Lecture 18

Renormalization of Non-Abelian Gauge Theories

We start with a generic Yang-Mills theory with fermions. Its lagrangian is given by

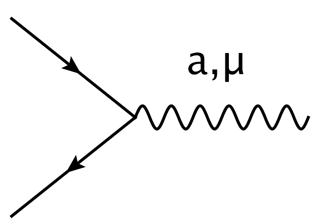
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}(i \not{D} - m)\psi \ , \tag{18.1}$$

where we have defined the non-abelian gauge field strength $F^a_{\mu\nu}$ by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc}\, A^b_\mu A^c_\nu \ , \tag{18.2}$$

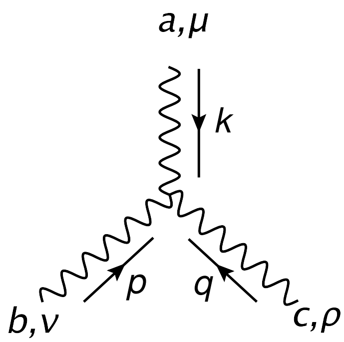
with A^a_μ the gauge field, f^{abc} the structure constant of the corresponding and g the gauge coupling.

We quickly recall the Feynman rules the Feynman rules from lecture 16. These are shown below. The first one corresponds to the interactions of the fermions with the non-abelian gauge bosons. It is very similar to the interaction of charged fermions with photons in QED, with the sole addition of the generator t^a.



$$= \qquad i\, g \gamma^\mu\, t^a$$

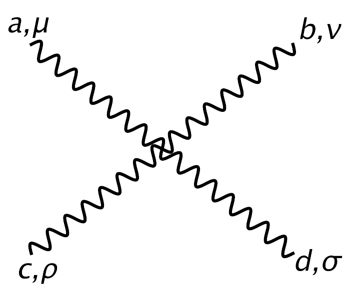
In addition, non-abelian gauge theories have Feynman rules governing the interactions among the gauge bosons themselves. From (18.1) we see there are terms involving three gauge bosons, depicted below:



$$= \quad g\, f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$$

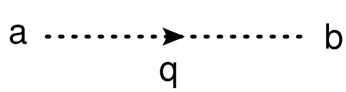
These terms come from one of the derivatives, for instance in $F^a_{\mu\nu}$, hitting the last term in (18.2). That is why the momenta in the diagram enter. This is also a Feynman rule coming in at order g.

Finally, there is a Feynman rule associated with the interaction of *four* gauge bosons, coming from the product of the last term in $F^a_{\mu\nu}$ with the similar term in $F^{a\mu\nu}$. This is given by



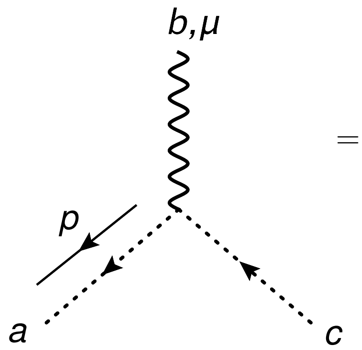
$$\begin{aligned} = \quad & -ig^2\left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\right. \\ & + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) \\ & \left. + f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right] \end{aligned}$$

In addition to these, we now need to consider the Feynman rules for ghost fields. As we saw in lecture 17, these are



$$= \quad \frac{i}{q^2 + i\epsilon}\,\delta^{ab}$$

for the ghost propagator, and

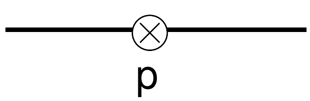


$$= \quad g\, f^{abc}\, p^{\mu}$$

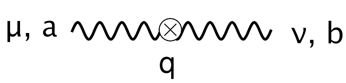
for the ghost–gauge boson vertex. Now, in order to renormaliza the theory, we need to define the necessary counterterms. Just as we did when computing the β function of QED or ϕ^4 scalar theory, we are going to assume that the renormalization scale μ is high compared with the masses of the fermions involved,

$$\mu \gg m \ , \tag{18.3}$$

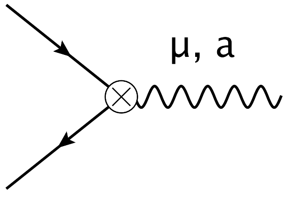
where here m stands for all the fermion masses. Then, the relevant coiunterterms are the ones associated to the renormalization of the fermion lines (δ_2), the gauge boson line (δ_3) and the fermion–gauge boson vertex (δ_1). They introduce the counterterm Feynman diagrams below:



$$= \quad i\not{p}\,\boldsymbol{\delta_2}$$



$$= \quad -i\left(q^2 g^{\mu\nu} - q^{\mu} q^{\nu}\right)\, \delta^{ab}\, \boldsymbol{\delta_3}$$



$$= \quad i\, g\, \gamma^{\mu}\, t^{a}\, \boldsymbol{\delta_1}$$

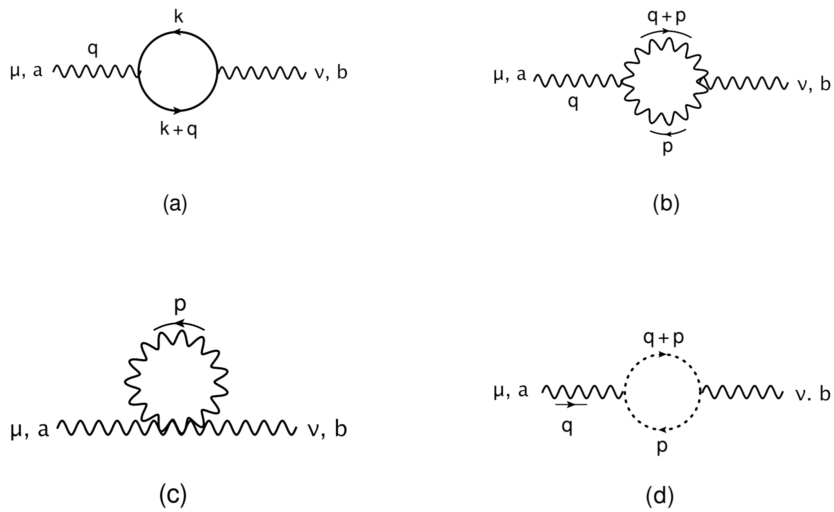


Figure 18.1: One loop contributions to the gauge boson two point function in non abelian gauge theories. Diagram (a) is the fermion loop, similar to the ones present in abelian theories. The last three diagrams are present only in non abelian gauge theories. In particular, diagram (d) is a ghost loop.

where, as mentioned earlier, the fermion mass is neglected in the fermion counterterm. Armed with these, we can compute the β function of non abelian gauge theories using the Callan-Symanzik equation:

$$\beta(g) = \mu \frac{\partial}{\partial \mu} \left(\frac{1}{2}\, g \sum_i \delta Z_i - \delta g \right) , \tag{18.4}$$

which in our case translates to

$$\beta(g) = g\, \mu \frac{\partial}{\partial \mu} \left(\frac{1}{2}\delta_3 + \delta_2 - \delta_1 \right) , \tag{18.5}$$

where the extra factor of δ_2 comes from the fact that there are two fermion lines in the interaction, and we used $\delta g = g\, \delta_1$. In what follows, we will compute the one loop β function in non abelian gauge theories, by computing the corresponding counterterms.

18.1 Gauge Boson Self-energy: δ_3

To compute the counterterm δ_3 to one loop accuracy, we will consider the contributions to the gauge boson two point function shown in Figure 18.1.

The sum of these contributions plus the counterterm at some renormalization scale μ^2 will fix δ_3.

Diagram (a): The first diagram in Figure 18.1 is analogous to the one in QED. In fact we can borrow the result from the electron loop, with the only difference being the group generators entering in the vertices. Using the result in dimensional regularization the contribution from diagram (a) is

$$I_{(a)} = i(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\mathrm{Tr}[\mathrm{t^a t^b}]\,\frac{-8\mathrm{g}^2}{(4\pi)^{\mathrm{d}/2}}\,\Gamma[\epsilon/2]\int_0^1 \mathrm{dx}\,\frac{\mathrm{x}(1-\mathrm{x})}{\Delta^{\epsilon/2}}\ , \tag{18.6}$$

where we defined

$$\Delta \equiv m^2 - x(1-x)\,q^2\ . \tag{18.7}$$

For fermions in a generic representation r the trace is[1]

