Lecture 22

Anomalies in the Functional Integral

In the previous lecture, we see that it ispossible to have symmetries that are classically preserved to be explicitely broken by quantum corrections. In particular, we saw that the axial current –classically conserved in the limit of massless fermions– is anomalous in the full quantum theory. In this lecture, we will approach the problem by looking at the response of the functional integral under symmetry transformations. Since we will be looking at the quantum properties of currents, we will first reexamine how symmetries and their associated currents appear in the functional integral formalism. Then, we will be ready to state the problem of anomalies as one of the (non-)invariance of the measure in the functional integral.

22.1 Symmetries in the Functional Integral

We can write the generating function of a theory in terms of sources as

$$Z[j, f_\mu] = \int \mathcal{D}\phi \, e^{i\int d^4x \left\{\mathcal{L}(\phi(x)) + j(x)\phi(x) + f_\mu(x) J^\mu(x)\right\}} \;, \tag{22.1}$$

where there is a scalar source $j(x)$ linearly coupled to the field $\phi(x)$, and we also defined a vector source $f_\mu(x)$ coupled to the current $J^\mu(x)$. In this way, we can define

$$\bar{J}^\mu(x) \equiv (-i)\frac{\delta \ln Z}{\delta f_\mu}\Big|_{f_\mu=0} = \langle 0|J^\mu(x)|0\rangle \;, \tag{22.2}$$

the vacuum expectation value of the current. More generally, we can define the expectation value of a product of currents. For instance, if we consider a theory with fermions and a vector boson, we can define

$$Z[v_\mu, a_\mu] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu\, e^{i\int d^4x\,\left\{\mathcal{L}(\psi,\bar{\psi},A_\mu)+v_\mu(x)J_V^\mu(x)+a_\mu(x)J_A^\mu(x)\right\}}\;, \tag{22.3}$$

where we introduced the vector and axial currents J_V^μ and J_A^μ, respectively. From it we can obtain

$$\langle 0|T J_A^\mu(x)\, J_V^\nu(y)\, J_V^\alpha(z)|0\rangle = (-i)^2\,\frac{\delta^2}{\delta v_\nu(y)\,\delta v_\alpha(z)}\,\bar{J}_A^\mu(x)\;, \tag{22.4}$$

where T corresponds to time ordering. This three point amplitude is analogous to the triangle diagram we computed in the previous lecture. If we were to obtain a current conservation equation as in

$$\partial_\mu \bar{J}_i^\mu(x) = 0, \tag{22.5}$$

with $i = V, A$ this would be a current conservation law valid in the presence of quantum effects.

We want to know what are the conditions for (22.5) to be satisfied. For this purpose, we will implement Noether's theorem in the functional integral. We start by considering an infinitesimal transformation of a field $\phi(x)$ as given by

$$\phi(x) \longrightarrow \phi(x) + \epsilon(x)F(\phi)\;, \tag{22.6}$$

where $F(\phi)$ is a function of the field that contains the information about the symmetry transformation (e.g. generators). Although we are considering $\epsilon(x)$ as an infinitesimal function of x, our results will equally apply for both local and global symmetry transformations. The change in the lagrangian $\mathcal{L}$ is given by

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu(\delta\phi)\\ &= \frac{\partial\mathcal{L}}{\partial\phi}\,\epsilon\,F(\phi) - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big)\,\epsilon F(\phi) + \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\epsilon F(\phi)\Big)\;,\end{aligned} \tag{22.7}$$

where in the first line we use that $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$, and in the second line we substituted $\delta\phi = \epsilon F(\phi)$. The first two terms in the second line of (22.7) vanish by virtue of the equation of motion. Thus, we have

$$\mathcal{L}(\phi') = \mathcal{L}(\phi) + \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \, \epsilon F(\phi) \Big) \ . \tag{22.8}$$

