Lecture 24

Extended Field Configurations

Although our fourmulation of Quantum Field Theory (QFT), whether through canonical or functional integral quantization methods, has been quite general, up to this point all our actual calculations have required the use of perturbation theory. This was typically implemented in expansions in powers of the couplings, or loop expansions. In addition, so far we have considered expansions around a constant vacuum –both in time and space– independently of this being a trivial or non trivial (i.e. spontaneously broken) vacuum state. We will now consider the existence of solutions beyond perturbation theory. These will be extended field configurations that will depend on the position and which stability will be the result of boundary conditions or *topological conservation laws*. These stable field configurations will require energies above the ground state. To introduce the basic concepts involved in these types of solutions of QFT, we will start with the simplest example: a $2D$ $(1+1)$ system.

24.1 Solitons in 2D

We start with a theory of a scalar field in 1 spatial dimension. The lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\ , \tag{24.1}$$

where for now we need not specify the potential. The energy of the system can is then given by

$$E = \int dx\, \left\{\frac{1}{2}\,(\partial_0\phi)^2 + \frac{1}{2}\,(\partial_1\phi)^2 + V(\phi)\right\}, \tag{24.2}$$

where we can consider the first term as the kinetic energy of the field $\phi(x)$, with the second and third terms the potential energy. The second term represents the energy stored in the spatial deformation of $\phi(x)$. The condition we need to impose for the energy to be bounded from below is that $V(\phi)$ has a minimum. We conventionally choose this to be

$$V_{\text{min.}}(\phi) = 0 \; . \tag{24.3}$$

Thus, the ground state is such that

- ϕ_0 is independent of t and x ,
- ϕ_0 is one of the zeroes of $V(\phi)$,

where ϕ_0 is the value of the field at the ground state. With these conditions we have that the energy of the ground state is

$$E_0 = 0 \; . \tag{24.4}$$

Having established the conditions for the ground state, we are now going to search for solutions that are *time independent* but, to go beyond the ground state, we consider spatially varying ones. In this case, imposing $\partial_0\phi = 0$, the energy is now

$$E = \int dx \left\{ \frac{1}{2}(\partial_1\phi)^2 + V(\phi) \right\} \; . \tag{24.5}$$

To find these *finite energy* non trivial field configurations we make use of a trick due to Bogomol'nyi. We rewrite (24.5) as

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} dx \left(\frac{d\phi}{dx} \mp \sqrt{2V(\phi)} \right)^2 \pm \int_{\phi(-\infty)}^{\phi(+\infty)} \sqrt{2V(\phi)}\, d\phi \; . \tag{24.6}$$

One can readily verify that the last two expressions for the energy are equivalent. Interestingly though, the last term in (24.6) only depends on the values of the field $\phi(x)$ at $x = \pm\infty$. Thus, if $\phi(-\infty) = \phi(+\infty)$ this integral vanishes. In general we can state that

$$E \geq \int_{\phi(-\infty)}^{\phi(+\infty)} \sqrt{2V(\phi)}\, d\phi \; . \tag{24.7}$$

The equality is obtained when the first term in (24.6) vanishes[1], i.e. when

$$\frac{d\phi}{dx} = \pm\sqrt{2\,V(\phi)} \; , \tag{24.8}$$

[1] This guarantees that the solution will have finite energy.

is satisfied. Here we can already see that integrating both sides of (24.8) and imposing that the integral in (24.7) vanishes will result in the ground state. So in order to find a non trivial solution with $\partial_1\phi(x) \neq 0$, we need to have

$$\phi(-\infty) \neq \phi(+\infty) \ . \tag{24.9}$$

This is also a good place to notice that the solution resulting from imposing (24.8) corresponds to

$$E_{\text{sol.}} = \phi(+\infty) - \phi(-\infty) \ , \tag{24.10}$$

is stable against continuous perturbations that maintain the values of the fields at the boundaries $x = \pm\infty$, since $E_{\text{sol.}}$ only depends on the latter. As we will see below, this is at the heart of the topological conservation law associated with these solutions.
The expression in (24.8) can be integrated to obtain

$$x - x_0 = \pm \int_0^{\phi(x)} \frac{df}{\sqrt{2V(f)}} \ , \tag{24.11}$$

where x_0 is an arbitrary integration constant and f just an integration variable. In order to go further and invert (24.11) to obtain the solution for $\phi(x)$ satisfying these conditions, we need to specify the potential $V(\phi)$. We consider the example