$$\mathrm{Tr}[\mathrm{t^a t^b}] = \mathrm{C(r)}\ . \tag{18.8}$$

but in order to use (18.6) to compute the β function, we are only interested in its dependence on the renormalization scale μ. In order to see how this comes about in $I_{(a)}$, we will set the renormalization condition at

$$q^2 = -\mu^2\ , \tag{18.9}$$

so that we can write

$$\Delta = m^2 + x(1-x)\mu^2 \equiv \mu^2\left(\kappa + x(1-x)\right)\ , \tag{18.10}$$

where in the second equality we defined the arbitrary dimensionless constant κ by the relation $m^2 = \kappa\,\mu^2$. Since

$$\frac{\Gamma[\epsilon/2]}{\Delta^{\epsilon/2}} = \left\{\frac{2}{\epsilon} - \ln\mu^2 - \ln\left[\kappa x(1-x)\right] + \cdots\right\}\ , \tag{18.11}$$

where the dots indicate terms that vanish in the $\epsilon \to 0$ limit. Then, it is clear that for the purpose of computing the β function we only need the coefficient of the term $-\ln\mu^2$. If we generally allow for there to be an n_f number of fermion flavors, then the fermion loop contribution relevant to the β function can be expressed as

[1] For instance, in the fundamental representation of $SU(N)$ we have $C(N) = 1/2$.

$$\boxed{I_{(a)} = i(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\frac{-g^2}{(4\pi)^{d/2}}\,\frac{\mathbf{4}}{\mathbf{3}}\,\mathbf{C(r)}\,\frac{\Gamma[\epsilon/2]}{\mu^{\epsilon/2}}}\;. \tag{18.12}$$

Diagram (b): This next diagram is generated by the triple gauge boson couplings coming from $\mathcal{L}_{3G}$ in the previous lecture. Using the Feynman rules derived from it for the triple gauge boson interaction we can write this contribution as

$$I_{(b)} = \frac{1}{2}\,g^2\,f^{acd}\,f^{bcd}\int \frac{d^4p}{(2\pi)^4}\,\frac{-i}{p^2}\,\frac{-i}{(p+q)^2}\,N^{\mu\nu}\;, \tag{18.13}$$

where we have defined the tensor

$$\begin{aligned} N^{\mu\nu} &= \Big(g^{\mu\rho}(q-p)^\sigma + g^{\rho\sigma}(2p+q)^\mu + g^{\sigma\mu}(-p-2q)^\rho\Big) \\ &\times \Big(\delta^\nu_\rho(p-q)_\sigma + g_{\rho\sigma}(-2p-q)^\nu + \delta^\nu_\sigma(2q+p)_\rho\Big)\;. \end{aligned} \tag{18.14}$$

The next steps are standard: Feynman parametrization, shift in the integration variable

$$p^\mu \to \ell^\mu = p + x\,q^\mu\;, \tag{18.15}$$

where x is the Feynman parameter introduced, and defining

$$\Delta \equiv -x(1-x)q^2\;, \tag{18.16}$$

we obtain

$$\begin{aligned} I_{(b)} &= \frac{i\,g^2}{(4\pi)^d}\,C_2(G)\,\delta^{ab}\int_0^1 dx\,\frac{1}{\Delta^{\epsilon/2}}\Big\{\Gamma[-1+\epsilon/2]\,g^{\mu\nu}\,q^2\left[\frac{3}{2}(d-1)x(1-x)\right] \\ &+ \Gamma[\epsilon/2]\,g^{\mu\nu}\,q^2\left[\frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)\right] \\ &- \Gamma[\epsilon/2]\,q^\mu q^\nu\left[(1-\frac{d}{2})(1-2x)^2 + (1+x)(2-x)\right]\Big\}\;, \end{aligned} \tag{18.17}$$

where we used that

$$f^{acd}\,f^{bcd} = (t^c_G)_{ad}\,(t^c_G)_{db} = C_2(G)\,\delta^{ab}\;, \tag{18.18}$$

with $C_2(G)$ the Casimir of the adjoint representation. It is clear from (18.17) that, unlike $I_{(a)}$, this contribution is not transverse by itself.

