Lecture 20

Spontaneous Breaking of Non Abelian Gauge Symmetries

We will now consider the case when the spontaneously broken non abelian symmetry is gauged. As we saw for the case of abelian gauge symmetry, the spontaneous breaking of the symmetry will be realized in the sense of the Anderson-Higgs mechanism, i.e. the NGBs would not be in the physical spectrum, and the gauge bosons associated with the *broken* generators will acquire mass. We will derive these results carefully in what follows.

20.1 The Generalized Anderson-Higgs Mechanism

We consider a lagrangian invariant under the gauge transformations

$$\phi(x) \to e^{i\alpha^a(x)\,t^a}\,\phi(x)\ , \tag{20.1}$$

where t^a are the generators of the group G, and the gauge fields transform as they should. If we consider infinitesimal gauge transfomations and write out the field $\phi(x)$ in its groups components, we have

$$\phi_i(x) \to \left(\delta_{ij} + i\alpha^a(x)\,(t^a)_{ij}\right)\phi_j(x) \tag{20.2}$$

In general, we consider representations where the $\phi_i(x)$ fields in (20.2) are complex. But for the purpose of our next derivation, it would be advantegeous to consider their real components. So if the original representation had dimension n, we now have $2n$ componentes in the real fields $\phi_i(x)$. If this is the case, then the generators in (20.2) *must be imaginary*, since the $\alpha^a(x)$ are real paramenter functions. This means we can write them as

$$t^a_{ij} = i\,T^a_{ij}\ , \tag{20.3}$$

where the T^a_{ij} are real. Also, since the t^a are hermitian, we have

$$\left(t^a_{ij}\right)^\dagger = t^a_{ij}\ , \tag{20.4}$$

we see that

$$T^a_{ij} = -T^a_{ji}\ , \tag{20.5}$$

so the T^a are antisymmetric. In general, the lagrangian of the gauge invariant theory for a scalar field in terms of the real scalar degrees of freedom would be[1]

$$\mathcal{L} = \frac{1}{2}\Big(D_\mu\phi_i\Big)\Big(D^\mu\phi_i\Big) - V\left(\phi_i\right)\ , \tag{20.6}$$

where the repeated i indices are summed. We can write the covariant derivatives above as

$$D_\mu\phi(x) = \left(\partial_\mu - igA^a_\mu(x)t^a\right)\phi(x) = \left(\partial_\mu + gA^a_\mu(x)T^a\right)\phi(x)\ , \tag{20.7}$$

where we omitted the group indices for the fields and the generators. We are interested in the situation when the potential in (20.6) induces spontaneous symmetry breaking. To see how this affects the gauge boson spectrum we must examine in detail the scalar kinetic term:

$$\begin{aligned}\frac{1}{2}\left(D_\mu\phi_i\right)\left(D^\mu\phi_i\right) &= \frac{1}{2}\partial_\mu\phi_i\,\partial^\mu\phi_i + \frac{1}{2}g^2A^a_\mu A^{b\mu}\left(T^a\phi\right)_i\left(T^b\phi\right)_i \\ &+ gA^a_\mu\left(T^a\phi\right)_i\partial^\mu\phi_i\ ,\end{aligned} \tag{20.8}$$

where we used the notation

$$\left(T^a\phi\right)_i = T^a_{ij}\phi_j\ , \tag{20.9}$$

[1]Here we concentrate on the scalar sector of $\mathcal{L}$ since it is here that SSB of the gauge symmetry arises. We can imagine adding fermion terms to $\mathcal{L}$ coupling them both to the gauge bosons through the covariant derivative, as well as Yukawa couplings between the fermions and the scalars. Of course, all these terms must also respect gauge invariance.

and as usual repeated group indices i, j are summed. If the potential $V(\phi_i)$ has a non trivial minimum then, the vacuum expectation value (VEV) of the fields ϕ_i at the minimum is

$$\langle 0|\phi_i|0\rangle = \langle\phi_i\rangle \equiv \left(\phi_0\right)_i , \tag{20.10}$$

which says that we are signling out directions in field space which may have non trivial VEVs. Then the terms in $\mathcal{L}$ quadratic in the gauge boson fields, i.e. the gauge boson mass terms, can be readily read off (20.8):

$$\mathcal{L}_m = \frac{1}{2} M^2_{ab}\, A^a_\mu A^{b\mu} , \tag{20.11}$$

where the gauge boson mass matrix is defined by

$$M^2_{ab} \equiv g^2 \left(T^a\phi_0\right)_i \left(T^b\phi_0\right)_i . \tag{20.12}$$

Since the T^a's are real, the non zero eigenvalues of M^2_{abv} are definite positive. We can clearly see now that if

$$T^a\phi_0 = 0, \tag{20.13}$$

then the associated gauge boson A^a_μ remains massless. That is, the*unbroken* generators, which as we saw in the previous lecture, *do not have NGBs associated with them*, do not result in a mass term for the corresponding gauge boson. On the other hand, if

$$T^a\phi_0 \neq 0 , \tag{20.14}$$

then we see that this results in a gauge boson mass term. The generators satisfying (20.14) are of course the *broken generators* which result in massless NGBs. However, just as we saw for the abelian case, these NGBs can be removed from the spectrum by a gauge transformation. To see how this works we consider the last term in (20.8), the mixing term. This is

$$\mathcal{L}_{\text{mix.}} = g A^a_\mu \left(T^a\phi_0\right)_i \partial^\mu\phi_i . \tag{20.15}$$

Thus, we see that if the associated generator is broken, i.e. (20.14) is satisfied, then there is mixing of the corresponding gauge boson with the massless ϕ_i fields, the NGBs. It is clear that, just as in the abelian case, we can eliminate this term by a suitable gauge transformation on A^a_μ. This would still leave the mass term unchanged, but would

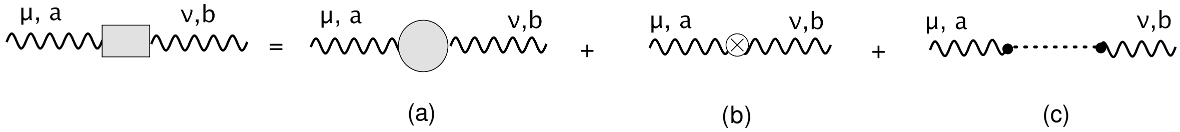
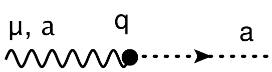


Figure 20.1: Contributions to the gauge boson two point function in the presence of spontaneous gauge symmetry breaking. Diagram (a) includes the tree level as well as loop diagrams, all of which are transverse contributions. Diagram (b) is the contribution from the gauge boson mass term. Diagram (c) depicts the contribution from the massless NGBs.

completely eliminate the NGBs mixing in (20.15) from the spectrum. But even if we leave the NGBs in the spectrum, and we still have to deal with the mixing term (20.15), we can still see that the gauge boson two point function remains transverse, a sign that gauge invariance is still respected despite the appearance of a gauge boson mass. This is depicted in Figure 20.1.

