Lecture 17

Quantization of Non-Abelian Gauge Theories

Just as we did for the case of abelian gauge theories, we address the quantization of non-abelian gauge theories by the Fadeev-Popov method. We start with a pure gauge theory with the action

$$S[A_\mu] = -\frac{1}{4}\,F^a_{\mu\nu}\,F^{a\mu\nu}\ , \tag{17.1}$$

where, as usual, we defined

$$A_\mu = A^a_\mu\,t^a \tag{17.2}$$

and the t^a are the generators of the group $\mathbf{G}$. The functional integral of interest is

$$Z[J_\mu] = \int \mathcal{D}A_\mu\, e^{iS[A_\mu]+i\int d^4x J_\mu(x)A^\mu(x)}\ , \tag{17.3}$$

where we coupled the gauge field to an external vector source $J_\mu(x)$. Just as for the abelian case, in order to reduce the gauge redundancy resulting from the freedom to choose a gauge, we write the measure of the functional integral as

$$\mathcal{D}A_\mu = \mathcal{D}\bar{A}_\mu\,\mathcal{D}\alpha\ , \tag{17.4}$$

where the $\bar{A}_\mu(x)$ fields represent physically inequivalent gauge field configurations, i.e. we cannot access a given field configuration $\bar{A}^1_\mu(x)$ belonging to the set $\{\bar{A}_\mu(x)\}$, by using a gauge transformation on another member of the set, $\bar{A}^2_\mu(x)$. Then the gauge redundancy in the measure of the functional integral is due to all the possible gauge transformations, which are represented by the $\mathcal{D}\alpha$ factor. Implementing the Fadeev-Popov method, the functional integral now reads

$$Z[J_\mu] = \int \mathcal{D}\bar{A}_\mu\,\mathcal{D}\alpha\,\mathcal{F}\left[G[A^\alpha_\mu]\right]\ \det\left[\frac{\delta G[A^\alpha_\mu]}{\delta\alpha}\right]\, e^{iS[A_\mu]+i\int d^4x J_\mu(x)A^\mu(x)}\ , \tag{17.5}$$

where $G[A^\alpha_\mu]$ is the gauge fixing functional, and A^α_μ is the gauge field *after* a gauge transformation is applied:

$$A^\alpha_\mu(x) = V(x)\left[A_\mu(x) + \frac{i}{g}\partial_\mu\right]V^\dagger(x)\ , \tag{17.6}$$

where g is the gauge coupling, and $V(x)$ is a unitary transformation in the gauge group $\mathbf{G}$, i.e.

$$V(x) = e^{i\alpha^a(x)t^a} , \tag{17.7}$$

given in terms of the group generators t^a. The functional $\mathcal{F}\left[G[A_\mu^\alpha]\right]$ in (17.5) fixes the gauge in the functional integral for an arbitrary choice of the gauge fixing functional $G[A_\mu^\alpha]$.[1]

Since the action is gauge invariant we have that

$$S[A_\mu] = S[A_\mu^\alpha] . \tag{17.8}$$

Also, the part of the measure that varies with the gauge transformation is just absorbed in $\mathcal{D}\alpha$. I.e.

$$\mathcal{D}\alpha\ \mathcal{D}A_\mu = \mathcal{D}\alpha\ A_\mu^\alpha , \tag{17.9}$$

where we dropped the bars above the gauge fields for simplicity of notation.

17.1 Gauge Fixing

The next step, is to determine the gauge fixing function. For practical use, it is advantageous to write the functional $\mathcal{F}\left[G[A_\mu^\alpha]\right]$ as a Gaussian factor:

$$\mathcal{F}\left[G[A_\mu^\alpha]\right] = e^{-\frac{i}{2\xi}\int d^4x\, G[A_\mu^\alpha]G[A_\mu^\alpha]} , \tag{17.10}$$

with ξ a real parameter. With this choice the effect of the gauge fixing factor is to add a term to the lagrangian:

$$\mathcal{L}_{\text{eff.}} = \mathcal{L} - \frac{1}{2\xi}(G[A])^2 , \tag{17.11}$$

where care must be taken on the right hand side since $A_\mu = A_\mu^a(x)t^a$ carry group indices. Choosing a Lorentz invariant gauge fixing functional as

$$G[A_\mu^a] = \partial^\mu A_\mu^a , \tag{17.12}$$

we then obtain

$$Z[J_\mu] = \int \mathcal{D}A_\mu \det\left[\frac{\delta G[A_\mu^\alpha]}{\delta\alpha}\right] e^{iS[A_\mu]-i\int d^4x\frac{(\partial^\mu A^a(x))^2}{2\xi}+i\int d^4x J_\mu(x)A^\mu(x)} , \tag{17.13}$$

We must still deal with the determinant. We first compute its argument, i.e. the operator of which we are computing the determinant. This is done by looking at the response

[1] In the derivation for the abelian case we used the delta function, so instead of $\mathcal{F}\left[G[A_\mu^\alpha]\right]$, we would have a factor of $\delta\left[G[A_\mu^\alpha]\right]$. The procedure above is general insofar the gauge fixing function is arbitrary.

of the gauge fixing function to a gauge transformation. We start by writing the gauge transformation for the case of infinitesimally small scalar functions $\alpha^a(x)$:

$$\begin{aligned} A_\mu^\alpha(x) &= e^{i\alpha^a(x)t^a}\left[A_\mu(x)+\frac{i}{g}\partial_\mu\right]e^{-i\alpha^a(x)t^a} \\ &= A_\mu(x)+\frac{1}{g}\partial_\mu\alpha^a(x)t^a+i\alpha^a(x)t^aA_\mu^b(x)t^b-iA_\mu^b(x)t^b\alpha^a(x)t^a \\ &= A_\mu(x)+\frac{1}{g}\partial_\mu\alpha^a(x)t^a+i\alpha^a(x)A_\mu^b(x)\left[t^b,t^a\right] \\ &= A_\mu(x)+\frac{1}{g}\partial_\mu\alpha^a(x)t^a-f^{abc}\alpha^a(x)A_\mu^b(x)t^c\ . \end{aligned} \tag{17.14}$$

Thus, using the antisymmetry of the structure constants, $f^{abc}=-f^{cba}$, we arrive at

$$(A_\mu^a(x))^\alpha = A_\mu^a(x)+\frac{1}{g}\partial_\mu\alpha^a(x)+f^{abc}A_\mu^b(x)\,\alpha^c(x)\ . \tag{17.15}$$

We notice that the last two terms above can be written as a covariant derivative acting on $\alpha^a(x)$, a scalar field transforming in the *adjoint* representation of **G**. This is

$$D_\mu\alpha^a(x)=\partial_\mu\alpha^a(x)-ig\left(t_G^b\right)_{ac}A_\mu^b(x)\,\alpha^c(x) \tag{17.16}$$

where t_G^c are the generators of the adjoint representation, i.e.

$$(t_G^c)_{ac}=if^{abc}\ . \tag{17.17}$$

Then, the gauge-transformed field can be written as

$$(A_\mu^a(x))^\alpha = A_\mu^a(x)+\frac{1}{g}D_\mu\alpha^a(x)\ . \tag{17.18}$$

We are now ready to write the gauge fixing functional. This is defined to be

$$G[A^\alpha]=\partial^\mu\left(A_\mu^a(x)\right)^\alpha\ , \tag{17.19}$$

a generalization of the gauge fixing functional we used in the abelian case. Thus,

$$G[A^\alpha]=\partial^\mu A_\mu^a(x)+\frac{1}{g}\,\partial^\mu D_\mu\alpha^a(x) \tag{17.20}$$

To compute the determinant we need first to perform the functional derivative (exercise), which results in

$$\boxed{\frac{\delta G[A^\alpha]}{\delta\alpha^a(x)}=\frac{1}{g}\,\partial^\mu D_\mu\ .} \tag{17.21}$$