Diagram (c): This diagram is generated by the quartic gauge boson interaction from $\mathcal{L}_{4G}$ defined in the previous lecture. Using the quartic gauge boson Feynman rule, we have

$$\begin{aligned} I_{(c)} = \frac{-i\,g^2}{2} \int \frac{d^4p}{(2\pi)^4} \frac{-i\,g_{\rho\sigma}}{p^2} \quad \delta^{cd} \quad & \Big\{ f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) \\ + \quad & f^{ace} f^{bde} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) \\ + \quad & f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}\right) \Big\} , \end{aligned} \tag{18.19}$$

But since the structure constants are antisymmetric, the first term in (18.19) does not contribute, i.e.

$$\delta^{cd}\, f^{cde} = 0 \ . \tag{18.20}$$

The remaining two terms will also be proportional to the quadratic Casimir of the adjoint representation, $C_2(G)$, after using (18.18). Collecting all terms we arrive at

$$I_{(c)} = -g^2\, C_2(G)\, \delta^{ab} \int \frac{d^4p}{(2\pi)^4} \frac{g^{\mu\nu}}{p^2} (d-1) \ . \tag{18.21}$$

In order to proceed further with the usual steps, it is advantageous to artificially introduce a dependence on the external momentum q^μ by writing

$$1 = \frac{(q+p)^2}{(q+p)^2} \ , \tag{18.22}$$

which allows us to introduce a Feynman parametrization through the momentum shift

$$\ell^\mu \equiv p^\mu + x\, q^\mu \ , \tag{18.23}$$

where x is the Feynman parameter. The rest of the calculation is as always: Wick rotation, dimensional regularization, etc. The result is

$$\begin{aligned} I_{(c)} = \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int_0^1 \frac{dx}{\Delta^{\epsilon/2}} \Big\{ \quad & - \quad \Gamma[-1+\epsilon/2]\, g^{\mu\nu}\, q^2\, \frac{1}{2}\, d\,(d-1)\, x\,(1-x) \\ & - \quad \Gamma[\epsilon/2]\, g^{\mu\nu}\, q^2\, (d-1)\, (1-x)^2 \Big\} \ . \end{aligned} \tag{18.24}$$

Once again, we see that this contribution is not transverse by itself, nor is its summ to diagram (b).

Diagram (d): Finally, we need to compute the ghost loop contribution to the gauge boson propagator. Using the ghost–gauge boson Feynman rule derived in the previous lecture and showed earlier in this one, we must be careful with the momentum flow and the group indices in the structure constants. This gives

$$I_{(d)} = (-1)\, g^2\, f^{dac}\, f^{cbd} \int \frac{d^4p}{(2\pi)^4}\, (q+p)^\mu\, p^\nu \frac{i}{(q+p)^2}\, \frac{i}{p^2}\ . \tag{18.25}$$

Notice that we considered the group index of the top ghost propagator to be d (assuming that the "conservation" of this index was already imposed by the delta function that appears in the ghost propagator), and to be c for the bottom propagator. The Feynman rule tells us that the gauge boson index at each vertex goes in the middle of the structure constant, whereas the first index corresponds to the one that carries the momentum factor. Finally, notice the factor of (-1) in front of the whole diagram: it is a loop of anticommuting fields, so there is an overall sign in front of it, which it will turn out to be crucial in the end. Following the same steps as for diagrams (b) and (c) we obtain

$$\begin{aligned} I_{(d)} = \frac{ig^2}{(4\pi)^{d/2}}\, C_2(G)\, \delta^{ab} \int_0^1 \frac{dx}{\Delta^{\epsilon/2}} \Big\{ & -\ \Gamma[-1+\epsilon/2]\, g^{\mu\nu}\, q^2\, \frac{1}{2}\, x\,(1-x) \\ & +\ \Gamma[\epsilon/2]\, q^\mu\, q^\nu\, x\,(1-x) \Big\}\ . \end{aligned} \tag{18.26}$$