In order to obtain diagram (c) we need to derive the Feynman rule resulting from the mixing term $\mathcal{L}_{\rm mix}$ (20.15). In momentum space this becomes



$$= g\big(T^a\phi_0\big)_i\, q^\mu \ , \tag{20.16}$$

where the NGB momentum is flowing out of the vertex (its sign changes if it is flowing into the vertex). The contributions to diagram (a) are transverse as they come from either the leading order propagator or the loop corrections to it, both already shown to be transverse. Then the two point function for the gauge boson in the presence of spontaneous symmetry breaking is

$$\begin{aligned}\Pi_{\mu\nu} &= \Pi^{(a)}_{\mu\nu} + iM^2_{ab}\, g_{\mu\nu} + g\big(T^a\phi_0\big)_i\, q_\mu\, \frac{i\delta_{ab}}{q^2}\, g\big(T^b\phi_0\big)_i\, (-q_\nu) \\ &= \Pi^{(a)}_{\mu\nu} + iM^2_{ab}\Big(g_{\mu\nu} - \frac{q_\mu\, a_\nu}{q^2}\Big) \ , \end{aligned}\tag{20.17}$$

where to obtain the second line we used (20.12). Then, just as we saw for the abelian case, we see that the gauge boson two point function is transverse even in the presence of gauge boson masses.

Example 1: $SU(2)$

In this first example we gauge the $SU(2)$ of the first example in the previous lecture. The lagrangian

$$\mathcal{L} = \left(D_\mu\phi\right)^\dagger D^\mu\phi - V(\phi^\dagger\phi) - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \ , \tag{20.18}$$

with the covariant derivative on the scalar field is[2]

$$D_\mu\phi(x) = \left(\partial_\mu - igA^a_\mu(x)t^a\right)\phi(x) \ , \tag{20.19}$$

where the $SU(2)$ generators are given in terms of the Pauli matrices as

$$t^a = \frac{\sigma^a}{2} \ , \tag{20.20}$$

with $a = 1, 2, 3$. Since they transform according to

$$\phi(x)_j \to e^{i\alpha^a(x)t^a_{jk}}\,\phi_k(x) \ , \tag{20.21}$$

with $j, k = 1, 2$, then that are *doublets* of $SU(2)$. Since each of the $\phi_j(x)$ are complex scalar fields, we have *four* real scalar degrees of freedom. We will consider the vacuum

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \ , \tag{20.22}$$

such that, as required by imposing a non trivial minimum, we have

$$\langle\phi^\dagger\phi\rangle = \frac{v^2}{2} \ , \tag{20.23}$$

where the factor of 2 above is chosen for convenience. We are particularly interested in the gauge boson mass terms. These can be readily obtained by substituting the vacuum value of the field in the kinetic term. This is

$$\begin{aligned} \mathcal{L}_{\rm m} &= \left(D_\mu\langle\phi\rangle\right)^\dagger D^\mu\langle\phi\rangle \ , \\ &= \frac{g^2}{2}\,A^a_\mu A^{b\mu}\begin{pmatrix}0 & v\end{pmatrix} t^a t^b \begin{pmatrix} 0 \\ v \end{pmatrix} \ , \end{aligned} \tag{20.24}$$

[2] We have gone back to complex scalar fields for the remaining of the lecture.

where we used (20.22) in the second line. But for the case of $SU(2)$ we can use the fact that

$$\{\sigma^a, \sigma^b\} = 2\delta^{ab} \ , \tag{20.25}$$

which translates into

$$\{t^a, t^b\} = \frac{1}{2}\delta^{ab} \ , . \tag{20.26}$$

Then, if we write

$$\begin{aligned} A^a_\mu \, A^{b\mu} \, t^a \, t^b &= \frac{1}{2} \, A^a_\mu \, A^{b\mu} \, t^a \, t^b + \frac{1}{2} \, A^b_\mu \, A^{a\mu} \, t^b \, t^a \\ &= \frac{1}{2} \, A^a_\mu \, A^{b\mu} \, \{t^a, t^b\} = \frac{1}{4} \, A^a_\mu \, A^{a\mu} \ , \end{aligned} \tag{20.27}$$

where in the last euality we used (20.26). Then we obtain

$$\mathcal{L}_{\rm m} = \frac{1}{8} g^2 \, v^2 \, A^a_\mu \, A^{a\mu} \ , \tag{20.28}$$

which results in a gauge boson mass of

$$M_A = \frac{g\, v}{2} \ . \tag{20.29}$$

Notice that *all three* gauge bosons obtain this same mass. It is interesting to compare this result with what we obtained in the previous lecture for the spontaneous breaking of a *global* $SU(2)$ symmetry using the same vacuum as in (20.22). In that case, we saw that all generators were broken, i.e. there are three massless NGBs in the spectrum and the $SU(2)$ is completely (spontaneously) broken in the sense that none of its generators leaves the vacuum invariant. In the case here, where the $SU(2)$ symmetry is gauged, we see that all three gauge bosons get masses. This is in fact the same phenomenon: none of the gauge symmetry leaves the $SU(2)$ vacuum (20.22) invariant. However, the end result is three massive gauge bosons, not three massless NGBs. We argued in our general considerations above that, just as for the abelian case before, the NGBs can be removed by a gauge transformation. Let us see how this can be implemented. We consider the following parametrization of the $SU(2)$ doublet scalar field:

$$\phi(x) = e^{i\pi^a(x)t^a/v} \begin{pmatrix} 0 \\ \\ \frac{v+\sigma(x)}{\sqrt{2}} \end{pmatrix} , \tag{20.30}$$

where $\sigma(x)$ and $\pi^a(x)$ with $a = 1, 2, 3$ are real sclalar fields satisfying

$$\langle\sigma(x)\rangle = 0 = \langle\pi^a(x)\rangle , \tag{20.31}$$

so that this choice of parametrization is consistent with the vacuum (20.22). Clearly, the potential will not depend on the $\pi^a(x)$ fields

$$V(\phi^\dagger\phi) = -\frac{m^2}{2}\,\phi^\dagger\phi + \frac{\lambda}{2}\left(\phi^\dagger\phi\right)^2 , \tag{20.32}$$

The minimization results in[3]

$$\langle\phi^\dagger\phi\rangle = \frac{m^2}{2\lambda} , \tag{20.33}$$

which results in

$$v^2 = \frac{m^2}{\lambda} . \tag{20.34}$$

Replacing this in the potential (20.32) we obtain

$$m_\sigma = \sqrt{2\,\lambda}\,v . \tag{20.35}$$

And of course, the implicit result of having

$$m_{\pi^1} = m_{\pi^2} = m_{\pi^3} = 0 . \tag{20.36}$$

But how do we get rid of the massless NGBs ? If we define the following gauge transformation

$$U(x) \equiv e^{-i\pi^a(x)t^a/v} \tag{20.37}$$

[3]Notice the different factor in the denominator of the second term. This is due to the factor of $\sqrt{2}$ in the definition of the vacuum.