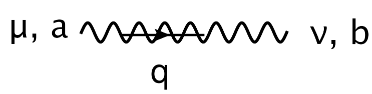
The operator that results does not depend on the gauge parameters $\alpha^a(x)$, as it is expected. However, unlike in the abelian case, it does depend on the gauge fields through the covariant derivative D_μ. Thus, the determinant of this operator depends on the field variables being integrated in the functional integral, i.e. $A_\mu(x)$, and therefore we cannot simply take it outside and absorb it in the normalization. Specifically, we see that

$$\det\left[\frac{\delta G[A^\alpha]}{\delta\alpha^a(x)}\right] = \frac{1}{g}\det\left[\partial^\mu D_\mu\right] \ . \tag{17.22}$$

The functional integral is now

$$Z[J_\mu] = \int \mathcal{D}A_\mu \det\left[\frac{1}{g}\,\partial^\mu D_\mu\right] \; e^{iS[A_\mu]-i\int d^4x\frac{(\partial^\mu A^a(x))^2}{2\xi}+i\int d^4x J_\mu(x)A^\mu(x)} \ . \tag{17.23}$$

From (17.23) we can read off the propagator for the non abelian gauge boson: it will be exactly the same as the one we derived for the photon in QED via the functional integral:



$$= \qquad \frac{-i}{q^2+i\epsilon}\left(g^{\mu\nu}-(1-\xi)\frac{q^\mu q^\nu}{q^2}\right)\delta^{ab}$$

where the delta function for the non abelian indices is the only difference from the photon propagator and, just as in the case of the photon, we obtain the Feynman gauge for $\xi = 1$.

17.2 Fadeev-Popov Ghost Fields

To proceed further, we notice that the determinant can be interpreted as resulting from integrating out *fermionic* degrees of freedom (remember that if they were bosonic , the result would be $\det[\mathcal{O}]^{-1/2}$). In particular, we can write

$$\det\left[\frac{1}{g}\,\partial^\mu D_\mu\right] = \int \mathcal{D}c\,\mathcal{D}\bar{c}\; e^{i\int d^4x\,\bar{c}\,(-\partial^\mu D_\mu)\,c} \ , \tag{17.24}$$

where in the right hand side we absorbed the coupling g by redefining $c \to \sqrt{g}\,c$. Here c and $\bar{c}$ are anticommuting Grasmann variables. However, there *must be bosonic in nature* since they are scalar under Lorentz transformations. Therefore they violate the statistics theorem. But we need not worry, these states are not physical. They are called Fadeev-Popov ghost fields and since they only couple to gauge fields and do not couple to external sources they can only appear in loops.

The ghost fields are scalars transforming in the adjoint representation of $\mathbf{G}$. Therefore

$$D_\mu c^a = \partial_\mu c^a + g f^{abc} A^b_\mu\, c^c \ . \tag{17.25}$$

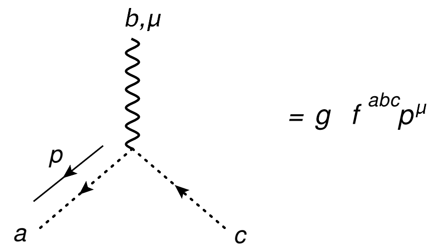


Figure 17.1: Feyman rule for the interaction of ghost fields with the gauge field.

Thus it is convenient to wrtite the covariant derivative as the operator

$$D_\mu^{ac} = \delta^{ac}\partial_\mu + gf^{abc}A_\mu^b \ . \tag{17.26}$$

Then the determinant in (17.23) results in a new term in the gauge lagrangian involving the ghost fields

$$\mathcal{L}_{\rm gauge} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}\left(\partial^\mu A_\mu^a\right)^2 + \bar{c}^a\left(-\partial^\mu D_\mu^{ac}\right)c^c \ . \tag{17.27}$$