We can rewrite this expression using the fact that ϵ is position dependent. Then we have

$$\mathcal{L}(\phi') = \mathcal{L}(\phi) + \partial_\mu \epsilon(x) \, F(\phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \text{total derivatives} \ , \tag{22.9}$$

where we have added the possibility of an additional total derivative, which then does not change the invariance of the action. Then, we can write the current as

$$J^\mu(x) = F(\phi) \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \ , \tag{22.10}$$

which results in

$$\mathcal{L}(\phi') = \mathcal{L}(\phi) + \partial_\mu \epsilon(x) \, J^\mu(x) \ , \tag{22.11}$$

and from which we could define the current also as

$$J^\mu(x) = \frac{\partial \mathcal{L}(\phi')}{\partial(\partial_\mu \epsilon)} \ . \tag{22.12}$$

We recover the classical statement of the invariance of the lagrangian by integrating by parts the second term in (22.11) to conclude that

$$\mathcal{L}(\phi') = \mathcal{L}(\phi) \qquad \text{if} \qquad \partial_\mu J^\mu = 0. \tag{22.13}$$

We want now to do something similar to understand the relationship between the conservation of the current defined in the functional integral by (22.2) and the invariance of the theory as fully defined by the generating functional. So starting from (22.1) for the generating functional, we consider its variation in response to the deformation of the source $f_\mu(x)$. This is given by

$$\begin{aligned} \delta \ln Z(f_\mu] &= \ln Z[f_\mu + \delta f_\mu] - \ln Z[f_\mu] \\ &= i \int d^4x \, \bar{J}^\mu(x) \, \delta f_\mu(x) \ , \end{aligned} \tag{22.14}$$

where we can view the second equality as the inverse of (22.2). If we now choose the vector source so that

$$\delta f_\mu = \partial_\mu \epsilon(x)\ , \tag{22.15}$$

then we have that

$$\begin{aligned}\delta_\epsilon \ln Z[f_\mu] &= \ln Z[f_\mu + \partial_\mu\epsilon] - \ln Z[f_\mu] \\ &= i\int d^4x\, \bar{J}^\mu(x)\partial_\mu\epsilon(x) \\ &= -i\int d^4x\, \partial_\mu \bar{J}^\mu(x)\, \epsilon(x)\ .\end{aligned} \tag{22.16}$$

where to obtain the last line we integrated by parts in the second. Then, we see that the invariance of the generating functional appears to hinge on the conservation of the current defined by (22.2). That is,

$$Z[f_\mu + \partial_\mu\epsilon] = Z[f_\mu] \qquad \leftrightarrow \qquad \partial_\mu \bar{J}^\mu(x) = 0\ . \tag{22.17}$$

Let us now check what are the conditions for (22.17) to be satisfied. The generating functional with the changed source can be written as

$$Z[f_\mu + \partial_\mu\epsilon] = \int \mathcal{D}\phi\, e^{i\int d^4x\left\{\mathcal{L}(\phi) + \left(f_\mu + \partial_\mu\epsilon\right)J^\mu(x)\right\}}\ . \tag{22.18}$$

But making use of (22.11), we can write this as

$$Z[f_\mu + \partial_\mu\epsilon] = \int \mathcal{D}\phi\, e^{i\int d^4x\left\{\mathcal{L}(\phi') + f_\mu(x)J^\mu(x)\right\}}\ . \tag{22.19}$$

But the expression above is *almost* identical to $Z[f_\mu]$. The only extra assumption needed is that

$$\boxed{\mathcal{D}\phi = \mathcal{D}\phi'}\ , \tag{22.20}$$

is satisfied. That is, if the Jacobian for the functional integration measure of the field under the symmetry transformation is unity, then we have that

$$Z[f_\mu + \partial_\mu \epsilon] = Z[f_\mu] \ , \tag{22.21}$$

which then implies that the expectation value of the current satisfies

$$\boxed{\partial_\mu \bar{J}^\mu(x) = 0} . \tag{22.22}$$

Thus, we have identified the condition that must be satisfied for the *absence* of anomalies in a given theory. Quantum anomalies will not be present if the integration measure of the fields in the functional integral is invariant under the symmetry transformations. Conversely, quantum anomalies will be present if the Jacobian of the transformation of the measure is not unity. Below we will consider the same example we studied via loop diagrams in the previous lecture: axial transformations and the anomalies in axial currents.