Clearly, this is not transverse by itself either. But the hope is that the summ of the three contributions from diagrams (b), (c) and (d) (diagram (a), a QED-like contribution, is already transverse) will result in an overall transverse gauge boson two point function. We will do this in a moment, but for now let us concentrate on the parts of the diagrams that are proportional to $\Gamma[-1+\epsilon/2]$. When we introduce dimensional regularization, we pointed out that these factors are typically appearing in diagrams that would have quadratic divergences if computed in a cutoff scheme such as Pauli-Villars. If we add all these contributions the answer will be proportional to

$$\Gamma[-1+\epsilon/2]\, g^{\mu\nu}\, q^2\, x(1-x)\, (-1+\frac{\epsilon}{2}(2-d)\ , \tag{18.27}$$

where we used that $\epsilon = 4-d$. But using

$$\Gamma[-1+\epsilon/2]\, (-1+\frac{\epsilon}{2}) = \Gamma[\epsilon/2]\ , \tag{18.28}$$

the result is now proportional to

$$\Gamma[\epsilon/2]\,(2-d)\,g^{\mu\nu}\,q^2\,x\,(1-x)\ . \tag{18.29}$$

Then, it looks like having the ghost field loop, with its factor of (-1) in front coming from the fact that they are anticommuting fields, is what turned this seemingly quadratic divergence in the gauge boson two point function to a tamer divergence, associated with logarithmic cutoff dependence. A quadratic divergence would have signaled the appearance of a (divergent) gauge boson mass, with the consequent loss of gauge invariance. We will see this below in the restoration of transversality. Adding all three diagrams we obtain

$$\begin{aligned} I_{(b)}+I_{(c)}+I_{(d)} &= i\left(g^{\mu\nu}\,q^2-q^\mu q^\nu\right)\frac{(-g^2)}{(4\pi)^{d/2}}\,C_2(G)\delta^{ab} \\ &\times \int_0^1 dx\,\frac{\Gamma[\epsilon/2]}{\Delta^{\epsilon/2}}\left\{2+(1-d/2)\,(1-2x)^2\right\}\ , \end{aligned} \tag{18.30}$$

which is manifestly transverse. Thus, we arrive at the conclusion that the ghost loop contribution to the gauge boson two point function was crucial for the preservation of gauge invariance, including the anticommuting character of ghost fields.

We can now finally compute the counterterm δ_3 by adding all four diagrams in Figure 18.1 plus the corresponding counterterm diagram contributing to the gauge boson two point function, and impose the renormalization condition that the summ of all these contributions vanish at the renormalization scale

$$q^2=-\mu\ . \tag{18.31}$$

This results in

$$\boxed{\delta_3=\frac{g^2}{16\pi^2}\left\{\mathbf{\frac{5}{3}\,C_2(G)-\frac{4}{3}\,n_f\,C(r)}\right\}\frac{\Gamma[\epsilon/2]}{(\mu^2)^{\epsilon/2}}}\ . \tag{18.32}$$

The first term in the expression above comes from the addition of diagrams (b), (c) and (d), whereas the second term is the contribution from the n_f fermion loops in (18.12). Even before we complete the computation of all counterterms and obtain the *beta* function, we can already draw some interesting conclusions from (18.32). From the Callan-Symanzik equation (18.5) we can see that the contribution from δ_3 to the β function of a non abelian gauge theory has two contributions of opposite sign. This contribution is

$$\frac{1}{2}\,g\,\frac{\partial\delta_3}{\partial\ln\mu}=\frac{g^3}{16\pi^2}\left\{-\frac{5}{3}\,C_2(G)+\frac{4}{3}\,n_f\,C(r)\right\}\ . \tag{18.33}$$

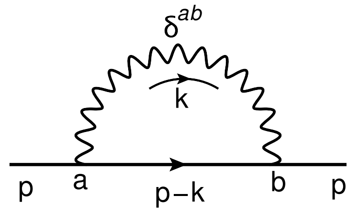


Figure 18.2: One loop contribution to the fermion self energy. The group indices at the vertices indicate factors of the generators, t^a and t^b. The factor of δ^{ab} appears in the non abelian gauge boson propagator.

We see that the term coming from the QED-like fermion loops fives a *positive* contribution to $\beta(g)$, just as in QED. On the other hand, the first term, coming entirely from the non abelian character of the gauge theory give a *negative* contribution. Thus, depending on the number of fermion flavors n_f, the contribution to $\beta(g)$ could be postive or negative. Of course, this sign is of great consequence for the infrared and ultraviolet behavior of the theory. But before we conclude on the sign of the β function, we still need to compute the other two counterterms entering in (18.5).