under which the fields transform as

$$\begin{aligned} \phi(x) \quad &\to \quad \phi'(x) = U(x)\,\phi(x) = \begin{pmatrix} 0 \\ \frac{v+\sigma(x)}{\sqrt{2}} \end{pmatrix} , \\ A_\mu \quad &\to \quad A'_\mu = U(x)\,A_\mu\,U^{-1}(x) - \frac{i}{g}\left(\partial_\mu U(x)\right)U^{-1}(x) , \end{aligned} \tag{20.38}$$

where we used the notation $A_\mu = A^a_\mu t^a$. It is clear from the first transformation above, that $\phi'(x)$ does not depend on the $\pi^a(x)$ fields. Thus, the gauge transformation (20.38) has removed them from the spectrum completely. However, the number of degrees of reedom is the same in boths gauges. We had three transverse gauge bosons (i.e. 6 degrees of freedom) and four real scalar fields. In this new gauge we have three massive gauge bosons (i.e. 9 degrees of freedom) plus one real scalar, $\sigma(x)$. The total number of degrees of freedom is always the samee. The gauge were the NGBs diissapear of the spectrum is called the *unitary gauge*.

<u>Example 2: $SU(3)$</u>

We now consider the caso with $G = SU(3)$, and the vacuum is chosen to be

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} . \tag{20.39}$$

The $SU(3)$ generators t^a, with $a = 1, \cdots, 8$, where explicitely written in the previous lecture. They generaly satisy the group algebra and normalization

$$\begin{aligned} \left[t^a, t^b\right] &= i f^{abc} t^c \\ \mathrm{Tr}\left[\mathrm{t^a\,t^b}\right] &= \frac{1}{2}\delta^{ab} . \end{aligned}$$

In addition to these, we can write the anticommutator as

$$\left\{t^a, t^b\right\} = \frac{1}{3}\delta^{ab} + d^{abc}\,t^c , \tag{20.40}$$

where the d^{abc} are totally *symmetric* constants[4] . We want to compute the gauge boson masses. Just as we did for the $SU(2)$ case in (20.24), the gauge boson mass terms come from the kinetic terms with ϕ replaced by the vacuum (20.39).

$$\begin{aligned} \mathcal{L}_{\mathrm{m}} &= \left(D_\mu\langle\phi\rangle\right)^\dagger D^\mu\langle\phi\rangle = \frac{g^2}{2}\,A^a_\mu\,A^{b\mu}\begin{pmatrix}0 & 0 & v\end{pmatrix} t^a\,t^b \begin{pmatrix}0\\0\\v\end{pmatrix} \\ &= \frac{g^2}{2}\,A^a_\mu\,A^{b\mu}\begin{pmatrix}0 & 0 & v\end{pmatrix}\frac{1}{2}\left\{t^a,t^b\right\}\begin{pmatrix}0\\0\\v\end{pmatrix} \end{aligned} \tag{20.41}$$

wher in the second line we used the same trick as in $SU(2)$ to write the anticommutator. Then, using (20.40) we obtain

$$\mathcal{L}_{\mathrm{m}} = \frac{g^2}{4}\,A^a_\mu\,A^{b\mu}\begin{pmatrix}0 & 0 & v\end{pmatrix}\left\{\frac{1}{3}\,\delta^{ab} + d^{abc}\,t^c\right\}\begin{pmatrix}0\\0\\v\end{pmatrix}\,. \tag{20.42}$$

In order to proceed further in the computation if the gauge boson masses we need to know the values of the d^{abc} constants for $SU(3)$. The non zero values are

$$\begin{aligned} d^{118} = d^{228} = d^{338} = -d^{888} &= \frac{1}{\sqrt{3}} \\ d^{448} = d^{558} = d^{668} = d^{778} &= -\frac{1}{2\sqrt{3}} \\ d^{146} = d^{157} = -d^{247} = d^{256} = d^{344} = d^{355} = -d^{366} = -d^{377} &= \frac{1}{2}\,. \end{aligned} \tag{20.43}$$

However, when it comes to the masses of the gauge bosons for $a,b = 1,2,3$ there is a simpler way to compute them. We know from the previous lecture that these geerators can be written as

$$t^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0\ 0 & 0 \end{pmatrix}\,, \tag{20.44}$$

with the σ^i the Pauli matrices for $i = 1,2,3$. We also know that they annihilate the vacuum defined in (20.39), that is

[4]In general, this expression is valid for $SU(N)$ groups, with the 3 replaced by N. For the particular case of $N = 2$ the d^{abc} vanish.

$$t^i \langle\phi\rangle = 0 \ , \tag{20.45}$$

for $i = 1, 2, 3$. Thus, just by looking at the first line in (20.41) we can see that the gauge bosons A^1_μ, A^2_μ and A^3_μ will be massless, that is[5]

$$M_{A^1} = M_{A^2} = M_{A^3} = 0 \ . \tag{20.46}$$

Once again, it is interesting to compare with the spontaneous breaking of the global $SU(3)$ symmetry studied in the previous lecture. There we saw that the first three generators of $SU(3)$ still left the vacuum invariant, and therefore there were no massless NBGs associated with them. Here, the fact that the $SU(2)$ subgroup of $SU(3)$ defined by the generators in (20.44) still preserve the vacuum (20.39) results in those gauge bosons remaining massless. It is in this sense that the $SU(2)$ remains *unbroken*.

Conversely, we expect that the *broken* generators, i.e. those that do not annihilate the vacuum (20.39), will be associated with massive gauge bosons, as they were associated with the existence of massless NGBs in the case of the global symmetry. To obtain their masses we first observe that the *off diagonal* mass terms in (20.42), i.e. those for $a \neq b$ are zero, even for $a, b = 4, 5, 6, 7, 8$. To see this, we write the corresponding mass terms as

$$\mathcal{L}_{\rm m}^{a\neq b} = \frac{g^2}{4}\, A^a_\mu\, A^{b\mu} \begin{pmatrix} 0 & 0 & v \end{pmatrix} d^{abc}\, t^c \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \ . \tag{20.47}$$

The terms contributing are those with d's in the last line of (20.43). For these, the generators t^c appearing in (20.47) correspond to $c = 4, 5, 6$ and 7. But these generators, when acting on the vacuum in (20.47) do not leave it invariant. They will result in a column vector that is orthogonal to the vacuum in (20.39), i.e. will give zero when it hits the row vector $(0 \quad 0 \quad v)$. Then, all these non diagonal mass terms vanish. Finally we consider the diagonal mass terms, i.e. $a = b = 4, 5, 6, 7, 8$. Their contributions come both from the δ^{ab} as well as from the d's in the second line in (20.43). Let us explicitely compute the case of $a = b = 4$. We have

$$\mathcal{L}_{\rm m}^{a=b=4} = \frac{g^2}{4}\, A^4_\mu\, A^{4\mu} \begin{pmatrix} 0 & 0 & v \end{pmatrix} \left\{\frac{1}{3} + d^{448}\, t^8\right\} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \ . \tag{20.48}$$