As mentioned above, the ghost fields only interact with gauge bosons. To derive the corresponding Feynman rule, we focus on the last term in (17.27)

$$\mathcal{L}_{\rm ghosts} = \bar{c}^a\left(-\delta^{ab}\partial^2 - gf^{abc}\partial^\mu A_\mu^b\right)c^c \ , \tag{17.28}$$

where we should be careful to notice that the derivative in the second term applies to both the gauge and ghost fields. From this we derive the ghost propagator

$$D_G^{ab}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\delta^{ab}\,\frac{i}{k^2}\,e^{-ik\cdot(x-y)} \ , \tag{17.29}$$

and the ghost-gauge field interaction

$$\mathcal{L}_{G-A} = -gf^{abc}\bar{c}^a\partial^\mu(A_\mu^b c^c) \ . \tag{17.30}$$

This results in the Feynman rule illustrated in Figure 17.1. The momentum is on the line with the a superscript as a result of integration by parts in (17.30).

The appearance of the ghost fields is a reflection of gauge invariance, as imposed by the quantization procedure in the functional integral. The Fadeev-Popov trick of introducing the ghosts allows us to derive the Feyman rules from the generating functional $Z[J]$ as usual. Thus, the presence of the ghost fields guarantees that gauge invariance is preserved. This can be seen, for instance, in processes with external gauge bosons. In the absence of ghosts, there will be contributions coming from unphysical polarizations. These are cancelled by the contributions from ghost fields. This is seen very clearly in the non-interacting theory. In this case, the ghost determinant is simply

$$\det\left[-\partial^2\right] \ . \tag{17.31}$$

On the other hand, the non-interacting gauge bosons will result in their own bosonic determinant (one for each spatial component A_μ)

$$\det\left[-\partial^2\right]^{-2} , \tag{17.32}$$

where the -2 results from $-d/2$ in four dimensions. We see that in this simple example, the ghost determinant does cancell two of the contributions from two of the spatial components of the gauge fields. These are the two unphysical polarizations, spacelike and timelike. To see how these cancellation occurs in the more general case, i.e. in the presence of interactions, it is useful to introduce a new global symmetry, BRST. We do this in what follows.

17.3 BRST Invariance

The complete lagrangian of the non-abelian gauge theory, including fermions, gauge bosons, ghosts and gauge fixing is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}\left(\partial A^a_\mu\right)^2 + \bar{c}^a(-\partial^\mu D^{ac}_\mu)c^c + \bar{\psi}\left(\not{D} - m\right)\psi \ . \tag{17.33}$$

The lagrangian above can be written in a different way by introducing a scalar auxiliary field B^a transforming in the adjoint of $\mathbf{G}$. This is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{c}^a(-\partial^\mu D^{ac}_\mu)c^c + \bar{\psi}\left(\not{D} - m\right)\psi + \frac{1}{2}\xi(B^a)^2 + B^a\partial^\mu A^a_\mu \ . \tag{17.34}$$

First, we see that the field $B^a(x)$ does not have a kinetic term. This means it does not propagate. This is the reason why we call it an *auxiliary* field. In order to convince ourselves that (17.34) is equivalent to (17.33) we integrate out the auxiliary field in the former. Then we have to perform the functional integral

$$\int \mathcal{D}B\ e^{i\int d^4x\left\{(1/2)\xi(B^a)^2 + B^a\partial^\mu A^a_\mu\right\}} , \tag{17.35}$$

In order to "complete the square" to decouple the auxiliary field from the gauge containing the gauge field so we we can have a quadratic form to integrate, we make the susbstitution

$$B^a \to B^a - \frac{1}{\xi}\partial^\mu A^a_\mu \ . \tag{17.36}$$

Then, we obtain

$$\int \mathcal{D}B\ e^{i\int d^4x\left\{(1/2)\xi(B^a)^2 + B^a\partial^\mu A^a_\mu\right\}} = N e^{-i\int d^4x\,\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2} , \tag{17.37}$$

where the normalization corresponds to the actual integral over the $B^a(x)$. Thus we prove that (17.34) is equivalent to (17.33), our original lagrangian for a non-abelian gauge theory.