18.2 Fermion Self Energy: δ_2

There is only one diagram contributing to one loop order to the self energy of the fermion in a non abelian gauge theory. This is shown in Figure 18.2. Once this diagram is computed (exercise) , it needs to be added to the counterterm controbution to the fermion two point function containing $]delta_2$. Imposing the appropriate renormalization condition at the scale $p^2 = -\mu^2$ which in the absence of a fermion mass counterterm simple means that the summ of the two contributions must vanish at this scale. Then, we obtain

$$\boxed{\delta_2 = -\frac{g^2}{(4\pi)^{d/2}}\, C_2(r)\, \frac{\Gamma[\epsilon/2]}{(\mu^2)^{\epsilon/2}}}\,. \tag{18.34}$$

In (18.34) we see the appearance of the quadratic Casimir for the fermion representation r (as opposed to $C_2(G)$, for the adjoint representation).

18.3 Vertex Counterterm: δ_1

There are two one loop diagrams contributing to the vertex renormalization of the fermio–gauge boson interaction, they are shown in Figure 18.3. Imposing the renormalization condition at $q^2 = -\mu^2$, with q^μ the momentum of the gauge boson coming out of the

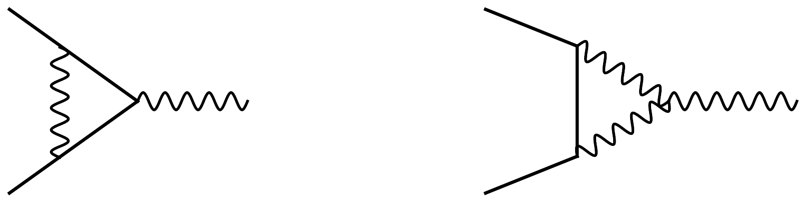


Figure 18.3: One loop contributions to the fermion–gauge boson vertex. Diagram (a) is similar to the QED case, whereas diagram (b) is only present in non abelian gauge theories.

vertex, to the summ of these two diagrams and the counterterm diagram including δ_1, we obtain (exercise)

$$\boxed{\delta_1 = -\frac{g^2}{(4\pi)^{d/2}}\,\frac{\Gamma[\epsilon/2]}{(\mu^2)^{\epsilon/2}}\,\left[C_2(r) + C_2(G)\right]}\;. \tag{18.35}$$

We are now ready to compute the β function of a non abelian gauge theory. For this, we need to apply (18.5) using the expressions (18.35), (18.34) and (18.32) for δ_1 , δ_2 and δ_3, respectively. The result is

$$\boxed{\beta(g) = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}\,C_2(G) - \frac{4}{3}\,n_f\,C(r)\right]}\;. \tag{18.36}$$

This is the *one loop* β function of a non abelian gauge theory with n_F fermions[2]. We can clearly see now that there is a competition between the first term coming exclusively from the non abelian features of the theory, and the fermion loop contributions -the more QED like behavior of the β function. As we thoroughly discussed when we introduced the concept of the β function of an interacting theory, its sign is crucial in understanding its behavior towards higher or lower energy scales, its renormalization flow. A positive β function results in growing couplings in the UV, such as in the case of QED. On the other, hand e negative sign results in an increasingly small coupling in the UV, even resulting in a free theory in the limit $\mu \to \infty$. This is the regime known as asymptotic freedom. Conversely, a negative β function implies a growing coupling towards the IR, signalling that perturbation theory might fail at low energies. This typically is associated with a phase transition as, for example is the case in QCD, an $SU(3)$ gauge theory.

[2]We could also have scalars. See exercise.

Example: $SU(N)$

For the case the group is $SU(N)$, if r is the fundamental representation, and G the adjoint, we have

$$C(r) = \frac{1}{2} \qquad\qquad C_2(G) = N \ , \tag{18.37}$$

Thus, the theory will be asymptotically free if $\beta(g) < 0$, which is satisfied if the number of fermion flavors is

$$n_f < \frac{11}{2} N \ . \tag{18.38}$$

Then, if we considering $N = 3$, the gauge theory of Quantum Chromodynamics (QCD), the number of quarks should satisfy

$$n_q \leq 16 \ , \tag{18.39}$$

in order for QCD to remain asymptotically free. In the standard model of particle physics the number of quarks flavors is $n_f = 6$, so there is no danger. We should also keep in mind, that the expression in (18.36) is the one loop β function. In this way, even if it vanishes at this order, there could be higher order terms giving a non zero contributions. But if the theory is perturbative, then the corrections to the vanishing of β come suppressed by

$$\frac{g^2}{16\pi^2} \ , \tag{18.40}$$

with respect to (18.36). We can refer to this behavior as "quasi-conformal", in that the β function is "almost" vanishing.