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 \ , \tag{24.12}$$

where λ is a coupling, and v defines the minimum of the potential, defined here to be zeroes of $V(\phi)$ such that

$$\phi_0 = \pm v \ . \tag{24.13}$$

Then, we can perform the integral in (24.11)

$$x - x_0 = \pm \int_0^{\phi}(x) \frac{df}{\sqrt{\frac{\lambda}{2}}(f^2 - v^2)} = \mp\sqrt{\frac{2}{\lambda}}\, \text{arctanh}\left(\frac{\phi}{v}\right) \ , \tag{24.14}$$

which translates into

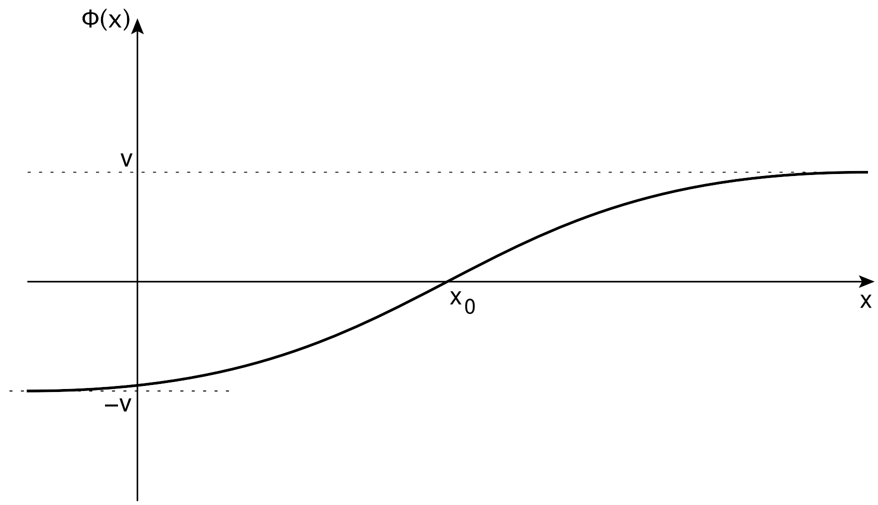

Figure 24.1: One of the kink solutions in (24.15), corresponding to the + sign.

$$\boxed{\phi(x - x_0) = \mp v \tanh\left(\sqrt{\frac{\lambda}{2}} v(x - x_0)\right)} \ . \tag{24.15}$$

The solution above is sketched in Figure 24.1. This non trivial spatial configuration of the field is called kink or a soliton. It corresponds to the + sign in (24.15). There will be another similar kink solution associated with the − sign. In both cases the kink solution connects the two possible vacua, $\phi_0 = -v$ and $\phi_0 = +v$, from two different points in space. The point x_0, the "center" of the kink, can be anywhere in x as dictated by translation invariance. The energy stored in this solution is concentrated around x_0. This is clear by considering the energy density

$$\epsilon(x) = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{\lambda}{4}\left(\phi(x)^2 - v^2\right)^2 \ , \tag{24.16}$$

which, using (24.8), is just twice the second term. The energy density profile is sketched in Figure 24.2. We can estimate the typical length scale L over which the energy is concentrated. Using dimensional analysis we can write

$$E = \int_{-\infty}^{+\infty} dx\, \epsilon(x) \sim L\left(\frac{v}{L}\right)^2 + L\,\lambda\, v^4 \ , \tag{24.17}$$

