{N} \delta^{(3)}(\mathbf{x}-\mathbf{x}_j)a^\dagger(\mathbf{x}_j)a^\dagger(\mathbf{y})a(\mathbf{y})a^\dagger(\mathbf{x}_1), \ldots, a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1}), \ldots, a^\dagger(\mathbf{x}_N)|0\rangle \\ = & \sum_{j=1}^{N} \delta^{(3)}(\mathbf{x}-\mathbf{x}_j)a^\dagger(\mathbf{y})a(\mathbf{y})a^\dagger(\mathbf{x}_1), \ldots, a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_j)a^\dagger(\mathbf{x}_{j+1}), \ldots, a^\dagger(\mathbf{x}_N)|0\rangle \end{aligned} \quad (5.47)$$

In the last step we carry through the $a^\dagger(\mathbf{x}_j)$ (obtained from the first factor $a^\dagger(\mathbf{x})$ by using the δ function) to go back to the place where we had taken it from due to the use of the commutation relation

$$[a(\mathbf{x}), a^\dagger(\mathbf{x}_j)] = \delta^{(3)}(\mathbf{x}-\mathbf{x}_j)\ . \quad (5.48)$$

In doing so we used the fact that $\mathbf{x} \neq \mathbf{y}$, so that the operators evaluated in $\mathbf{x}$ and $\mathbf{y}$ commute. We can now do the same for $a^\dagger(\mathbf{y})a(\mathbf{y})$. This results in

$$a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})|\mathbf{x}_1, \ldots, \mathbf{x}_N\rangle = \sum_{j=1}^{N} \delta^{(3)}(\mathbf{x}-\mathbf{x}_j) \sum_{k\neq j}^{N} \delta^{(3)}(\mathbf{y}-\mathbf{x}_k)|\mathbf{x}_1, \ldots, \mathbf{x}_N\rangle\ . \quad (5.49)$$

If we now multiply by $V(\mathbf{x}, \mathbf{y})/2$ and integrate over $\mathbf{x}$ and $\mathbf{y}$ as indicated in (5.45) we then obtain the right-hand side of (5.44), proving that (5.45) is the correct expression for the second-quantized interaction potential in position space.

To conclude, we want to see how $\hat{V}$ looks in momentum space. Starting from (5.45) and using (5.35) we have

$$\begin{aligned} \hat{V} &= \frac{1}{2}\int d^dx d^dy\, V(\mathbf{x}-\mathbf{y})\, a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x}) \\ &= \frac{1}{2L^{2d}}\int d^dx d^dy \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} e^{-i(\mathbf{p}_1\cdot\mathbf{x}+\mathbf{p}_2\cdot\mathbf{y}-\mathbf{p}_3\cdot\mathbf{y}-\mathbf{p}_4\cdot\mathbf{x})} V(\mathbf{x}-\mathbf{y}) a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2} a_{\mathbf{p}_3} a_{\mathbf{p}_4}\ , \end{aligned} \quad (5.50)$$

where we have imposed translation invariance so that the potencial depends on the difference of the positions, i.e. we have $V(\mathbf{x}-\mathbf{y})$. We change variables to $\mathbf{z}=\mathbf{x}-\mathbf{y}$ and $\mathbf{y}$. In terms of these we have now

$$\hat{V} = \frac{1}{2L^{2d}} \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2} a_{\mathbf{p}_3} a_{\mathbf{p}_4} \int d^d z V(\mathbf{z}) e^{-i(\mathbf{p}_1-\mathbf{p}_4)\cdot\mathbf{z}}\, L^d\, \delta_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4}\ , \tag{5.51}$$

where we used (5.38) to get the last factor. If we define

$$\mathbf{q} = \mathbf{p}_3 - \mathbf{p}_2\ , \tag{5.52}$$

we can rewrite $\hat{V}$ as

$$\hat{V} = \frac{1}{2} \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}} a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2} a_{\mathbf{p}_2+\mathbf{q}} a_{\mathbf{p}_1-\mathbf{q}} V_{\mathbf{q}}\ , \tag{5.53}$$

where we defined the Fourier transform of the potential as

$$V_{\mathbf{q}} \equiv \frac{1}{L^d} \int d^d z\, V(\mathbf{z})\, e^{-i\mathbf{q}\cdot\mathbf{z}}\ . \tag{5.54}$$