18.4 Gauge Invariance and Counterterm Relations

We have the lagrangian invariant under non abelian gauge symmetry given by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{c}^a \left(\partial^\mu D^{ac}_\mu\right) c^c + \bar{\psi} \left(i\not{D} - m\right) \psi \ , \tag{18.41}$$

where, once again we have the covariant derivative of the matter in a given representation r

$$D_\mu\psi(x) = \left(i\partial\!\!\!/ - ig\, A^a_\mu(x) t^a\right)\psi(x)\ , \tag{18.42}$$

and the covariant derivative of a the ghosts corresponds to the one for a scalar field in the adjoin representation, i.e.

$$D^{ac}_\mu c^c(x) = \left(\partial_\mu\,\delta^{ac} + g\, f^{abc}\, A^b_\mu(x)\right)\, c^c(x)\ . \tag{18.43}$$

In the lagrangian in (18.41), we fixed the gauge by $\xi=\infty$, the unitary gauge. This makes the gauge fixing term dissapear from $\mathcal{L}$. We would like to proceed to define the theory in terms of renormalized parameters (fields, couplings, masses) and the counterterms. We will see that, in general there are a wealth of counterterms we can define. However, as we will show below, there are relations that reduce the number of independent counterterms. We start from the unrenormalized lagrangian. In terms of the unrenormalized parameters, we write it as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu}\right)^2 + \bar\psi_0(i\partial\!\!\!/ - m_0)\psi_0 - \bar c^a_0\,\partial^2\, c^a_0 \\ &+ g_0 A^a_{0\mu}\bar\psi_0\gamma^\mu t^a\psi_0 - g_0 f^{abc}(\partial_\mu A^a_{0\nu})A^{b\mu}_0 A^{c\nu}_0 \\ &- g_0^2\left(f^{eab}A^a_{0\mu}A^b_{0\nu}\right)\left(f^{ecd}A^{c\mu}_0 A^{d\nu}_0\right) - g_0\bar c^a_0 f^{abc}\partial^\mu A^b_{0\mu} c^c_0\ .\end{aligned} \tag{18.44}$$

If we now define the renormalized fields by

$$\begin{aligned}A^a_\mu &\equiv Z_3^{-1/2}\, A^a_{0\mu}\ ,\\ \psi &\equiv Z_2^{-1/2}\,\psi_0\ ,\\ c^a &\equiv (Z^c_2)^{-1/2}\, c^a_0\ ,\end{aligned} \tag{18.45}$$

we can rewrite the lagrangian in terms of them as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}Z_3\,\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + Z_2\,\bar\psi(i\partial\!\!\!/ - m_0)\psi - Z^c_2\,\bar c^a\,\partial^2\, c^a \\ &+ g_0 Z_3^{1/2}\, Z_2\, A^a_\mu\bar\psi\gamma^\mu t^a\psi - g_0\, Z_3^{3/2}\, f^{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} \\ &- g_0^2 Z_3^2\,\left(f^{eab}A^a_\mu A^b_\nu\right)\left(f^{ecd}A^{c\mu}A^{d\nu}\right) - g_0 Z_3^{1/2}\, Z^c_2\,\bar c^a f^{abc}\partial^\mu A^b_\mu c^c\ .\end{aligned} \tag{18.46}$$

Finally, we define the most general set of counterterms by

$$\begin{aligned}
\delta_2 &\equiv Z_2 - 1 & \delta_3 &\equiv Z_3 - 1 \\
\delta_1 &\equiv \frac{g_0}{g}\, Z_3^{1/2}\, Z_2 - 1 & \delta_2^c &\equiv Z_2^c - 1 \\
\delta_1^{3G} &\equiv \frac{g_0}{g}\, Z_3^{3/2} - 1 & \delta_1^{4G} &\equiv \frac{g_0}{g}\, Z_3^2 - 1 \\
\delta_1^c &\equiv \frac{g_0}{g}\, Z_3^{1/2}\, Z_2^c - 1 & \delta m &\equiv Z_2\, m_0 - m \ ,
\end{aligned} \tag{18.47}$$