22.2 Chiral Anomaly in the Functional Integral

We consider a theory of massless fermions with a generating functional given by

$$Z = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{i\int d^4x\, \bar{\psi} i \not{D} \psi} \ , \tag{22.23}$$

where the covariant derivative is that of a vector gauge theory (e.g. QED, QCD). We focus on infinitesimal chiral transformations defined by

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{i\alpha(x)\gamma_5}\, \psi(x) \simeq \big(1 + \alpha(x)\gamma_5\big)\, \psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) &= \bar{\psi}(x)\, e^{i\alpha(x)\gamma_5} \simeq \bar{\psi}(x)\big(1 + \alpha(x)\gamma_5\big) \ , \end{aligned} \tag{22.24}$$

with $\alpha(x)$ an infinitesimal function. Then, the variation in the action is readily obtained as

$$\begin{aligned} \int d^4x\, \bar{\psi}'\, i\not{D}\psi' &= \int d^4x \left\{ \bar{\psi} i \not{D} \psi - \partial_\mu \alpha(x)\, \bar{\psi}\gamma^\mu\gamma_5\psi \right\} \\ &= \int d^4x \left\{ \bar{\psi} i \not{D} \psi + \alpha(x)\, \partial_\mu\big(\bar{\psi}\gamma^\mu\gamma_5\psi\big) \right\} \ , \end{aligned} \tag{22.25}$$

which clearly shows the classical conservation of the current in the event of the invariance of the lagrangian. That is,

$$\frac{\delta\mathcal{L}}{\partial\alpha} = 0 \qquad \Rightarrow \qquad \partial_\mu J^\mu_A(x) = 0 \;, \tag{22.26}$$

with the usual definition of the axial current: $J^\mu_A(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$. But this was the easy part.

Now we need to compute the Jacobian of the chiral transformations defined in (22.24). We start by expanding the fermionic fields in a basis of eigenstates of the operator $i\not{D}$. As we will see below, this will allow us to perform the functional integration. We write the fermionic fields as

$$\psi(x) = \sum_m a_m \phi_m(x) \;, \tag{22.27}$$

where the coefficients a_m are Grassmann variables, and the scalar eigenfunctions satisfy

$$i\not{D}\,\phi_m(x) = \lambda_m\,\phi_m(x) \;, \tag{22.28}$$

with the λ_m the eigenvalues. Analogously, we can write[1]

$$\bar{\psi}(x) = \sum_m \hat{a}_m \hat{\phi}_m(x) \;, \tag{22.29}$$

and the eigenfunctions satisfying

$$\hat{\phi}_m(x) i\not{D} = \lambda_m\,\hat{\phi}_m(x) \;. \tag{22.30}$$

The functional integration will then be carried out by integrating the Grassmann coefficients a_m and $\hat{a}_m$, i.e.

$$\mathcal{D}\psi\,\mathcal{D}\bar{\psi} = \prod_m da_m\,d\hat{a}_m \;. \tag{22.31}$$

We first consider the chiral transformation for $\psi(x)$ as shown in the first line of (22.24). In terms of $\psi'(x)$ expansion defined by

$$\psi'(x) = \sum_m a'_m\,\phi_m(x) \;, \tag{22.32}$$

[1]This may appear confusing. The $\hat{\phi}_m(x)$ are the same as the $\phi_m(x)$. They belong to the basis of eigenfunctions of $i\not{D}$. The hat notation is to differentiate the variables and eigenfunctions of the expansions of $\psi(x)$ and $\bar{\psi}(x)$. This is reflected on the fact that the eigenvalues λ_m are the same for both.

we have

$$\sum_m a'_m \phi_m(x) = \big(1 + i\alpha(x)\gamma_5\big) \sum_n a_n \phi_n(x) \ . \tag{22.33}$$

But using the orthonormality condition of the eigenfunctions

$$\int d^4x\, \phi^\dagger_m(x)\, \phi_n(x) = \delta_{mn} \ , \tag{22.34}$$

we can isolate an expression for $a'_m(x)$ given by

$$a'_m(x) = \sum_n \int d^4x \phi^\dagger_m(x) \Big(1 + i\alpha(x)\gamma_5\Big) \phi_n(x) a_n \ , \tag{22.35}$$

which we can write in compact form as

$$a'_m = \sum_n \Big(\delta_{mn} + C_{mn}\Big) a_n \ , \tag{22.36}$$

where we defined

$$C_{mn} \equiv \int d^4x\, \phi^\dagger_m(x), i\alpha(x)\gamma_5\, \psi_n(x) \ . \tag{22.37}$$