We can interpret this result as having $\hat{V}$ annihilating two particles of momenta $\mathbf{p}_1-\mathbf{q}$ and $\mathbf{p}_2+\mathbf{q}$ in the initial state, and then creating two particles of momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. To put it in a more familiar form, we can simply shift the initial momenta by $\mathbf{q}$ as in

$$\begin{aligned} \mathbf{p}_1 &\to \mathbf{p}_1 + \mathbf{q} \\ \mathbf{p}_2 &\to \mathbf{p}_2 - \mathbf{q}\ , \end{aligned} \tag{5.55}$$

which allows us to rewrite (5.53) as

$$\hat{V} = \frac{1}{2} \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}} a^\dagger_{\mathbf{p}_1+\mathbf{q}} a^\dagger_{\mathbf{p}_2-\mathbf{q}} a_{\mathbf{p}_2} a_{\mathbf{p}_1} V_{\mathbf{q}}\ . \tag{5.56}$$

which corresponds to the annihilation of two particles of initial momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ and the creation of two particles with momenta $\mathbf{p}_1+\mathbf{q}$ and $\mathbf{p}_2-\mathbf{q}$. This process can be schematically represented by the diagram of Figure 5.1 below.

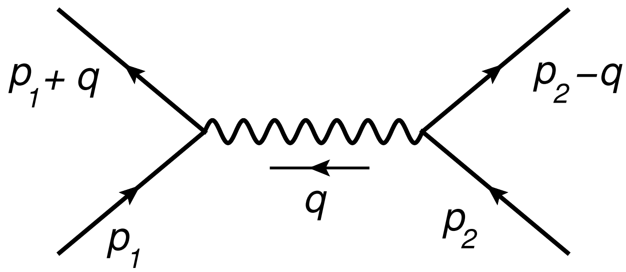

Figure 5.1: Diagram corresponding to the action of $\hat{V}$ in momentum space, from equation (5.56).

We can see that, as expected, imposing translation invariance resulted in an interaction that conserves momentum. The wavy line in the diagram above should be understood as simply signifying the action of the two-particle potential. At this point it does not imply that there is a virtual particle propagating between the two fermion lines. However, these diagrams are very useful and there are related with the ones we would obtain had we taken the non-relativistic limit of a given relativistic interaction. We will later see how these diagrams are used in perturbation theory in many body theory.

5.3 Coherent States

We have seen that many particle hamiltonians can be conveniently expressed in terms of annihilation and creation operators. Then, when building a funcional integral of a many-body theory it will be advantageous to formulate it in terms of eigenstates of these operators. These are the so-called coherent states. Here we will review some of their properties before we use them to build a functional integral representation of the partition function of a many-body system.

5.3.1 Coherent States of the Harmonic Oscillator

We are looking for eigenstates of the annihilation operator a. We know that the eigenstates of the hamiltonian satisfy

$$a|n\rangle = \sqrt{n}|n-1\rangle \ . \tag{5.57}$$

We define a coherent state $|\alpha\rangle$ by

$$a|\alpha\rangle = \alpha|\alpha\rangle \ . \tag{5.58}$$

Clearly, the state $|\alpha\rangle$ can be written as a linear combination of the eigenstates of the hamiltonian as

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \ , \tag{5.59}$$

which results in

$$a|\alpha\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n} |n-1\rangle \ . \tag{5.60}$$