which is minimized for

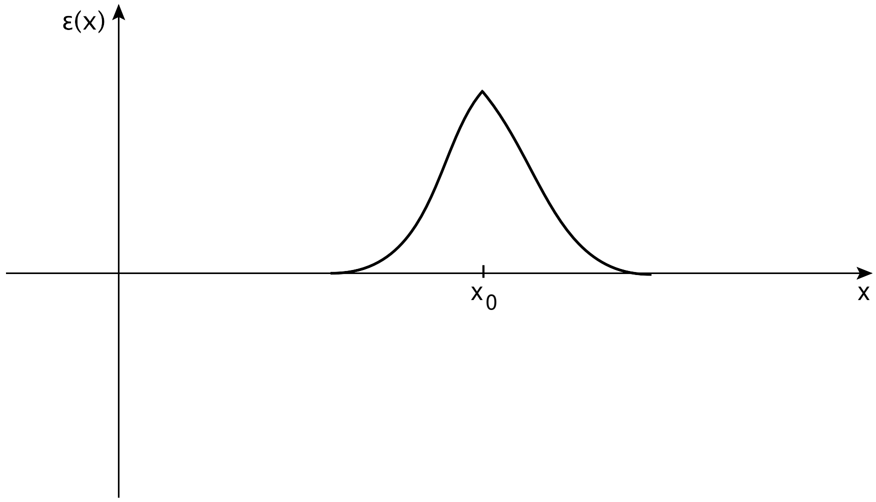

Figure 24.2: Energy density profile of the kink solutions (24.15).

$$L \simeq \frac{1}{\sqrt{\lambda}v} = \frac{1}{m_\eta} \ , \tag{24.18}$$

with m_η the mass of the scalar field *particle excitation* we usually obtain by expanding around the ground state. That is from writing

$$\phi(x) = v + \eta(x) \ , \tag{24.19}$$

with $\langle\eta(x)\rangle = 0$. So we see that the length scale associated with the kink solutions is essentially the Compton wavelength of the particle excitation of the field. From (24.18) we can see that in the non perturbative limit of large λ the kink solutions tend to behave as particles with vanishing L.

24.2 Topological Charge

Next, we need to address the question of the stability of these solutions. A useful way to classify them is by defining the current

$$J^\mu \equiv \frac{1}{2v}\,\epsilon^{\mu\nu}\,\partial_\nu\phi(x) \ , \tag{24.20}$$

where $\epsilon^{\mu\nu}$is the two dimensional Levi-Civita tensor with the convention $\epsilon^{01} = 1$. This is a conserved current by construction, i.e.

$$\partial_\mu J^\mu = 0 \ . \tag{24.21}$$

More interestingly, the associated charge is given by

$$\begin{aligned} Q &= \int dx\, J^0 = \frac{1}{2v} \int_{-\infty}^{+\infty} dx\, \frac{d\phi}{dx} \\ &= \frac{1}{2v} \left[\phi(+\infty) - \phi(-\infty)\right] \ . \end{aligned} \tag{24.22}$$

We can write this result for the *topological* charge Q in a general way as

$$Q = n\, 2v \ , \tag{24.23}$$

with $n = 0, +1, -1$. The kink solution (24.15) with the $+$ sign would have $Q = 2v$, or $n = 1$. On the other hand, the solution with the $-$ sign (the antikink) would have $Q = -2v$ or $n = -1$. Finally, we can consider the ground state solutions $\phi_0 = \pm v$ as having $Q = 0$ (or $n = 0$) since for them $\phi(+\infty) = \phi(-\infty)$.