Thus the Jacobian for the transformation from $\psi(x)$ to $\psi'(x)$ (or from $a_m \to a'_m$) is

$$\Delta \equiv \det\Big(\mathbb{1} + \mathbb{C}\Big) \ . \tag{22.38}$$

Similarly, the *same* Jacobian will emerge when making the chiral transformation on $\bar{\psi}(x) \rightarrow \bar{\psi}'(x)$. The way Δ appears in the transformation of the functional integral measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ is a bit counterintuitive. In order to clarify this we consider the simple case of integration over two Grassmann variables, θ_1 and θ_2. The most general function we can write is

$$f(\theta_1, \theta_2) = f_0 + f_1\, \theta_1 + f_2\, \theta_2 + f_{12}\, \theta_1\, \theta_2 \ , \tag{22.39}$$

where the f's are arbitrary coefficients. If now we consider a linear transformation of the Grassmann variables

$$\begin{aligned} \theta_1' &= D_{11}\,\theta_1 + D_{12}\,\theta_2 \\ \theta_2' &= D_{21}\,\theta_1 + D_{22}\,\theta_2 \end{aligned} \tag{22.40}$$

or in matrix form

$$\begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix} = \mathbb{D} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} , \tag{22.41}$$

We now define the change in the measure of the Grassmann integrals by

$$\int d\theta_1'\, d\theta_2'\, f(\theta') \equiv \int \mathbb{K}\, d\theta_1\, d\theta_2\, f(\mathbb{D}\,\theta) \ . \tag{22.42}$$

But using the properties of Grassmann integration

$$\int d\theta_i = 0, \qquad\qquad \int d\theta_i\,\theta_i = 1, \tag{22.43}$$

we see that the only contributing term is the f_{12} one. Se we can rewrite (22.42) as

$$\int d\theta_1'\, d\theta_2'\, f_{12}\,\theta_1'\,\theta_2' = \mathbb{K} \int d\theta_1\, d\theta_2\, f_{12}\, \big(D_{11}\,\theta_1 + D_{12}\,\theta_2\big)\big(D_{21}\,\theta_1 + D_{22}\,\theta_2\big) \ . \tag{22.44}$$

Then we have

$$f_{12} \int d\theta_1'\, d\theta_2'\,\theta_1'\,\theta_2' = \mathbb{K}\, f_{12}\, \big(D_{11}D_{22} - D_{12}D_{21}\big) \int d\theta_1\, d\theta_2\,\theta_1\,\theta_2 \ , \tag{22.45}$$

where the minus sign appearing in the right hand side is the result of the anticommutation rule $\{\theta_1, \theta_2\} = 0$. Then, we obtain that

$$1 = \mathbb{K}\, \big(D_{11}D_{22} - D_{12}D_{21}\big) \ , \tag{22.46}$$

or

$$\boxed{\mathbb{K} = \Big(\det\mathbb{D}\Big)^{-1}} \ , \tag{22.47}$$

Going back to our funcional integration, we can generalize (22.47) for integration over $\theta_1, \ldots \theta_m, \ldots$, corresponding to the tranformations $a_m \to a'_m$ associated with the chiral transformations $\psi \to \psi'$. Similarly for the $\hat{a}_m \to \hat{a}'_m$ transformations associated with $\bar{\psi} \to \bar{\psi}'$. Using the result in (22.47) we see that the change in the masure of the functional integral will be

$$\mathcal{D}\psi'\,\mathcal{D}\bar{\psi}' = \Delta^{-1}\,\mathcal{D}\psi\,\Delta^{-1}\,\mathcal{D}\bar{\psi} = \Delta^{-2}\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\ , \tag{22.48}$$

where Δ was defined in (22.38).