The lagrangian in (17.34) possesses a *global* symmetry, under the following transformations:

$$\delta A_\mu^a(x) = \theta\, D_\mu^{ac} c^c(x) \tag{17.38}$$

$$\delta\psi(x) = ig\,\theta\, c^a(x) t^a\, \psi(x) \tag{17.39}$$

$$\delta c^a(x) = -\frac{1}{2}\, g\,\theta\, f^{abc}\, c^b(x)\, c^c(x) \tag{17.40}$$

$$\delta\bar{c}^a(x) = \theta\, B^a(x) \tag{17.41}$$

$$\delta B^a(x) = 0\ , \tag{17.42}$$

where θ is a Grassmann (anticommuting) infinitesimal constant. These so called BRST transformations leave (17.34) invariant. Thus non abelian gauge theories possess a global symmetry – the BRST symmetry – in addition to gauge invariance.

To check that (17.34) is invariant under the BRST transformations (17.38)–(17.42), we first recall that an infinitesimal gauge transformation is given by

$$\begin{aligned}\delta A_\mu^a(x) &= \frac{1}{g}\left(\partial_\mu\alpha^a(x) + g f^{abc} A_\mu^b(x)\alpha^c(x)\right)\\ &= \frac{1}{g}\, D_\mu^{ac}\alpha^c(x)\ .\end{aligned} \tag{17.43}$$

Thus, if in (17.38) we identify

$$g\,\theta\, c^c(x) \leftrightarrow \alpha^c(x)\ , \tag{17.44}$$

then we see that (17.38) is just a gauge transformation with the gauge parameter given by $g\,\theta\, c^c(x)$. This is correct since the product of the two scalar anticommuting variables $\theta\, c^c(x)$ results in a scalar *commuting* variable (as it should since we identify it as the gauge paramenter $\alpha^c(x)$). Similarly, the BRST variation in (17.39) corresponding to the fermion field change, corresponds to the same gauge transformation using (17.44). Thus, the first two shifts due to a *global* BRST transformation correspond to the usual shifts of the gauge and fermion fields under a specific gauge transformation involving the ghost fields as given by (17.44). The first and third terms in (17.34) are invariant under the BRST symmetry transformation.

The fourth term is obviously invariant due to (17.42). We can now consider the second and fifth terms. For the second term we have that its variation under a BRST transformation is

$$\delta\left[\bar{c}^a(-\partial^\mu D_\mu^{ac})c^c\right] = \delta\bar{c}^a(-\partial^\mu D_\mu^{ac})c^c + \bar{c}^a\,\delta\left[(-\partial^\mu D_\mu^{ac})c^c\right]\ . \tag{17.45}$$

The first term in the right hand side above is in fact

$$\delta\bar{c}^a(-\partial^\mu D_\mu^{ac})c^c = \theta B^a(-\partial^\mu D_\mu^{ac})c^c\ , \tag{17.46}$$

where we used (17.41). But this *cancels* against the BRST variation of the last term in (17.34), which is given by

$$\begin{aligned} \delta\left[B^a \partial^\mu A^a_\mu\right] &= B^a\, \partial^\mu \delta A^a_\mu \\ &= \theta B^a \partial^\mu D^{ac}_\mu c^c\ , \end{aligned} \tag{17.47}$$

where we used (17.42) to set $\delta B^a = 0$ in the first equality, and we used (17.38) in the second. Finally, the second term in (17.45) vanishes, given that

$$\delta\left[D^{ac}_\mu c^c\right] = 0\ . \tag{17.48}$$

We leave the proof of this equality as an exercise.

17.4 BRST Operator and Separation of States

We then have proved that the lagrangian of a non-abelian gauge theory, appropriately supplemented with the ghost terms, equation (17.34), is invariant under the BRST transformations (17.38)–(17.42). Thus, even after fixing the gauge in (17.33) or (17.34), (i.e. for fixed ξ) we still have a global symmetry.