Since the charge Q is conserved, this means that these distinct field configurations, i.e. with $n = 1$, $n = -1$ and $n = 0$ are stable in the sense that we cannot deform one into another by *finite energy* deformations. To see this more clearly, let us consider once more the kink solution depicted in Figure 24.1. For instance, in order to deform it into the ground state with $\phi_0 = +v$, we need all the points on the negative x axis to be lifted by a finite amount from the negative values of $\phi(x)$ to the value $+v$. But this needs to be done for all points from $x = 0$ to $x = -\infty$, which has an infinite amount of energy cost. We can say that the $n = 1$ kink is stable in the sense that it cannot decay to the ground state or the $n = -1$ antikink due to this energy cost. This stability is what makes this type of conservation law different than the ones derived from Noether's theorem.

Let us introduce some more formal aspects of what we have found. The topological conservation law divides finite energy solutions en distinct sectors (e.g. with $n = 1$, $n = -1$ and $n = 0$) according to their topological charges. We can establish a mapping between the discrete set of points at infinity

$$S = \left\{ x = +\infty, x = -\infty \right\} \ , \tag{24.24}$$

and the zeroes (actually the minima) of the potential $V(\phi)$

$$M_0 = \left\{ \phi / V(\phi) = 0 \right\} \ . \tag{24.25}$$

The reason is that for the energy of these solutions to remain finite the values of $\phi(x)$ at $x = \pm\infty$ must be zeroes of $V(\phi)$. Or

$$\lim_{x\to\pm\infty} \phi(x) = \phi \in M_0 \ . \tag{24.26}$$

We can define *mappings* from S to M_0. If we define $\phi_\pm \equiv \phi(\pm\infty)$ then the configurations with $n = 1$ are described by the mapping

$$\phi_+ \to +v \qquad\qquad \phi_- \to -v \ . \tag{24.27}$$

That is, the position $x = +\infty$ is mapped to the field value $+v$ in M_0, whereas $x = -\infty$ is mapped to $-v$. On the other hand, the $n = -1$ configuration, the antikink, corresponds to the mapping

$$\phi_+ \to -v \qquad\qquad \phi_- \to +v \ . \tag{24.28}$$

Finally, the configurations with $n = 0$, the ground state, correspond to

$$\begin{array}{ccc} \phi_+ \to +v & & \phi_- \to +v \ , \\ & \text{or} & \\ \phi_+ \to -v & & \phi_- \to -v \ . \end{array} \tag{24.29}$$

The mappings described in (24.27), (24.28) and (24.29) are topologically distinct since the correspond to field configurations with different values of n or of the topological charge Q. As we saw earlier, it is not possible to continuously deform these mappings into one another.

In the case study here, in $2D$, we saw that the mappings are between two discrete sets: S and M_0. This will change when going to higher dimensions.

24.3 Higher Dimensions and Derrick's Theorem

Here we consider the four dimensional case, $(3 + 1)$, but as we will see later our results will also be valid for three dimensions or $(2 + 1)$. Starting from the same lagrangian as in (24.1), we can only see that the topology of the spatial infinity. What was a discrete set with two possible values in the $(1 + 1)$ case, now is a continuous set described by S_2 the sphere at $r \to \infty$[2]. This is already an important difference. But more urgently, the

[2]In $(2 + 1)$ the spatial infinity is of course the circle S_1, also continuous.

question is to see if there are any stabe finite energy solutions that are non trivial. We then retrace the steps we followed for the $(1+1)$ case above. First we write the energy as

$$E = \int d^3x \left\{ \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right\} \; . \tag{24.30}$$

Next, we impose that the field configuration in question satisfies that at the surface at $r \to \infty$ it goes to zeros of $V(\phi)$. This gets rid of any possible divergent contribution coming from the potential. In more formal language we impose that

$$\phi^\infty(\hat{r}) \equiv \lim_{R\to\infty} \phi(R\,\hat{r}) \in M_0 \; , \tag{24.31}$$

where, just as in our $2D$ example, we defined M_0 as the set of zeroes (minima) of $V(\phi)$. However, even with (24.31) imposed, and with the assumption of a time independent configuration that would vanish the first term in (24.30), we still have a problem with the second term. This is

$$\nabla\phi\cdot\nabla\phi = \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial\phi}{\partial\varphi}\right)^2 \; , \tag{24.32}$$