All is left now is to compute Δ. We start by noticing that

$$\Delta = \det\big(1+\mathbb{C}\big) = e^{\mathrm{Tr}\ln(1+\mathbb{C})}\ , \tag{22.49}$$

where the trace refers to all indices, both the ones in the eigenbasis expansion as well as Dirac indices. Given that $\mathbb{C}$ is infinitesimal (is proportional to $\alpha(x)$, the infinitesimal gauge paramenter), we can use

$$\ln\big(1+\mathbb{C}\big) \simeq \mathbb{C} + \ldots\ , \tag{22.50}$$

where the dots indicate higher orders in powers of $\alpha(x)$. We then arriva at

$$\Delta = e^{\mathrm{Tr}\sum_{\rm n}\mathrm{C}_{\rm nn}+\ldots}\ , \tag{22.51}$$

or

$$\ln\Delta = i\,\mathrm{Tr}\int \mathrm{d^4x}\,\alpha(\mathrm{x})\sum_{\rm n}\phi^\dagger_{\rm n}(\mathrm{x})\,\gamma_5\,\phi_{\rm n}(\mathrm{x})\ , \tag{22.52}$$

where now the trace is only over Dirac indices since we already took it over the eigenbasis indices. At first sight it looks like (22.52) vanishes. This is because $\mathrm{Tr}\gamma_5 = 0$. If this were true then we would have $\Delta = 1$, and there would be no nontrivial transformation of the measure in the functional integral, and therefore no axial anomaly. However, this is not case since the sum over the eigenfunctions in (22.52) diverges. In fact, we can see that the product of the two eigen functions evaluated at the same spacetime point x can be thought of as a two point function. So we know that as spacetime separation goes to zero or momentum goes to infinity there will be divergences. For instance we can write the sum in (22.52) as

$$\lim_{x\to y}\sum_n \phi^\dagger(x)\,\gamma_5\,\phi_n(y) \qquad \text{or} \qquad \sum_n \phi^\dagger(x)\,\gamma_5\,\phi_n(y)\,\delta(x-y)\ . \tag{22.53}$$

Either way we can see that this sum is not well defined and needs to be regularized.

To regularize the sum over the products of the eigenfunctions in (22.52), we are going to introduce a regularization function defined by

$$\sum_n \phi_n^\dagger(x)\,\gamma_5\,\phi_n(x) = \lim_{M\to\infty} \sum_n \phi_n^\dagger(x)\,\gamma_5\,R\Big[\big(i\not{D}/M\big)^2\Big]\,\phi_n(x)\ , \tag{22.54}$$

where $R[s]$ must satisfy

$$R[0] = 1, \qquad R[\infty] = 0\ . \tag{22.55}$$

We have chosen the argument of R in (22.54) to be a power of $i\not{D}$ since the $\psi_n(x)$ are its eigenfunctions. For instance, we can choose

$$R[s] = e^{-s}\ , \tag{22.56}$$

which when applied to the eigenfunctions $\phi_n(x)$ will give

$$e^{(i\not{D})^2/M^2}\,\phi_n(x) = e^{-\not{D}^2/M^2}\,\phi_n(x) = e^{\lambda_n^2/M^2}\,\phi_n(x)\ . \tag{22.57}$$

The last factor above is of the required form given that at high momentum we have $i\not{D} \simeq i\not{\partial} \sim k$ so that

$$\lambda_n^2 \simeq k^2 = -k_E^2 < 0\ , \tag{22.58}$$

so that R has the form (22.56) when performing the integration on the euclidean momentum k_E. Going back to (22.54), we write its trace over the Dirac indices as

$$\begin{aligned}
\mathrm{Tr}\sum_{\mathrm{n}} \phi_{\mathrm{n}}^\dagger(\mathrm{x})\,\gamma_5\,\phi_{\mathrm{n}}(\mathrm{x}) &= \lim_{M\to\infty}\,\mathrm{Tr}\sum_{\mathrm{n}} \phi_{\mathrm{n}}^\dagger(\mathrm{x})\,\gamma_5\,\mathrm{e}^{(\mathrm{i}\not{D})^2/\mathrm{M}^2}\,\phi_{\mathrm{n}}(\mathrm{x}) \\
&= \lim_{M\to\infty}\,\langle x|\mathrm{Tr}\Big[\gamma_5\,\mathrm{e}^{(\mathrm{i}\not{D})^2/\mathrm{M}^2}\Big]|\mathrm{x}\rangle\ ,
\end{aligned} \tag{22.59}$$