Which results in a total of 8 counterterms. Now we can write the lagrangian in (18.46) entirely in terms of renormalized parameters and the counterterms defined in (18.47). We obtain

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2 + \bar\psi(i\partial\!\!\!/ - m)\psi - \bar c^a\,\partial^2\, c^a \\
&+ g\, A_\mu^a \bar\psi\gamma^\mu t^a\psi - g\, f^{abc}(\partial_\mu A_\nu^a)A^{b\mu}A^{c\nu} \\
&- g^2\left(f^{eab}A_\mu^a A_\nu^b\right)\left(f^{ecd}A^{c\mu}A^{d\nu}\right) - g\,\bar c^a f^{abc}\partial^\mu A_\mu^b c^c \\
&- \frac{1}{4}\delta_3\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2 + \bar\psi(i\delta_2\partial\!\!\!/ - \delta m)\psi - \delta_2^c\,\bar c^a\,\partial^2\, c^a \\
&+ g\,\delta_1\, A_\mu^a\bar\psi\gamma^\mu t^a\psi - g\,\delta_1^{3G}\, f^{abc}(\partial_\mu A_\nu^a)A^{b\mu}A^{c\nu} \\
&- g^2\,\delta_1^{4G}\left(f^{eab}A_\mu^a A_\nu^b\right)\left(f^{ecd}A^{c\mu}A^{d\nu}\right) - g\,\delta_1^c\,\bar c^a f^{abc}\partial^\mu A_\mu^b c^c \ ,
\end{aligned} \tag{18.48}$$

where the top three lines correspond to the original lagrangian in terms of renormalized parameters, and the last three lines are the counterterms. However, it is clear that not all 8 counterterms are independent. Even if we ignore the mass counterterm, we still have 7 left, but these are determined by only 4 paramters: Z_2, Z_3, Z_2^c and g_0/g. This means that there must be 3 relations among the 7 counterterms. These relations are a consequence of gauge invariance. A very useful way to see this, is in perturbation theory. In this case we can always use the fact that

$$Z_i \simeq 1 + \delta_{Z_i} \ , \qquad\qquad \frac{g_0}{g} \simeq 1 + \delta g \ , \tag{18.49}$$

where we assume that $\delta_{Z_i} \ll 1$ and $\delta g \ll 1$. Then, if we write the exact relations

$$\begin{aligned}
\delta_1 - \delta_2 &= \frac{g_0}{g} Z_2 Z_3^{1/2} - Z_2 \\
\delta_1^c - \delta_2^c &= \frac{g_0}{g} Z_2^c Z_3^{1/2} - Z_2^c \\
\delta_1^{3G} - \delta_3 &= \frac{g_0}{g} Z_3^{3/2} - Z_3 \\
\delta_1^{4G} - \delta_3 &= \frac{g_0^2}{g^2} Z_3^2 - Z_3 \,,
\end{aligned} \tag{18.50}$$

and then expand them using (18.49) we obtain

$$\boxed{\delta_1 - \delta_2 = \delta_1^c - \delta_2^c = \delta_1^{3G} - \delta_3 = \frac{1}{2}\left(\delta_1^{4G} - \delta_3\right)}\,. \tag{18.51}$$

This tells us, for instance, that the divergences in the vertex counterterms δ_1^{3G} ,δ_1^{4G} and δ_1^c are removed by the same procedure that removes them in δ_1, δ_2 and δ_3. We had derive the β function in (18.36) from the Callan-Symanzik equation applied to the *fermion-fermion* gauge boson. But (18.51) implies that we can equally derive the same result by using the three-gauge boson or even the four-gauge boson vertices. We will even get the same result using the gauge boson–ghost–ghost vertex.

The result in (18.51) also implies that if we have a non abelian gauge theory involving various types of fields, then if the vertex counterterm of the field i with the gauge boson is δ_1^i and its field renormalization counterterm is δ_2^i, then the quantity

$$\boxed{\delta_1^i - \delta_2^i}\,, \tag{18.52}$$

is universal for all fields.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 16.
- *Quantum Field Theory* , by M. Srednicki, Chapter 73.