where we are using polar coordinates where φ is the azimutal angle. The trouble is that its contribution to the energy

$$\int d^3x\,(\nabla\phi)^2 = \int r^2 dr\, d\Omega\,(\nabla\phi)^2 \; , \tag{24.33}$$

diverges for any non trivial field configurations. To see this we must assume that at least one term in (24.32) is non zero. Otherwise, we will have obtained the trivial configuration corresponding to the ground state $E = 0$. This is analogous to what we can see in Figure 24.1. For a field configuration to be non trivial we need the values of ϕ^∞ to have variation at $r \to \infty$. For instance, one of the angular derivatives could be non zero. But then these terms lead to a linear divergence[3]. So we conclude that in $(3+1)$ dimensions is not possible to have finite energy non trivial field configurations such as the kinks we found in $(1+1)$ dimensions. The same can be said of scalar theories in $(2+1)$ dimensions. This result is known as Derrick's theorem and it appears to tell us that extended field configurations can only be found in the lowest possible dimensional systems. As we will see in the following lectures, there is a way around this result that involves complicating the theory by introducing gauge fields.

[3]Incidentally, repeating the whole argument for $(2+1)$ dimensions, we will arrive at a logarithmic divergence. So finite energy non trivial field configurations cannot be obtained in this either.

24.4 Introducing Stabilizing Gauge Fields

Let us start by considering a gauge theory in $(2+1)$ dimensions. The extension to $(3+1)$ dimensions will be straightforward. The lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - V(\phi) \ , \tag{24.34}$$

invariant under gauge transformations defined in the gauge group G, and with the usual definition of the covariant derivative. The potential has a set of values that minimize it, $\{\phi_0\}$, which can be taken to be its zeroes by shifting $V(\phi)$ by an irrelevant constant.

To make our calculations simpler it is advantageous to use the temporal gauge:

$$A^a_0 = 0. \tag{24.35}$$

For instance, this means that we will have

$$D_0\phi = \partial_0\phi \ , \tag{24.36}$$

and

$$F^a_{0i} = \partial_0 A^a_i \ . \tag{24.37}$$

From (24.34) and using

$$H = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial_0\phi - \mathcal{L} \ , \tag{24.38}$$

we obtain

$$E \ = \ \int d^dx \left\{\frac{1}{4}(\partial_0 A^a_i)^2 + \partial_0\phi^\dagger\partial_0\phi + \frac{1}{4}F^a_{ij}F^a_{ij} + \vec{D}\phi^\dagger\cdot\vec{D}\phi + V(\phi)\right\} \ . \tag{24.39}$$

We want to find time independent, finite energy solutions that are not the trivial ground state. We start by dropping the first two terms in (24.39) since there are time derivativs. We also impose that the solutions must be such that (in $(2+1)$ dimensions)

$$\lim_{r\to\infty}\phi(r,\theta) \equiv \phi(\infty,\theta) \ , \tag{24.40}$$

are zeroes of $V(\phi)$. This, of course guarantees that the last term in (24.39) does not diverge, i.e. that

$$\int_0^\infty r dr\, V(\phi)\ , \tag{24.41}$$

is not divergent. We now concentrate on the conditions to make the fourth term in (24.39) finite. Remember that, in the absence of gauge fields, this term was divergent (logarithmically for $(2+1)$ dimensions). The point is that without the gauge fields we have

$$\int_0^\infty r\, dr\, \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2\ , \tag{24.42}$$

diverges, unless the angular derivative vanishes at $r\to\infty$. But this would result in trivial configuration, the ground state. But the presence of the gauge fields in the covariant derivative gives us a way to cancel this contribution. In other words, we consider the θ component of the covariant derivative

$$\hat{e}_\theta\cdot\vec{D} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} + ig\, A^a_\theta(x)t^a\phi(x)\ , \tag{24.43}$$

with g the gauge coupling, $x=(r,\theta)$, and t^a the group generators. We then impose that these two terms cancel in the $r\to\infty$ limit, i.e.