When substituting for the values of d^{448} and for t^8 (see previous lecture) we obtain

[5]It is clear that all the elements of the 3×3 mass matrix will vanish, both diagonal and non diagonal.

$$\mathcal{L}_{\mathrm{m}}^{a=b=4} = \frac{1}{8}\, g^2\, v^2\, A^4_\mu\, A^{4\mu} \ , \tag{20.49}$$

which results in

$$M_{A^4} = \frac{g\, v}{2} \ . \tag{20.50}$$

However, since all the d^{ab8} with $a = b \neq 8$ have the same value and only t^8 contributes, then it is clear that we will have

$$\boxed{M_{A^4} = M_{A^5} = M_{A^6} = M_{A^7} = \frac{g\, v}{2}} . \tag{20.51}$$

Finally, taking into account the different value of d^{888}, we obtain

$$\boxed{M_{A^8} = \frac{g\, v}{\sqrt{3}}} . \tag{20.52}$$

These completes the spectrum of gauge bosons: five massive, and 3 massless ones. Just as we expect from the fact that the spontaneous symmetry breaking induced by the vacuum (20.39) respects $SU(2)$. We say that the spontanous symmetry breaking pattern is

$$SU(3) \to SU(2) \ . \tag{20.53}$$

20.2 The Electroweak Standard Model

For an example of spontaneous breaking of a gauge symmetry we look no further than to the standard model (SM) of particle physics. What is broadly called the SM is a quantum field theory that describes all the interactions of all the known elementary particles, with the exception of a quantum description of gravity. These are the strong interactions as described by quantum chromodynamics (QCD), an unbroken $SU(3)$ gauge theory; and the electroweak interactions as described by the gauge symmetry $SU(2)_L \times U(1)_Y$. It is the latter that is spontaneously broken to give rise to three massive weak gauge bosons and a massless photon responsible for the electromagnetic interactions. Since the QCD gauge symmetry is not spontaneously broken by the Anderson-Higgs mechanism, we will ignore it in what follows, it is a "spectator" interaction that does not change when the electroweak symmetry is broken to electromagnetism.

20.2.1 Why $SU(2)_L \times U(1)_Y$?

The electroweak gauge group is a product of an $SU(2)$ gauge symmetry and a $U(1)$ one. How do we know this is the electroweak gauge symmetry group ? And, what is the meaning of the subscripts L and Y ? Although we will not attempt to go through all the fascinating history of experimental evidence that goes into building the electroweak SM, it is worthwhile to understand the main points that result in the building of this strange looking, yet amazingly successful description of fundamental physics.

Let us review the main evidences leading to the gauge structure of the electroweak theory.

- Weak Interactions (Charged): Weak decays, such as β decays $n \to p\, e^- \bar{\nu}_e$ or $\mu^- \to \nu_\mu\, \bar{\nu}_e\, e^-$ among many others, are mediated by *charged* currents. Let us look at the case of muon decay. It is very well described by a four fermion interaction, i.e. with a non renormalizable coupling G_F, the Fermi constant. In fact, all other weak interactions can be described in this way with the same Fermi constant (to a very good approximation, more later). The relevant Fermi lagrangian is

$$\mathcal{L}_{\text{Fermi}} = \frac{G_F}{\sqrt{2}} \left(\bar{\mu}_L \gamma_\mu \nu_L\right) \left(\bar{e}_L \gamma^\mu \nu_e\right) \,, \tag{20.54}$$

  where we already included the fact that the charged weak interactions only involve *left handed fermions*. That is, the phenomenolgically built Fermi lagrangian above tells us that the weak decay of a muon is described by the product of two *charged vector currents* coupling only left handed fermions. The fact that only left handed fermions participate in the charged weak interactions is an experimentally established fact, observed in *all charged weak interactions*. This is done by a variety of experimental techniques. Fors instance, in the case of muon decay, the angular distribution of the outgoing electron is very different if this is left or right handed. Precise measurements (performed over decades of increasingly accurate experiments) have concluded that the outgoing electron is left handed only. The different couplings involving left and right handed fermions require *parity violation*. Moreover, the charged weak interactions require *maximal parity violation*: only one handedness participate. If we assume that the non renormalizable four fermion interaction is the result of integrating out a gauge boson with a renormalizable interaction, this would point to the need of 2 charged gauge bosons.This is schematically shown in Figure 20.2. Assuming that $m_\mu \ll M_W$, we integrate out the massive vector gauge boson to obtain

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \,, \tag{20.55}$$

  where g is the renormalizable coupling of the gauge bosons to fermions in diagram

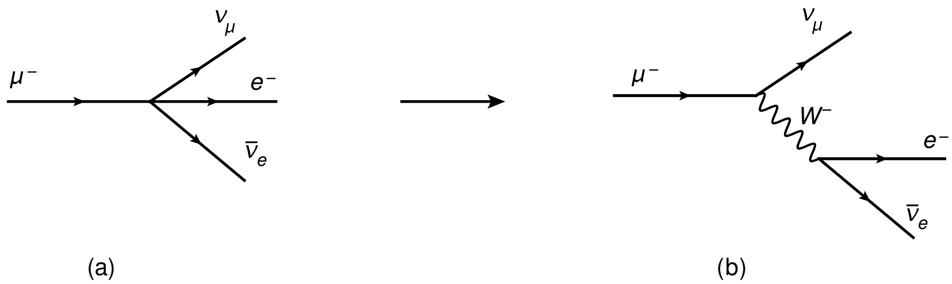


Figure 20.2: Diagram (a) is the Feyman diagrams associated with the four fermion Fermi lagrangian (20.54). Diagram (b) shows the corresponding exchange of a massive charged gauge boson, $W^{\pm}_{\mu}$.

(b). The charged vector gauge bosons, $W^{\pm}$ were discovered in the 1980s and studied with gerat detail ever since.

- Weak Neutral Currents: In addition to the charged currents described by (20.54), we have know since experimental evidence first appeared in the 1970s, that there are also *neutral weak currents*. These were first observed by neutrino scattering off nucleons. Normally, the charged currents would result in $\nu_e\, N \to e^-\, N'$, with N and N' protons and neutrons. This is just a crossed diagram of β decay. But the reaction $\nu\, N \to \nu\, N$ was also observed. Many other reactions involving neutral currents have been observed since then. They also violate parity. However, they do not do so maximally. This menas that the neutral current, or the vector gauge boson that we need to integrate out to obtain them at low energies, couple differently to lef and right handed fermions but, unlike the charged currents, the do couple to right handed fermions. The neutral vector gauge boson, Z^0, was also discovered in the 1980s and its properties studied with great precision.

- Electromagnetism: Of course, we know that the electromagnetic interactions are described by a quantum field theory, QED, mediated by a neutral *massless* vector gauge boson, the photon. One important feature to remember is that the photon coupling in QED is *parity invariant*. No parity violation is present in QED.