Comparing (5.60) with (5.58) we see that the coefficients of the expansion defined by (5.59) must satisfy the recursion relation

$$c_{n+1} = \alpha \frac{c_n}{\sqrt{n+1}} \ . \tag{5.61}$$

Thus, the coherent state $|\alpha\rangle$ can be expanded using (5.61) as

$$|\alpha\rangle = c_0 \left(|0\rangle + \alpha \frac{|1\rangle}{\sqrt{1!}} + \alpha^2 \frac{|2\rangle}{\sqrt{2!}} + \dots \right) \ . \tag{5.62}$$

Using that

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \ , \tag{5.63}$$

we can rewrite (5.62) as

$$\begin{aligned} |\alpha\rangle &= c_0 \left(1 + \alpha \frac{a^\dagger}{1!} + \alpha^2 \frac{(a^\dagger)^2}{2!} + \dots \right) |0\rangle \\ &= c_0 \, e^{\alpha a^\dagger} \, |0\rangle \ . \end{aligned} \tag{5.64}$$

Demanding that $\langle \alpha | \alpha \rangle = 1$ fixes c_0. We then finally obtain

$$|\alpha\rangle = e^{-|\alpha|^2/2} \, e^{\alpha a^\dagger} \, |0\rangle \tag{5.65}$$

Notice that since a is not hermitian, the eigenvalue α is in general a complex number.

5.3.2 Coherent States in Many-body Theory

Following the discussion above for the coherent states of the harmonic oscillator, we would like to build coherent states from the eigenstates in the occupation number representation of many-body theory. That is, starting with states defined by

$$|n_1, n_2, \ldots\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_1^\dagger)^{n_2}}{\sqrt{n_2!}} \ldots |0\rangle \ , \tag{5.66}$$

we want to define a coherent state by

$$|\phi\rangle = \sum_{n_1, n_2, \ldots} C_{n_1, n_2, \ldots} |n_1, n_2, \ldots\rangle \ , \tag{5.67}$$

which is a superposition of states with different number of particles. The state $|\phi\rangle$ should be an eigenstate of the annihilation operators a_j, where the index j refers to a given single particle state. That is, we want

$$a_j|\phi\rangle = \phi_j|\phi\rangle \ , \tag{5.68}$$

where the eigenvalue ϕ_j is a complex number, just as α in (5.67) was. In order to satisfy (5.68) we write

$$\boxed{|\phi\rangle = e^{\sum_i \phi_i a_i^\dagger} |0\rangle} \ . \tag{5.69}$$

To prove that the state written as in (5.69) satisfies (5.68) we expand $|\phi\rangle$. We have

$$a_j|\phi\rangle = a_j \left(1 + \sum_i \phi_i a_i^\dagger + \frac{1}{2!} \sum_i \phi_i a_i^\dagger \sum_k \phi_k a_k^\dagger + \ldots \right) |0\rangle \ . \tag{5.70}$$

Using the commutation rules (we are assuming bosons for now)

$$[a_j, a_i^\dagger] = \delta_{ji} \ , \tag{5.71}$$

we obtain

$$
\begin{aligned}
a_j|\phi\rangle &= \left(\phi_j + \phi_j \sum_k \phi_k a_k^\dagger + \ldots\right)|0\rangle \\
&= \phi_j \left(1 + \sum_k \phi_k a_k^\dagger + \ldots\right)|0\rangle \\
&= \phi_j \, e^{\sum_k \phi_k a_k^\dagger}|0\rangle \ ,
\end{aligned}
\tag{5.72}
$$

just as postulated in (5.68). Then, the sates $|\phi\rangle$ defined this way are coherent states and will be very useful when using operators in second quantization language. They will be the building blocks of the functional integral formulation of the partition function. Before we go into this, it will be useful to get to know some of the properties of these states.