$$\boxed{\lim_{r\to\infty} r\, A^a_\theta(x)t^a\phi(x) = -\lim_{r\to\infty}\frac{i}{g}\frac{\partial\phi(x)}{\partial\theta}}\ . \tag{24.44}$$

The condition above guarantees that the fourth term in (24.39) does not diverge. The right hand side is a condition on the value of the angular derivative of $\phi(\infty,\theta)$ and it has not r dependence. Thus, this implies that the θ component of the gauge field at $r\to\infty$ satisfies

$$A^a_\theta(\infty,\theta)\sim\frac{1}{r}\ . \tag{24.45}$$

This translates in turn into a behavior of the purge gauge term in (24.39). Since (24.45) imposes

$$F_{ij}\sim\frac{1}{r^2}\ , \tag{24.46}$$

then we will have

$$F_{ij}F_{ij} \sim \frac{1}{r^4} \ , \tag{24.47}$$

rendering the contribution of the pure gauge term to the energy harmless since the radial integral

$$\int_0^\infty r\,dr\,F_{ij}F_{ij} \tag{24.48}$$

is finite. Incidentally, it is clear from the arguments above that the same applies to $(3+1)$ dimensions, since now we would have the radial integral

$$\int_0^\infty r^2\,dr\,F_{ij}F_{ij} \ , \tag{24.49}$$

which is still finite. By the same token, we cannot go beyond $(3+1)$ dimensions given that the radial integral will now be divergent (e.g. diverges logarithmically in $(4+1)$ dimensions). In conclusion, the condition (24.44) must be satisfied by
$phi(\infty, \theta)$. This guarantees that we will have a finite energy stable non trivial solution. The $\phi(\infty, \theta)$ is a *mapping* of the circle at $r \to \infty$ into the coset group G/H, since they must be the zeroes (minima) of $V(\phi)$. Mappings that cannot be deformed into each other are topologically distinct, i.e. they have different topological charges. We illustrate this with an example. Let us consider the potential

$$V(\phi) = \frac{\lambda}{2}\left(\phi^*\phi - v^2\right)^2 \ , \tag{24.50}$$

where $\phi(x)$ transforms under the group $G = U(1)$. The zeroes of $V(\phi)$, ϕ_0 clearly satisfy

$$|\phi_0|^2 = v^2 \ , \tag{24.51}$$

which translates into

$$\phi_0 = v\,e^{i\sigma} \ , \tag{24.52}$$

where σ is a phase. The coset space G/H is the locus of the zeroes, i.e. a circle of radius v. We see that there is a mapping from the physical space at $r \to infty$ into the coset space G/H which is defined by

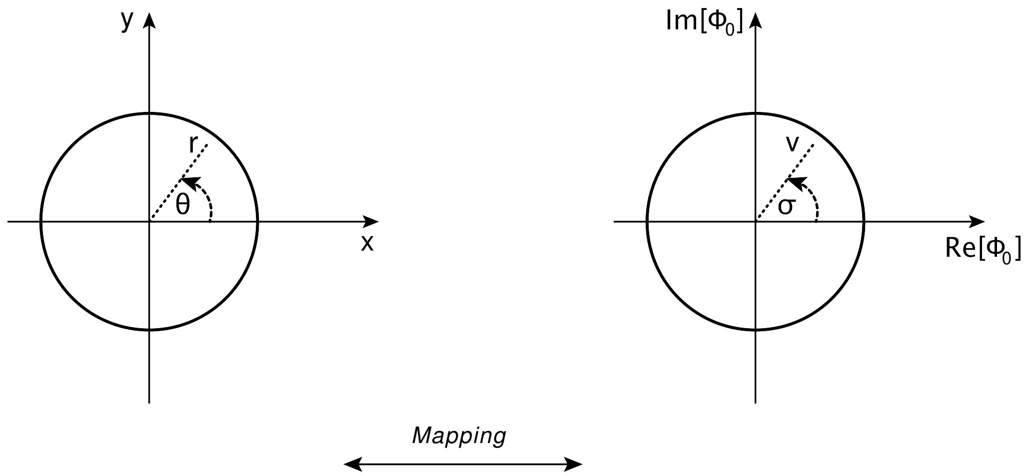

Figure 24.3: Mapping from the $2D$ space (x, y) to the coset space G/H. This corresponds to $S_1 \to S_1$, a mapping from the circle into the circle.