where in the second line we replaced the summ over the eigenfunctions evaluated at x by the corresponding eigenstates of $i\not{D}$ at the same position, $|x\rangle$ for notational simplicity. We next notice that the covariant derivative operator in the exponent above can be written as

$$
\begin{aligned}
\left(i\not{D}\right)^2 &= -\gamma^\mu\gamma^\nu\, D_\mu\, D_\nu \\
&= -\Big\{\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}\big[\gamma^\mu,\gamma^\nu\big]\Big\}\, D_\mu\, D_\nu \\
&= -D^2 - (-i)\frac{i}{2}\big[\gamma^\mu,\gamma^\nu\big]\,\frac{1}{2}\big[D_\mu, D_\nu\big] \\
&= -D^2 + \frac{g}{2}\,\sigma^{\mu\nu}\, F_{\mu\nu}\ ,
\end{aligned}
\tag{22.60}
$$

where in the las line we used

$$
\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]\ , \tag{22.61}
$$

and that the commutator of the covariant derivatives can be written as

$$
[D_\mu, D_\nu] = -i\, g\, F_{\mu\nu}\ . \tag{22.62}
$$

We can then rewrite (22.59) as

$$
\mathrm{Tr}\sum_{\mathrm{n}} \phi^\dagger_{\mathrm{n}}(\mathrm{x})\,\gamma_5\,\phi_{\mathrm{n}}(\mathrm{x}) = \lim_{\mathrm{M}\to\infty}\langle \mathrm{x}|\mathrm{Tr}\Big[\gamma_5\,\mathrm{e}^{(-\mathrm{D}^2+(\mathrm{g}/2)\sigma^{\mu\nu}\mathrm{F}_{\mu\nu})/\mathrm{M}^2}\Big]|\mathrm{x}\rangle\ . \tag{22.63}
$$

The presence of γ_5 in (22.63) above clearly requires that for the trace to be non vanishing we need at least *four* gamma matrices in the expansion of the exponential. The first term in the expansion of the exponential will only give two gamma matrices with one γ_5 so the trace vanishes. The second term already has four gamma matrices so it will give a non zero contribution as we will show below. Also shown below is the fact that all the higher order terms in the exponential expansion beyond second order will vanish in the limit $M\to\infty$. The relevant contribution is then

$$
\begin{aligned}
\mathrm{Tr}\sum_{\mathrm{n}} \phi^\dagger_{\mathrm{n}}(\mathrm{x})\,\gamma_5\,\phi_{\mathrm{n}}(\mathrm{x}) &= \lim_{M\to\infty}\langle x|\mathrm{Tr}\Big[\gamma_5\,\frac{1}{2!}\left(\frac{\mathrm{g}}{2}\,\frac{\sigma^{\mu\nu}\mathrm{F}_{\mu\nu}}{\mathrm{M}^2}\right)^2\Big]\,\mathrm{e}^{-\mathrm{D}^2/\mathrm{M}^2}|\mathrm{x}\rangle \\
&= \lim_{M\to\infty}\,\mathrm{Tr}\Big[\gamma_5\,\frac{1}{2!}\,\frac{\mathrm{g}^2}{4}\,\frac{1}{\mathrm{M}^4}\,\left(\sigma^{\mu\nu}\mathrm{F}_{\mu\nu}\right)^2\Big]\,\langle \mathrm{x}|\mathrm{e}^{-\partial^2/\mathrm{M}^2}|\mathrm{x}\rangle\ .
\end{aligned}
\tag{22.64}
$$

