Lecture 19

Spontaneous Breaking of Non Abelian Symmetries

Having introduced non abelian groups and non aberlian gauge symmetries, we are now in a position to consider the spontaneous breaking of both global and gauge symmetries. We will atart with the simpler case of the global symmetry and we will restate Goldstone's theorem in a more general way so as to include different symmetry breaking patterns, which will result in a different number of Nambu–Goldstone Bosons (NGBs). Then we will consider the spontaneous breaking of non abelian gauge symmetries, i.e. the most general version of the Higgs mechanism.

19.1 Spontaneous Breaking of a Global Non Abelian Symmetry

We start with the lagrangian for a scalar field ϕ,

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - \frac{\mu^2}{2} \phi^\dagger \phi - \frac{\lambda}{4} \left(\phi^\dagger \phi\right)^2 . \tag{19.1}$$

The lagrangian above is invariant under the transformation

$$\phi(x) \to e^{i\alpha^a t^a} \phi(x) , \tag{19.2}$$

where the t^a are the generators of the non abelian group G, and the arbitrary parameters α^a are constants. Here the scalar field $\phi(x)$ must carry a group index in order for (19.2) to make sense. We say the symmetry is spontaneously broken if we have

$$\mu^2 = -m^2 < 0 , \tag{19.3}$$

then the potential has a non trivial minimum at

$$\left(\phi^\dagger\phi\right)_0 = \langle\phi^\dagger\phi\rangle = \frac{m^2}{\lambda} \equiv v^2 \ . \tag{19.4}$$

However, we need to ask *how* is the symmetry spontaneously broken. In other words, Spontaneous Symmetry Breaking (SSB) means that the value of the field at the minimum, let us call it the vacuum expectation value (VEV) of the field $\langle\phi\rangle$, is not invariant under the symmetry transformation (19.2). That is,

$$\langle\phi\rangle \to e^{i\alpha^a t^a}\,\langle\phi\rangle = \left(1 + i\alpha^a t^a + \cdots\right)\langle\phi\rangle \ , \tag{19.5}$$

can be either equal to $\langle\phi\rangle$ or not. This tells us that if

$$t^a\langle\phi\rangle = 0 \ , \tag{19.6}$$

the ground state is invariant under the action of the symmetry (*unbroken symmetry directions*), whereas if

$$t^a\langle\phi\rangle \neq 0 \ , \tag{19.7}$$

the ground state is not invariant (*broken symmetry directions*). We see that some of the generators will annihilate the ground state $\langle\phi\rangle$, such as in (19.6), whereas others will not. In the first case, these directions in group space will correspond to preserved or unbroken symmetries. Therefore, there should not be massless NGBs associated with them. On the other hand, if the situation is such as in (19.7), then the ground state is not invariant under the symmetry transformations *defined by these generators*. These directions in group space defined *broken directions or generators* and there should be a massless NGB associated with each of them. Thus, as we will see in more detail below, the number of NGB will correspond to the total number of generators of G, minus the number of unbroken generators, i.e. the number of *broken generators*.

Example 1: $SU(2)$

As a first example, let us consider the case where the symmetry transformations are those associated with the group $G = SU(2)$. The *three* generators of $SU(2)$ are

$$t^a = \frac{\sigma^a}{2} \ , \tag{19.8}$$

with σ^a the three Pauli matrices. This means that the scalar fields appearing in the lagrangian (19.1) are *doublets* of $SU(2)$, i.e. we can represent them by a column vector

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \\ \phi_2(x) \end{pmatrix} , \tag{19.9}$$

and that the symmetry transformation can be written as[1]

$$\phi^i(x) = \left(\delta^{ij} + i\alpha^a t^a_{ij} + \cdots\right) \phi^j(x) , \tag{19.10}$$

where $i, j = 1, 2$ are the group indices for the scalar field in the fundamental representation. We now need to *choose* the vacuum $\langle\phi\rangle$. This is typically informed by either the physical system we want to describe or by the result we want to get. Let us choose

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{19.11}$$

Clearly this satisfies (19.4). This choice corresponds to having

$$\begin{array}{lcl} \langle \mathrm{Re}[\phi_1] \rangle & = & 0 \qquad\qquad \langle \mathrm{Im}[\phi_1] \rangle = 0 \\ \langle \mathrm{Re}[\phi_2] \rangle & = & v \qquad\qquad \langle \mathrm{Im}[\phi_2] \rangle = 0 , \end{array} \tag{19.12}$$

in (19.9). We can now test what generators annihilate the vacuum (19.11) and which ones do not. We have

$$t^1 \langle\phi\rangle = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} v \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{19.13}$$