1. The conjugate state satisfies

$$
\langle\phi|a_j^\dagger = \langle\phi|\bar{\phi}_j \ , \tag{5.73}
$$

with $\bar{\phi}_j$ the complex conjugate of ϕ_j. This means that it can be expanded in term of *annihilation* operators as in

$$
\boxed{\langle\phi| = \langle 0|\, e^{\sum_k \bar{\phi}_k a_k}} \ . \tag{5.74}
$$

2. The following identity is satisfied:

$$
\boxed{a_j^\dagger|\phi\rangle = \partial_{\phi_j}|\phi\rangle} \ , \tag{5.75}
$$

where we defined

$$
\partial_{\phi_j} = \frac{\partial}{\partial\phi_j} \ . \tag{5.76}
$$

To prove (5.75) we use the expansion for $|\phi\rangle$ in (5.69). Then the right-hand side of (5.75) is

$$\begin{aligned}
\partial_{\phi_j}|\phi\rangle &= \partial_{\phi_j} e^{\sum_k \phi_k a_k^\dagger}|0\rangle \\
&= \partial_{\phi_j}\left(1+\sum_k \phi_k a_k^\dagger + \frac{1}{2}\sum_{k,\ell}\phi_k a_k^\dagger \phi_\ell a_\ell^\dagger + \ldots\right)|0\rangle \qquad (5.77)\\
&= a_j^\dagger\left(1+\sum_k \phi_k a_k^\dagger + \ldots\right)|0\rangle \\
&= a_j^\dagger|\phi\rangle\ , \qquad (5.78)
\end{aligned}$$

which proves (5.75). Finally, it is straightforward to prove that the complex conjugate of (5.75) is

$$\boxed{\langle\phi|a_j = \partial_{\bar{\phi}_j}\langle\phi|}\ . \qquad (5.79)$$

3. Coherent States are not orthogonal.

For instance, take two coherent states $|\phi\rangle$ and $|\theta\rangle$. Their inner product is

$$\langle\theta|\phi\rangle = \langle 0|e^{\sum_k \bar{\theta}_k a_k}|\phi\rangle\ , \qquad (5.80)$$

where we used (5.74). Using the repeated action of a_k on $|\phi\rangle$ as defined by (5.68) we have

$$\langle\theta|\phi\rangle = e^{\sum_k \bar{\theta}_k \phi_k}\langle 0|\phi\rangle\ . \qquad (5.81)$$

But since $\langle 0|\phi\rangle = 1$ (check this!) we have that

$$\boxed{\langle\theta|\phi\rangle = e^{\sum_k \bar{\theta}_k \phi_k}}\ . \qquad (5.82)$$

In particular, the norm of a coherent state is not unity. It is given by

$$\langle\phi|\phi\rangle = e^{\sum_k \bar{\phi}_k \phi_k}\ . \qquad (5.83)$$

4. Resolution of the identity in Fock space.

   The following identity is satisfied by coherent states:

$$\boxed{\int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} \, e^{-\sum_k \bar{\phi}_k \phi_k} \, |\phi\rangle\langle\phi| = \mathbb{1}_{\mathcal{F}}} \, . \tag{5.84}$$

   In the expression above, the right-hand side corresponds to the identity in Fock space. The integrals are over all possible values of ϕ_i and of $\bar{\phi}_i$. This identity will be very important in the implementation of the functional integral in many-body theory, so let us prove it carefully.

   First, the operators a_j and $a_j^\dagger$ form an irreducible representation of Fock space. Then, by Schur's lemma, we know that if an operator commutes with them then it must be proportional to $\mathbb{1}_{\mathcal{F}}$. In other words,

$$[\hat{\mathcal{O}}, a_j] = 0 = [\hat{\mathcal{O}}, a_j^\dagger] = 0 \Rightarrow \hat{\mathcal{O}} \propto \mathbb{1}_{\mathcal{F}} \, . \tag{5.85}$$

   Let us now compute the action of a_j on the operator defined on the left-hand side of (5.84). For notational simplicity we define

$$d\left(\bar{\phi}, \phi\right) \equiv \prod_i \frac{d\bar{\phi}_i, d\phi_i}{\pi} \, . \tag{5.86}$$