The elements described above suggest that we need: 4 gauge bosons for a unified description of the weak and electromagnetic interactions. Three of them are massive, one (the photon) must remain massless after spontaneous symmetry breaking. The SM gauge group is then $G = SU(2) \times U(1)$ which matches the number of gauge bosons. However, we know that two of these only couple to left handed fermions. whereas one of the massive ones (the neutral) couples differently to left and right handed fermions. Finally, the photon must remain massless and its couplings parity invariant. The choice of gauge group is then

$$\boxed{G = SU(2)_L \times U(1)_Y}, \tag{20.56}$$

where the three gauge bosons couple to left handed fermions only, and the $U(1)_Y$ is not identified with the $U(1)_{\rm EM}$, the abelian gauge symmetry responsible for electromagnetism. As we will see below, two of the $SU(2)_L$ gauge bosons will result in the $W^{\pm}_{\mu}$. On the other hand to obtain the Z^0 and the photon we will need to carefully choose the pattern of spontaneous symmetry breaking of the SM gauge group in (20.56) down to $U(1)_{\rm EM}$.

20.2.2 Spontaneous Breaking of $SU(2)_L \times U(1)_Y \to U(1)_{\rm EM}$

All matter in the SM (i..e. fermions and scalars) will have some transformation property under the SM gauge group G. This means that we need to assign to each particle a charge under the abelian group factor, the $U(1)_Y$. This is called *hypercharge*, since it is not quite the electric charge.
We first consider a scalar field Φ in the fundamental representation of $SU(2)$ and with hypercharge $U(1)_Y$,

$$Y_\Phi = 1/2\ . \tag{20.57}$$

The scalar doublet can be written as

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}\ , \tag{20.58}$$

where ϕ^+ and ϕ^0 are complex scalar fields, resulting in four real scalar degrees of freedom[6]. Under a $SU(2)_L \times U(1)_Y$ gauge transformation, the Higgs doublet transforms as

$$\Phi(x) \to e^{i\alpha^a(x)t^a}\, e^{i\beta(x)Y_\Phi}, \Phi(x)\ , \tag{20.59}$$

where t^a are the $SU(2)_L$ generators (i.e. Pauli matrices divided by 2), $\alpha^a(x)$ are the three $SU(2)_L$ gauge parameters, $\beta(x)$ is the $U(1)_Y$ gauge parameter, and it is understood that the $U(1)_Y$ factor of the gauge transformation contains a factor of the identity $I_{2\times2}$ after the hypercharge Y_Φ. Thus, the covariant derivative on Φ is given by

$$D_\mu\Phi(x) = \left(\partial_\mu - igA^a_\mu(x)t^a - ig'B_\mu(x)Y_\Phi I_{2\times2}\right)\Phi(x)\ . \tag{20.60}$$

[6]At this point, the labels "+" and "0" are just arbitrary, since we have not even defined electric charges But these label will be consistent in the future, after we have done this.

Here, $A_\mu^a(x)$ is the $SU(2)_L$ gauge boson, $B_\mu(x)$ the $U(1)_Y$ gauge boson, and g and g' are their corresponding couplings. The lagrangian of the scalar and gauge sectors of the SM is then

$$\mathcal{L} = \big(D_\mu\Phi\big)^\dagger D^\mu\Phi - V(\Phi^\dagger\Phi) - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \ , \tag{20.61}$$

where $F_{\mu\nu}^a$ is the usual $SU(2)$ field strength built out of the gauge fields $A_\mu^a(x)$ and $B_{\mu\nu}$ is the $U(1)_Y$ field strength given by the abelian expression

$$B_{\mu\nu} = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) \ . \tag{20.62}$$

As usual, we consider the potential

$$V(\Phi^\dagger\Phi) = -m^2\big(\Phi^\dagger\Phi\big) + \lambda\big(\Phi^\dagger\Phi\big)^2 \ , \tag{20.63}$$

which is minimized for

$$\langle\Phi^\dagger\Phi\rangle = \frac{m^2}{2\lambda} \equiv \frac{v^2}{2} \ . \tag{20.64}$$

In order to fulfil this, we choose the vacuum

$$\langle\Phi\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \ . \tag{20.65}$$

Just as in the previous examples of SSB of non abelian gauge symmetries, the next question is what is the symmetry breaking pattern, i.e. what gauge bosons get what masses, if any. In particular, we want one of the four gauge bosons in G to remain massless after imposing the vacuum $\langle\Phi\rangle$ in (20.65). This means that there must be a generator or, in this case, a linear combination of generators of G that annihilates $\langle\Phi\rangle$, leaving the vacuum invariant under a G transformation. This combination of generators must be associated with the massless photon in $U(1)_{\rm EM}$, the remnant gauge group after the spontaneous breaking. One trick to identify this combination of generators is to consider the gauge transformation defined by

$$\begin{aligned} \alpha^1(x) = \alpha^2(x) &= 0 \\ \alpha^3(x) &= \beta(x) \ . \end{aligned} \tag{20.66}$$

The exponent in the gauge transformation has the form

$$\begin{aligned} i\alpha^3(x)t^3 + i\beta(x)Y_\Phi I_{2\times 2} &= i\frac{\beta(x)}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= \frac{i\beta(x)}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \end{aligned} \tag{20.67}$$

Then we see that this combination

$$\boxed{\left(t^3 + Y_\Phi\right)\langle\Phi\rangle = 0}\,, \tag{20.68}$$

indeed annihilates the vacuum, leaving it invariant. Thus, we suspect that this linear combination of $SU(2)_L \times U(1)_Y$ generators must be associated with the massless photon. We will come back to this point later.
We now go to extract the gauge boson mass terms from the scalar kinetic term in (20.61). This is

$$\begin{aligned} \mathcal{L}_{\rm m} &= \left(D_\mu\langle\Phi\rangle\right)^\dagger D^\mu\langle\Phi\rangle \\ &= \frac{1}{2}\,(0 \quad v)\left(gA^a_\mu t^a + g'Y_\Phi B_\mu\right)\left(gA^{b\mu}t^b + g'Y_\Phi B^\mu\right)\begin{pmatrix} 0 \\ v \end{pmatrix} . \end{aligned} \tag{20.69}$$

For the product of the two $SU(2)$ factors we will use the trick in (20.27). Then, the only terms we need to be careful about are the mixed ones: one $SU(2)$ times one $U(1)_Y$ contribution. There are two of them, and each has the form

$$\frac{1}{2}\,(0 \quad v)\, g\, g'\frac{\sigma^3}{2}Y_\Phi\begin{pmatrix} 0 \\ v \end{pmatrix} = -\frac{1}{2}\frac{v^2}{4}\, g\, g'\, A^3_\mu\, B^\mu \ , \tag{20.70}$$

where in the second equality we used $Y_\Phi = 1/2$. We then have

$$\mathcal{L}_{\rm m} = \frac{1}{2}\frac{v^2}{4}\left\{g^2A^1_\mu A^{1\mu} + g^2A^2_\mu A^{2\mu} + g^2A^3_\mu A^{3\mu} + g'^2B_\mu B^\mu - 2gg'A^3_\mu B^\mu\right\} . \tag{20.71}$$