Similarly, we have

$$t^2 \langle\phi\rangle = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -iv \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \tag{19.14}$$

and

[1] We put the group indices in the fields upstairs for future notational simplicity. There is no actual meaning to them being "up" or "down" indices, but the summation convention still holds.

$$t^3 \langle\phi\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -v \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{19.15}$$

So we conclude that with the choice of vacuum (19.11), all $SU2)$ generators are broken. This means that all the continuous symmetry transformations generated by (19.2) change the chosen vacuum $\langle\phi\rangle$. Thus, Goldstone's theorem predicts there must be *three* massless NGBs. In order to explicitly see who are these NGBs, we write the lagrangian (19.1) in terms of the real scalar degrees of freedom as in

$$\phi(x) = \begin{pmatrix} \mathrm{Re}[\phi_1(x)] + i\,\mathrm{Im}[\phi_1(x)] \\ \\ v + \mathrm{Re}[\phi_2(x)] + i\,\mathrm{Im}[\phi_2(x)] \end{pmatrix} , \tag{19.16}$$

which amounts to expanding about the vacuum (19.11) as long as (19.12) is satisfied. Substituting in (19.1) we will find that there are three massless states, namely, $\mathrm{Re}[\phi_1(x)]$, $\mathrm{Im}[\phi_1(x)]$ and $\mathrm{Im}[\phi_2(x)]$, and that there is a massive state corresponding to $\mathrm{Re}[\phi_2(x)]$ with a mass given by m. this looks very similar to waht we obtain in the abelian case, of course. Also analogously to the abelian case, we could have parametrized $\phi(x)$ as in

$$\phi(x) = e^{i\pi^a(x)t^a/f} \begin{pmatrix} 0 \\ v + c\,\sigma(x) \end{pmatrix} , \tag{19.17}$$

where $\sigma(x)$ and $\pi^a(x)$, with $a = 1, 2, 3$ are real scalar fields, and the scale f and the constant c are to be determined so as to obtain canonically normalized kinetic terms for them in $\mathcal{L}$. In fact, choosing

$$f = \frac{v}{\sqrt{2}}, \qquad\qquad c = \frac{1}{\sqrt{2}} , \tag{19.18}$$

we arrive at

$$\mathcal{L} = \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma + \frac{1}{2}\partial^\mu\pi^a\partial_\mu\pi^a - \frac{m^2}{2}\left(v + \frac{\sigma(x)}{\sqrt{2}}\right)^2 + \frac{\lambda}{4}\left(v + \frac{\sigma(x)}{\sqrt{2}}\right)^4 , \tag{19.19}$$

from which we see that the three $\pi^a(x)$ fields are massless and are therefore the NGBs. Furthermore, using $m^2 = \lambda v^2$, we can extract

$$m_\sigma = m = \lambda\, v \tag{19.20}$$

The choice of vacuum $\langle\phi\rangle$ resulting in this spectrum could have been different. For instance, we could have chosen

$$\langle\phi\rangle = \begin{pmatrix} v \\ 0 \end{pmatrix} . \tag{19.21}$$

But it is easy to see that this choice is equivalent to (19.11), and that it would result in an identical real scalar spectrum. Similarly, the aparently different vacuum

$$\langle\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ v \end{pmatrix} , \tag{19.22}$$

results in the same spectrum. All these vacuum choices spontaneously break $SU(2)$ *completely*, i.e. there are not symmetry transformations that respect these vacua. Below we will see an example of partial spontaneous symmetry breaking.

Example 2: $SU(3)$ If we now consider that $\mathcal{L}$ in (19.1) is invariant under $SU(3)$ global transformations, there are going to be $3^2 - 1 = 8$ generators. A convenient basis for them is provided by the Gellmann matrices:

$$\begin{aligned}
t^1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & t^2 &= \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
t^3 &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & t^4 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
t^5 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & t^6 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
t^7 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & t^8 &= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} ,
\end{aligned} \tag{19.23}$$

which are nothing but "stretched" Pauli matrices. In fact, it is useful to note that for $i = 1, 2, 3$ we have

$$t^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0\;0 & 0 \end{pmatrix} , \tag{19.24}$$

which means that $SU(2)$ is a subgroup of $SU(3)$, since its generators appears as block diagonal matrices in the generators of $SU(3)$.