In the first line of (22.64) we expanded only the term in the exponential containing $\sigma^{\mu\nu}$ (the only one relevant for the trace) and left the squared of the covariant derivative in the exponential. In the second line we replaced the squared of the covariant derivative by the simple squared derivative (Dalembertian). This is justified since, as we will se below, the dominant contributions come from high momenta, of order M, for which $i\,\not{D} \simeq i\,\not{\partial}$ is a good approximation. In order to proceed we need to compute the last factor in the second line of (22.64). for this, we write

$$\begin{aligned}
\langle x|e^{-\partial^2/M^2}|x\rangle &= \lim_{x\to y}\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, e^{k^2/M^2} \\
&= i\int \frac{d^4k_E}{(2\pi)^4}\, e^{-k_E^2/M^2} = \frac{i}{16\pi^4}\int k_E^3 dk_E d\Omega_{k_E}\, e^{-k_E^2/M^2} \\
&= \frac{i}{16\pi^2}\int_0^\infty k_E^2\, dk_E^2\, e^{-k_E^2/M^2} \\
&= \frac{i}{16\pi^2}\, M^4\ . \qquad (22.65)
\end{aligned}$$

Here we see that terms beyond the quadratic term in the expansion of $e^{(g^2/2)\sigma^{\mu\nu}F_{\mu\nu}/M^2}$ are going to be suppressed by powers of M^6 or larger, so they will not contribute in the $M\to\infty$ limit. Thus, the second order term in (22.64) is the only non zero contribution. We then obtain

$$\begin{aligned}
\mathrm{Tr}\sum_{\mathrm{n}}\phi_{\mathrm{n}}^\dagger(\mathrm{x})\,\gamma_5\,\phi_{\mathrm{n}}(\mathrm{x}) &= \lim_{M\to\infty}\frac{i}{16\pi^2}\,M^4\,\frac{g^2}{8}\frac{1}{M^4}\,\mathrm{Tr}\Big[\gamma_5\sigma^{\mu\nu}\sigma^{\alpha\beta}\Big]\,\mathrm{F}_{\mu\nu}\,\mathrm{F}_{\alpha\beta} \\
&= -\frac{i}{16\pi^2}\frac{g^2}{8}\,\mathrm{Tr}\Big[\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\Big]\,\mathrm{F}_{\mu\nu}\,\mathrm{F}_{\alpha\beta} \\
&= -\frac{g^2}{32\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\,F_{\mu\nu}\,F_{\alpha\beta}\ , \qquad (22.66)
\end{aligned}$$

where in the last line we used

$$\mathrm{Tr}\Big[\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\Big] = -4\mathrm{i}\,\epsilon^{\mu\nu\alpha\beta}\ .$$

Finally, substituting the result of (22.66) in (22.52) we obtain

$$\Delta = e^{-i\int d^4x\,\alpha(x)\,\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}\,F_{\mu\nu}F_{\alpha\beta}}\ . \qquad (22.67)$$

This means that under the chiral transformations of (22.24), the generating functional (22.23) turns into

$$\begin{aligned} Z &= \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \Delta^{-2}\, e^{i\int d^4x \left\{\bar{\psi} i\not{D}\psi + \alpha(x)\,\partial_\mu J_A^\mu\right\}} \\ &= \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{i\int d^4x \left\{\bar{\psi} i\not{D}\psi + \alpha(x)\,\left[\partial_\mu J_A^\mu + \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}\right]\right\}}\;, \end{aligned} \tag{22.68}$$

where in the second line we used (22.67) for Δ. It is now clear that imposing that the lagrangian be invariant with respect to the paramenter α results in the non conservation of the axial current. That is

$$\frac{\delta\mathcal{L}}{\delta\alpha} = 0\;, \tag{22.69}$$

implies that

$$\boxed{\partial_\mu J_A^\mu = -\frac{g^2}{16\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\, F_{\mu\nu}\, F_{\alpha\beta}}\;, \tag{22.70}$$

just as we computed diagrammatically in the previous lecture. Just as in the previous lecture, we saw that the effect of the anomaly comes from the UV behavior of the theory in the presence of the axial coupling, i.e. in the presence of the γ_5 in a vertex. But what we have seen here, is that the entire effect comes from the non invariance of the functional integral measure under the chiral transformations of the fermion fields. In coming lectures we will recast the meaning of the anomaly in light of some fundamental properties of the quantum fields theories that posses them.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Sections 19.1 and 19.2.
- *The Quantum Theory of Fields, Vol. II*, by S. Weinberg, Section 22.3.
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