From this expression we can clearly ee that A^1_μ and A^2_μ acquire masses just as we saw in the pure $SU(2)$ example. It will be later convenient to define the linear combinations

$$W^\pm_\mu \equiv \frac{A^1_\mu \mp iA^2_\mu}{\sqrt{2}} \ , \tag{20.72}$$

which allows us to write the first two terms in (20.71) as

$$\mathcal{L}_{\rm m}^{W} = \frac{g^2\,v^2}{4}\,W_\mu^+ W^{-\mu}\;. \tag{20.73}$$

These two states have masses

$$M_W = \frac{g\,v}{2}\;. \tag{20.74}$$

On the other hand, the fact that A_μ^3 and B_μ have a mixing term prevents us from reading off masses. We need to rotate these states to go to a bases without mixing, a diagonal basis. In order to clarify what needs to be done, we can write the last three terms in (20.71 in matrix form

$$\mathcal{L}_{\rm m}^{\rm neutral} = \frac{1}{2}\frac{v^2}{4}(A_\mu^3 \quad B_\mu)\left(\begin{array}{cc} g^2 & -g\,g' \\ -g\,g' & g'^2 \end{array}\right)\left(\begin{array}{c} A^{3\mu} \\ B^\mu \end{array}\right)\;, \tag{20.75}$$

where the task is to find the eigenvalues and eigenstates of the matrix above. It is clear that one of the eigenvalues is zero, since the determinant vanishes. Then the squared masses of the physical neutral gauge bosons are

$$\begin{aligned} M_\gamma^2 &= 0 \\ M_Z^2 &= \frac{v^2}{4}\,(g^2+g'^2) \end{aligned} \tag{20.76}$$

The eigenstates in terms of A_μ^3 and B_μ, the original $SU(2)_L$ and $U(1)_Y$ gauge bosons respectively, are

$$\boxed{A_\mu \equiv \frac{1}{\sqrt{g^2+g'^2}}\left(g' A_\mu^3 + g B_\mu\right)} \tag{20.77}$$

$$\boxed{Z_\mu \equiv \frac{1}{\sqrt{g^2+g'^2}}\left(g A_\mu^3 - g' B_\mu\right)}\,. \tag{20.78}$$

Alternatively, we could have obtained the same result by defining an orthogonal rotation matrix to diagonalize the interactions above. That is, rotating the states by

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix} , \tag{20.79}$$

results in diagonal neutral interactions if we have

$$\cos\theta_W \equiv \frac{g}{\sqrt{g^2 + g'^2}}, \qquad \sin\theta_W \equiv \frac{g'}{\sqrt{g^2 + g'^2}}, \tag{20.80}$$

where θ_W is called the Weinberg angle. It is useful to invert (20.79) to obtain

$$A^3_\mu = \sin\theta_W A_\mu + \cos\theta_W Z_\mu \tag{20.81}$$

$$B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu \ . \tag{20.82}$$

Using these expressions for A^3_μ and B_μ we can replace them in the covariant derivative acting on the scalar doublet Φ. Their contribution fo D_μ is

$$\begin{aligned} -igA^3_\mu t^3 - ig' Y_\Phi B_\mu &= -iA_\mu\big(g\sin\theta_W t^3 + g'\cos\theta_W Y_\Phi\big) - i\big(g\cos\theta_W t^3 - g'\sin\theta_W Y_\Phi\big)\, Z_\mu \\ &= -ig\sin\theta_W\Big(t^3 + Y_\Phi\Big)A_\mu - i\frac{g}{\cos\theta_W}\Big(t^3 - (t^3 + Y_\Phi)\sin^2\theta_W\Big)Z_\mu \ , \end{aligned} \tag{20.83}$$

where it is always understood that the hypercharge Y_Φ is always multiplied by the identity, and in the last identity we used the fact that

$$g'\cos\theta_W = g\sin\theta_W \ , \tag{20.84}$$

and trigonometric identities. We can conclude that is A_μ is to be identified with the photon field, then its coupling must be e times the charged of the particle it is coupling to (e.g. -1 for an electron. Thus we must impose that

$$\boxed{e = g\,\sin\theta_W}, \tag{20.85}$$

and that the charge operator, acting here on the field Φ coupled to A_μ is defined as

$$\boxed{Q = t^3 + Y_\Phi}. \tag{20.86}$$

Then we can read the photon coupling to the doublet scalar field Φ from

$$-i\,e\,A_\mu\,Q\,\Phi(x) = -i\,e\,A_\mu\,Q\,\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}\ . \tag{20.87}$$

Substituting $Y_\Phi = 1/2$ we have

$$Q\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} = \begin{pmatrix}\phi^+\\ 0\end{pmatrix}\ , \tag{20.88}$$

which tells us that the top complex field in the scalar doublet has charge equal to 1 (in units of e, the proton charge), whereas the bottom component has zero charge, justifying our choice of labels. On the other hand, we see that fixing Q to be the electromagnetic charge operator, completely fixes the couplings of Z_μ to the scalar Φ. This is now, from (20.83),

$$-i\frac{g}{\cos\theta_W}Z_\mu\Big(t^3 - Q\sin^2\theta_W\Big)\,\Phi\ . \tag{20.89}$$

We will see below that the choice of fixing the A_μ couplings to be those of electromagnetism, fixes completely the Z_μ to all elementary couplings, giving a wealth of predictions.

20.2.3 Gauge Couplings of Fermions

The SM is a *chiral gauge theory*, i.e. its gauge couplings differ for different chiralities. To extract the left handed fermion gauge couplings, we look at the covariant derivative

$$D_\mu\psi_L = \big(\partial_\mu - igA^a_\mu t^a - ig'Y_{\psi_L}B_\mu\big)\psi_L\ , \tag{20.90}$$

where Y_{ψ_L} is the left handed fermion hypercharge. On the other hand, since right handed fermions do not feel the $SU(2)_L$ interaction, their covariant derivative is given by

$$D_\mu\psi_R = \big(\partial_\mu - ig'Y_{\psi_R}B_\mu\big)\psi_R\ , \tag{20.91}$$

with Y_{ψ_R} its hypercharge. Using the covariant derivatives above, we can extract the neutral and charged couplings. We start with the neutral couplings, which in terms of the gauge boson mass eigenstates are the couplings to the photon and the Z.