Let us now consider the following choice of vacuum

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} , \tag{19.25}$$

which clearly satisfies the constraint $\langle\phi^\dagger\phi\rangle = v^2$. It is straightforward to verify that the generators in (19.24) annihilate this vacuum, i.e.

$$t^i \langle\phi\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , \tag{19.26}$$

for $i = 1, 2, 3$. This means that with this vacuum choice, the $SU(2)$ subgroup of $SU(3)$ leaves the vacuum invariant, i.e. this subgroup *is not* spontaneously broken. On the other hand, we can see that

$$t^{4,\cdots,8} \langle\phi\rangle \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , \tag{19.27}$$

resulting in 5 broken generators. Then there are 5 continuous symmetries spontaneously broken, resulting in 5 NGBs. This is what we call partial spontaneous breaking: the breaking pattern is

$$SU(3) \to SU(2) , \tag{19.28}$$

from which we can see that the number of NGBs can also be thought of as the number of original generators, minus the number of generators of the unbroken group.

19.2 Goldstone Theorem Revisited

We go back to considering the infinitesimal transformation (19.10), but we rewrite it as

$$\phi^i \to \phi^i + \Delta^i(\phi) , \tag{19.29}$$

where we defined

$$\Delta^i(\phi) \equiv i\alpha^a \, (t^a)_{ij}\phi^j \ . \tag{19.30}$$

If the potential has a non trivial minimum at $\Phi^i(x) = \phi_0^i$, then it is satisfied that

$$\left.\frac{\partial V(\phi^i)}{\partial \phi^i}\right|_{\phi_0} = 0 \ . \tag{19.31}$$

We can then expand the potential around the minimum as

$$V(\phi^i) = V(\phi_0^i) + \frac{1}{2}\left(\phi^i - \phi_0^i\right)\left(\phi^j - \phi_0^j\right)\left.\frac{\partial^2 V}{\partial\phi^i\partial\phi^j}\right|_{\phi_0} + \cdots \ , \tag{19.32}$$

where the first derivative term is omitted in light of (19.31). The second derivative term in (19.32) Above defines a matrix with units of square masses:

$$M_{ij}^2 \equiv \left.\frac{\partial^2 V}{\partial\phi^i\partial\phi^j}\right|_{\phi_0} \geq 0 \ . \tag{19.33}$$

where the last inequality results from the fact that ϕ^0 is a minimum. M_{ij}^2 is the mass squared matrix. We are now in the position to state Goldstone's theorem in this context.

Theorem:
"For each symmetry of the lagrangian that *is not* a symmetry of the vacuum ϕ_0, there is a zero eigenvalue of M_{ij}^2 ."

Proof:
The infinitesimal symmetry transformation in (19.29) leaves the lagrangian invariant. In particular, it also leaves the potential invariant, i.e.

$$V(\phi^i) = V\Big(\phi^i + \Delta^i(\phi)\Big) \ . \tag{19.34}$$

Expanding the right hand side of (19.34) and keeping only terms leading in $\Delta^i(\phi)$, we can write

$$V(\phi^i) = V(\phi^i) + \Delta^i(\phi)\,\frac{\partial V(\phi^i)}{\partial\phi^i} \ , \tag{19.35}$$

which, to be satisfied requires that

$$\Delta^i(\phi)\,\frac{\partial V(\phi)}{\partial \phi^i} = 0\ . \tag{19.36}$$

To make this result useful, we take a derivative on both sides and specified for $\phi^i = \phi_0^i$, i.e. we evaluate all the expression at the minimum of the potential. We obtain

$$\frac{\partial \Delta^i(\phi)}{\partial \phi^j}\bigg|_{\phi_0}\frac{\partial V(\phi)}{\partial \phi^i}\bigg|_{\phi_0} + \Delta^i(\phi_0)\,\frac{\partial^2 V(\phi)}{\partial \phi^j \partial \phi^i}\bigg|_{\phi_0} = 0\ . \tag{19.37}$$

But by virtue of (19.31), the first term above vanishes, leaving us with

$$\boxed{\Delta^i(\phi_0)\,\frac{\partial^2 V(\phi)}{\partial \phi^j \partial \phi^i}\bigg|_{\phi_0} = 0}\ . \tag{19.38}$$

There are two ways to satisfy (19.38):

1. $\Delta^i(\phi_0) = 0$.
   But this means that, under a symmetry transformation, the vacuum is invariant, since according to (19.29) this results in

$$\phi_0^i \to \phi_0^i\ . \tag{19.39}$$

2. $\Delta^i(\phi_o) \neq 0$.
   This requires that the second derivative factor in (19.38) must vanish, i.e.

$$M_{ij}^2 = 0\ . \tag{19.40}$$

   We then conclude that for each symmetry transformation that *does not leave the vacuum invariant* there must be a zero eigenvalue of the mass squared matrix M_{ij}^2. QED.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 20.