   Then we have that

$$\begin{aligned} a_j \int d\left(\bar{\phi}, \phi\right) e^{-\sum_k \bar{\phi}_k \phi_k} \, |\phi\rangle\langle\phi| &= \int d\left(\bar{\phi}, \phi\right) e^{-\sum_k \bar{\phi}_k \phi_k} \, \phi_j |\phi\rangle\langle\phi| \\ &= -\int d\left(\bar{\phi}, \phi\right) \left(\partial_{\bar{\phi}_j} e^{-\sum_k \bar{\phi}_k \phi_k}\right) \, |\phi\rangle\langle\phi| \, , \end{aligned} \tag{5.87}$$

   where in the first line in (5.87) we apply the defining property of coherents states (5.68). Integrating by parts we obtain

$$\begin{aligned} a_j \int d\left(\bar{\phi}, \phi\right) e^{-\sum_k \bar{\phi}_k \phi_k} \, |\phi\rangle\langle\phi| &= \int d\left(\bar{\phi}, \phi\right) e^{-\sum_k \bar{\phi}_k \phi_k} \, |\phi\rangle \left(\partial_{\bar{\phi}_j} \langle\phi|\right) \\ &+ \int \partial_{\bar{\phi}_j} \left(e^{-\sum_k \bar{\phi}_k \phi_k} \, |\phi\rangle\langle\phi|\right) d\left(\bar{\phi}, \phi\right) \, . \end{aligned} \tag{5.88}$$

The second term in (5.88) is a total derivative and we can do the integrals. The integrals we must perform are all of the type

$$\int \partial_{\bar{\phi}_j} \left(e^{-\bar{\phi}_j \phi_j} \, |\phi\rangle\langle\phi| \right) d\bar{\phi}_j d\phi_j = e^{-(\mathrm{Re}[\phi_j]^2 + \mathrm{Im}[\phi_j]^2)} \Big]_{-\infty}^{+\infty} |\phi\rangle\langle\phi| = 0 \ , \tag{5.89}$$

where we used that

$$d\bar{\phi}_j \, d\phi_j = d\mathrm{Im}[\phi_j] \, d\mathrm{Re}[\phi_j] \ . \tag{5.90}$$

Then we see that the second term in (5.88) vanishes. Furthermore, using (5.79) the expression in (5.88) becomes

$$a_j \int d\left(\bar{\phi}, \phi\right) e^{-\sum_k \bar{\phi}_k \phi_k} \, |\phi\rangle\langle\phi| = \int d\left(\bar{\phi}, \phi\right) e^{-\sum_k \bar{\phi}_k \phi_k} \, |\phi\rangle\langle\phi| a_j \ , \tag{5.91}$$

proving that the operator defined in the left hand-side of (5.84) commutes with a_j. It is straightforward to repeat this for proving that it also commutes with the $a_j^\dagger$'s. Then, it must be proportional to the identity in Fock space $\mathbb{1}_{\mathcal{F}}$. In order to compute the constant of proportionality c, we use the fact that its vacuum expectation value must be c, i.e.

$$\int \prod_i d\bar{\phi}_i d\phi_i \, e^{-\sum_k \bar{\phi}_k \phi_k} \, \langle 0|\phi\rangle\langle\phi|0\rangle = c \ . \tag{5.92}$$

We need to perform the integrals

$$\int_{-\infty}^{+\infty} d\bar{\phi}_i d\phi_i e^{-\bar{\phi}_i \phi_i} = \int_{-\infty}^{+\infty} d\mathrm{Re}[\phi_i] \, d\mathrm{Im}[\phi_i] \, e^{-(\mathrm{Re}[\phi_i]^2 + \mathrm{Im}[\phi_i]^2)} = \sqrt{\pi}\,\sqrt{\pi} = \pi \ . \tag{5.93}$$

which tells us that

$$c = \prod_i \pi \ , \tag{5.94}$$

which completes our proof of (5.84).

In the next lecture we will make use of this machinery to build the functional integral for the partition function in a many-body theory.

Additional suggested readings

- *Condensed Matter Field Theory*, Altland and Simons, Section 4.1.
- *Quantum Theory of Many Particle Systems*, A. L. Fetter and J. D. Walecka, Chapters 1 and 2.