<u>Neutral Couplings</u>: From (20.90), the neutral gauge couplings of a left handed fermions are

$$
\begin{aligned}
(-igt^3 A^3_\mu - ig' Y_{\psi_L} B_\mu)\psi_L &= ig \sin\theta_W \left(t^3 + Y_{\psi_L}\right) A_\mu \psi_L \\
&- i\frac{\left(g^2 t^3 - ig'^2 Y_{\psi_L}\right)}{\sqrt{g^2 + g'^2}}\, Z_\mu \psi_L \ ,
\end{aligned}
\tag{20.92}
$$

where on the right hand side we made use of (20.81) and (20.82). Now, we know that the photon coupling should be

$$
-ie\, Q_{\psi_L} \ , \tag{20.93}
$$

with Q_{ψ_L} the fermion electric charge operator. Thus, we must identify

$$
Q_{\psi_L} = t^3 + Y_{\psi_L} \ , \tag{20.94}
$$

as the fermion charge. We can use our knowledge of the fermion charges to fix their hypercharges. As an example, let us consider the left handed lepton doublet. For the lightest family, this is written in the notation

$$
L = \begin{pmatrix} \nu_{eL} \\ e^-_L \end{pmatrix} \ . \tag{20.95}
$$

The action of t^3 on L is

$$
\begin{aligned}
t^3\, L &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e^-_L \end{pmatrix} \\
&= \begin{pmatrix} (1/2)\, \nu_{e_L} \\ (-1/2)\, e^-_L \end{pmatrix} \equiv \begin{pmatrix} t^3_{\nu_{e_L}}\, \nu_{e_L} \\ t^3_{e_L}\, e^-_L \end{pmatrix} \ ,
\end{aligned}
\tag{20.96}
$$

where in the last equality we defined $t^3_{\nu_{e_L}} = 1/2$ and $t^3_{e_L}$ as the eigenvalues of the operator t^3 associated to the electron neutrino and the electron. Then, we have

$$
Q_L\, L = \begin{pmatrix} 1/2 + Y_L & 0 \\ -1/2 + Y_L & 0 \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e^-_L \end{pmatrix} = \begin{pmatrix} (1/2 + Y_L)\, \nu_{e_L} \\ (-1/2 + Y_L)\, e_L \end{pmatrix} \ . \tag{20.97}
$$

But we know that the eigenvalue of the charge operator applied to the neutrino must be zero, as well as that the eigenvalue of the electron must be -1. Thus, we obtain the hypercharge of the left handed lepton doublet

$$\boxed{Y_L = -\frac{1}{2}}, \tag{20.98}$$

which is fixed to give us the correct electric charges for the members of the doublet L. We can do the same with the right handed fermions. These, however do not have t^3 in the covariant derivative (see (20.91)). Then, for e^-_R, the right handed electron, we have that $t^3_{e_R} = 0$, which means that, since

$$Q_{e^-_R} = -1 \ , \tag{20.99}$$

then the right handed electron's hypercharge is equal to it:

$$\boxed{Y_{e^-_R} = -1}. \tag{20.100}$$

Similarly, the right handed electron neutrino has zero electric charge, which results in

$$\boxed{Y_{\nu_R} = 0}. \tag{20.101}$$

Now that we fixed all the lepton hypercharges by imposing that they have the QED couplings to the photon, we can extract their couplings to the Z as predictions of the electroweak SM. From (20.92) we have

$$\begin{aligned} -i\big(g\,\cos\theta_W\, t^3 - g'\,\sin\theta_W\, Y_\psi\big) Z_\mu\,\psi &= -i\frac{g}{\cos\theta_W}\left(\cos^2\theta_W\, t^3 - \sin^2\theta_W\, Y_\psi\right) Z_\mu\,\psi \\ &= -i\,\frac{g}{\cos\theta_W}\left(t^3 - \sin^2\theta_W\, Q_\psi\right) Z_\mu\,\psi \ , \end{aligned} \tag{20.102}$$

where the initial expressios makes use of $\cos\theta_W$ and $\sin\theta_W$ in terms of g and g', in the first equality we used that $\tan\theta_W = g'/g$ and, in the final equality, we used that in general $Q_\psi = t^2 + Y_\psi$, independently of the fermion chirality, as long as we generalize (20.94) for right handed fermions using $t^3_{\psi_R} = 0$. For instance, from (20.102) we can read off the lepton couplings of the Z boson. These are,

$$
\begin{aligned}
\nu_{e_L}: &\qquad -i\frac{g}{\cos\theta_W}\left(\frac{1}{2}\right) \\
e_L^-: &\qquad -i\frac{g}{\cos\theta_W}\left(-\frac{1}{2}+\sin^2\theta_W\right) \\
e_R^-: &\qquad -i\frac{g}{\cos\theta_W}\left(-\sin^2\theta_W\right) \\
\nu_{e_R}: &\qquad 0\,.
\end{aligned}
\tag{20.103}
$$

From the couplings above, we see that every lepton has a different predicted coupling to the Z. These are, of course, three level predictions. Measurements of these Z couplings have been performed with subpercent precision for a long time, and the SM predictions for the fermion gauge couplings have passed the tests every time. Another, interesting point, is that right handed neutrinos have *no gauge couplings* in the SM: no Z coupling, certainly no electric charge and no QCD couplings. Thus, from the point of view of the SM, the right handed neutrino need not exist.

Charged Couplings:

We complete here the derivation of the gauge couplings of leptons by extracting their charged couplings.These come from the $SU(2)_L$ gauge couplings, as we see from

$$
-ig\big(A^1_\mu t^1 + A^2_\mu t^2\big) = -i\frac{g}{\sqrt{2}}\begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix}\,,
\tag{20.104}
$$

which then involve only left handed fermions. Then, from the gauge part of the left handed doublet kinetic term

$$
\mathcal{L}_L = \bar{L}i\not{D}L\,,
\tag{20.105}
$$

we obtain their charged couplings

$$
\begin{aligned}
\mathcal{L}_L^{\text{ch.}} &= \begin{pmatrix}\bar{\nu}_{e_L} & \bar{e}_L\end{pmatrix}\gamma^\mu\frac{g}{\sqrt{2}}\begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix}\begin{pmatrix}\nu_{e_L} \\ e_L\end{pmatrix} \\
&= \frac{g}{\sqrt{2}}\left\{\bar{\nu}_{e_L}\gamma^\mu e_L\,W^+_\mu + \bar{e}_L\gamma^\mu\nu_{e_L}\,W^-_\mu\right\}\,,
\end{aligned}
\tag{20.106}
$$

where we can see that, as required by hermicity, the second term is the hermitian conjugate of the first. The Fermi lagrangian can be obtained from $\mathcal{L}_L^{\text{ch.}}$ by integrating out the $W^\pm$ gauge bosons.

We now briefly comment on the electroweak gauge coulpings of quarks. Just as for leptons, we concentrate on the first family. The left handed quark doublet is

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} , \tag{20.107}$$

We know that, independently of helicity, the charges of the up and down quarks are $Q_u = +2/3$ and $Q_d = -1/3$. Then we have

$$Q_{q_L} = \left(t^3 + Y_{q_L}\right) = \begin{pmatrix} +2/3 & 0 \\ 0 & -1/3 \end{pmatrix} , \tag{20.108}$$

which results in

$$\boxed{Y_{q_L} = \frac{1}{6}} . \tag{20.109}$$

The hypercharge assignments for the right handed quarks are again trivial and given by the quark electric charges. We have

$$\boxed{Y_{u_R} = +\frac{2}{3} , \qquad Y_{d_R} = -\frac{1}{3}} . \tag{20.110}$$

With these hypercharge assignments we can now write the quark couplings to the Z. Using (20.102) we obtain

$$\begin{aligned}
u_L : &\qquad -i\frac{g}{\cos\theta_W}\left(\frac{1}{2} - \sin^2\theta_W\frac{2}{3}\right) \\
d_L : &\qquad -i\frac{g}{\cos\theta_W}\left(-\frac{1}{2} + \sin^2\theta_W\frac{1}{3}\right) \\
u_R : &\qquad -i\frac{g}{\cos\theta_W}\left(-\sin^2\theta_W\frac{2}{3}\right) \\
d_R : &\qquad -i\frac{g}{\cos\theta_W}\left(\sin^2\theta_W\frac{1}{3}\right) .
\end{aligned} \tag{20.111}$$

Once again, we see that each type of quark has a different coupling to the Z. All of these predictions have been tested with great precision, confirming the SM even beyond leading order.