$$\lim_{r\to\infty} \phi(r, \theta) = \phi(\infty, \theta) = \phi_0\, e^{i\sigma} \ , \tag{24.53}$$

where the second equality is the condition that the $\phi(\infty, \theta)$ must be zeroes of $V(\phi)$. The condition (24.53) above corresponds to a mapping illustrated in Figure 24.3. Each point in (x, y) can be mapped to a point in the coset space G/H. However, this correspondence is fixed up to the windings around G/H. For each mapping we define a winding number n that tells us how many times we need to go around the circle in G/H to find the point in it corresponding to the one originating in (x, y). Thus, the winding number n characterizes the mapping. To be more concrete, let us write

$$\int_0^{2\pi} d\theta = 2\pi \ , \tag{24.54}$$

defining the span of the variable θ in space, whereas for the phase in G/H we have that $\sigma \to \sigma + 2\pi\, n$ results in the same point in the coset circle. Then, for each circle in (x, y) (i.e. for each 2π in the variable θ we have

$$2\pi\, n = \int_0^{2\pi} d\sigma = \int_0^{2\pi} d\theta\, \frac{d\sigma}{d\theta} \ , \tag{24.55}$$

which results in

$$n = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \frac{d\sigma}{d\theta}\ . \tag{24.56}$$

The expression above for the winding number will be of great importance in our understanding of the topological properties of these mappings.

As an application, let us consider the case when the gauge group is abelian. In particular, here we will consider $G = U(1)_{\rm EM}$, i.e. electromagnetism. We rewrite the condition (24.44) as

$$\lim_{r\to\infty} r\, A_\theta\, \phi(\infty,\theta) = -\frac{i}{e}\,\frac{d\phi(\infty,\theta)}{d\theta}\ . \tag{24.57}$$

The magnetic flux crossing a surface S is given by

$$\Phi = \int_S \vec{B}\cdot d\vec{S} = \oint_{C(S)} \vec{A}\cdot d\vec{\ell}\ , \tag{24.58}$$

For our two dimensional sistem this is

$$\Phi = \int_0^{2\pi} r\, d\theta\, A_\theta(r,\theta)\ . \tag{24.59}$$

But using (24.57), we have that

$$\lim_{r\to\infty} r\, A_\theta(r,\theta)\, v\, e^{i\sigma} = -\frac{i}{e}\, v\, e^{i\sigma}\, i\,\frac{d\sigma}{d\theta}\ . \tag{24.60}$$

Using (24.60) in (24.59) we arrive at

$$\Phi = \lim_{r\to\infty}\int_0^{2\pi} \frac{1}{e}\, d\theta\,\frac{d\sigma}{d\theta} = \frac{2\pi\, n}{e}\ . \tag{24.61}$$

Thus, we obtain the quantization of the magnetic flux as a result of the topological properties of the mapping from space to coset space. This corresponds to the description of type II superconductors, where the quantized magnetic flux is confined in the cross section of vortices. These simple examples contain several of the more formal elements that we will use in more complex situations, i.e. to describe monopole solutions in $(3+1)$ dimensions or instantons in euclidean 4D. We will introduce some of these formal elements of topology in the next lecture before me move on to these cases.

Additional suggested readings

- *Advanced Topics in Quantum Field Theory: A Lecture Course*, by M. Shifman, Chapter 2.
- *The Quantum Theory of Fields, Vol. II*, by S. Weinberg, Section 23.1.