The charged gauged couplings of left handed quarks are trivial to obtain: they are dictated by $SU(2)_L$ gauge symmetry and therefore there must be the same as those of the left handed leptons in (20.106). So we have

$$\mathcal{L}_q^{\text{ch.}} = \frac{g}{\sqrt{2}} \left\{ \bar{u}_L \gamma^\mu d_L \, W_\mu^+ + \bar{d}_L \gamma^\mu u_L \, W_\mu^- \right\} . \tag{20.112}$$

20.2.4 Fermion Masses

We have seen that SSB leads to masses for same of the gauge bosons, preserving gauge invariance. We now direct our attention to fermion masses. In principle these terms

$$\mathcal{L}_{\text{fm}} = m_\psi \bar{\psi}_L \psi_R + \text{h.c.} \ , \tag{20.113}$$

are forbidden by $SU(2)L \times U(1)_Y$ gauge invariance since thy are not invariant under

$$\begin{aligned} \psi_L &\rightarrow e^{i\alpha^a(x)t^a} \, e^{i\beta(x)Y_{\psi_L}} \, \psi_L \\ \psi_R &\rightarrow e^{i\beta(x)Y_{\psi_R}} \, \psi_R \ . \end{aligned}$$

But the operator

$$\bar{\psi}_L \, \Phi \, \psi_R \ , \tag{20.114}$$

is clearly invariant under the $SU(2)_L$ gauge transformations, and it would be $U(1)_Y$ invariant if

$$-Y_{\psi_L} + Y_\Phi + Y_{\psi_R} = 0 \ . \tag{20.115}$$

Since $Y_\Phi = 1/2$, this form of the oprator will work for the down type quarks and charged leptons. For instance, since $Y_L = -1/2$ and $Y_{e_R} = -1$, the operator

$$\mathcal{L}_{me} = \lambda_e \, \bar{L} \, \Phi \, e_R \ + \text{h.c.}, \tag{20.116}$$

is gauge invariant since the hypercharges satisfy (20.115). In (20.116) we defined the dimensionless coupling λ_e which will result in a Yukawa coupling of electrons to the Higgs boson. To see this, we write $\Phi(x)$ in the unitary gauge, so that

$$\begin{aligned}\mathcal{L}_{me} &= \lambda_e \left(\bar{\nu}_{e_L} \quad \bar{e}_L\right) \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} e_R + \text{h.c.} \\ &= \lambda_e \frac{v}{\sqrt{2}}\, \bar{e}_L e_R + \lambda_e \frac{1}{\sqrt{2}}\, h(x)\, \bar{e}_L e_R + \text{h.c.}\ , \end{aligned} \tag{20.117}$$

where the first term is the electro mass term resulting in

$$m_e = \lambda_e \frac{v}{\sqrt{2}}\ , \tag{20.118}$$

and the second term is the Yuawa interaction of the electron and the Higgs boson $h(x)$. We can rewrite (20.117) as

$$\mathcal{L}_{\text{me}} = m_e\, \bar{e}_L e_R + \frac{m_e}{v}\, h(x)\, \bar{e}_L e_R + \text{h.c.}\ , \tag{20.119}$$

from which we can see that the electron couples to the Higgs boson with a strength equal to its mass in units of the Higgs VEV v. Similarly, for quarks we have that the operator

$$\mathcal{L}_{md} = \lambda_e\, \bar{q}_L\, \Phi\, d_R \ + \text{h.c.}, \tag{20.120}$$

is gauge invariant since $Y_{q_L} = 1/6$ and $Y_{d_R} = -1/3$ satisfy (20.115). Them we obtain

$$\mathcal{L}_{\text{md}} = m_d\, \bar{d}_L d_R + \frac{m_d}{v}\, h(x)\, \bar{d}_L d_R + \text{h.c.}\ , \tag{20.121}$$

and where the down quark mass was defined as

$$m_d = \lambda_d \frac{v}{\sqrt{2}}\ . \tag{20.122}$$

As we can see, it will be always the case that fermions couple to the Higgs boson with the strength m_ψ/v. Thus, the heavier the fermion, the stronger its coupling to the Higgs. Finally, in order to have gage invariant operators with up type right handed quarks we need to use the operator

$$\mathcal{L}_{\text{mu}} = \lambda_u\, \bar{q}_L \tilde{\Phi}\, u_R + \text{h.c.}\ , \tag{20.123}$$

where we defined

$$\tilde{\Phi}(x) = i\sigma^2 \, \Phi(x)^* = \left(\begin{array}{c} \frac{v+h(x)}{\sqrt{2}} \\ \\ 0 \end{array} \right) , \qquad (20.124)$$

where in the last equality we are using the unitary gauge. It is straightforward[7] to prove that $\tilde{\Phi}(x)$ is an $SU(2)_L$ doublet with $Y_{\tilde{\Phi}} = -1/2$, which is what we need so as to make the operator in (20.123) invariant under $U(1)_Y$. Then we have

$$\mathcal{L}_{\rm mu} = m_u \, \bar{u}_L u_R + \frac{m_u}{v} \, h(x) \, \bar{u}_L u_R + \text{h.c.} \; , \qquad (20.125)$$

with

$$m_u = \lambda_u \, \frac{v}{\sqrt{2}} \; . \qquad (20.126)$$

The fermion Yukawa couplings are parameters of the SM. In fact, since there are three families of quarks their Yuakawa couplings are in general a non diagoinal three by three matrix. This fact has important experimental consequences. On the other hand, we could imagina having something similar if we introduce a right handed neutrino. This however, might be beyond the SM, since this state does not have any SM gauge quantum numbers. Overall, the SM is determined by the paremeters v, g, g' and $\sin\theta_W$ in the electroweak gauge sector, plus all the Yukawa couplings in the fermion sector leading to all the observed fermion masses and mixings.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 20.
- *The Quantum Theory of Fields, Vol. II*, by S. Weinberg, Chapter 21.
- *Quantum Field Theory* , by M. Srednicki, Chapters 84 and 86.
- *Quantum Field Theory in a Nutshell*, by A. Zee. Chapter IV.6.

[7] Only need to use that $\sigma^2\sigma^2 = 1$, and that $\sigma^2(\sigma^a)^*\sigma^2 = -\sigma^a$.