In order to better understand the meaning of the BRST symmetry, we introduce the BRST operator Q. Its action on a given field is to produce a BRST variation as defined by (17.38)–(17.42), such as

$$\theta Q\phi = \delta\phi\ . \tag{17.49}$$

So, for instance we have

$$Q A^a_\mu(x) = D^{ac}_\mu c^c(x)\ , \tag{17.50}$$

or

$$Q\psi = igc^a(x)\, t^a\, \psi(x)\ , \tag{17.51}$$

and so on. The crucial property of the BRST operator Q is that is *nilpotent*:

$$\boxed{Q^2\phi = 0\ .} \tag{17.52}$$

For instance,

$$\theta^2 Q^2 A^a_\mu = \delta\left[\delta A^a_\mu\right] = \delta\left[D^{ac}_\mu c^c\right] = 0\ , \tag{17.53}$$

where in the last equality we used (17.48). Similarly, we can prove that $Q^2\psi = 0$, $Q^2 c^a = 0$, etc.

To study the consequences of (17.52) it is useful to consider the theory in the hamiltonian picture as if we had proceeded with canonical quantization. The BRST symmetry of the lagrangian described by equations (17.38)–(17.42) implies that there is a conserved charge, Q, which commutes with the hamiltonian. In our case obviously the charge Q is the one defined by (17.49). Since Q commutes with H and is nilpotent, i.e. $Q^2 = 0$, the eigenstates of H are divided in three separate subspaces. The first one is the so called kernel $\mathcal{H}_0$, such that states $|\psi_0\rangle$ belonging to it satisfy

$$Q|\psi_0\rangle = 0 \ . \tag{17.54}$$

The second subspace is $\mathcal{H}_1$ and its states satisfy

$$Q|\psi_1\rangle \neq 0 \ . \tag{17.55}$$

Finally, the subspace $\mathcal{H}_2$, called the image of Q, has states that satisfy

$$|\psi_2\rangle = Q|\psi_1\rangle \ . \tag{17.56}$$

We see that the states belonging to $\mathcal{H}_2$ have zero norm

$$\langle\psi_2|\psi_2\rangle = 0 \ . \tag{17.57}$$

They also have zero inner product with the kernel states:

$$\langle\psi_2|\psi_0\rangle = \langle\psi_1|Q|\psi_0\rangle = 0 \ . \tag{17.58}$$

In order to identify which states in the non abelian gauge theory belong to each of the subspaces, we consider the limit of the free theory ($g \to 0$) and the action of Q on the various fields. First, we recall that in addition to the two transverse polarizations of the gauge bosons there are two *unphysical* polarizations: the longitudinal and the timelike polarizations. We can combine these two to create an orthogonal basis given by

$$\epsilon^+_\mu = \frac{1}{2|\mathbf{k}|}\left(k^0, +\mathbf{k}\right) \ , \qquad \epsilon^-_\mu = \frac{1}{2|\mathbf{k}|}\left(k^0, -\mathbf{k}\right) \ , \tag{17.59}$$

which we refer to as the forward and backward polarizations, respectively. Then, we see that in the $g \to 0$ limit the action of Q on the gauge boson is to turn it into a ghost field:

$$QA^a_\mu = \partial_\mu c^a \ . \tag{17.60}$$

This means that the original gauge boson was of the forward type, since its Lorentz structure must be proportional to k_μ. Also at $g = 0$, we see from (17.40) that Q *annihilates* a ghost field. Finally, we see from (17.41) that Q turns an anti-ghost field into the auxiliary field B^a. But if we derive the equations of motion for this field from the lagrangian (17.34) we obtain

$$\frac{\partial\mathcal{L}}{\partial B^a} = \xi B^a + \partial^\mu A^a_\mu = 0 \ , \tag{17.61}$$

since there are no derivate terms. Thus we have

$$\xi B^a = -\partial^\mu A^a_\mu \ , \tag{17.62}$$

which tells us that the auxiliary field B^a can be identified as the degrees of freedom in A^a_μ that satisfy $k^\mu\epsilon_\mu \neq 0$. These correspond to the *backward* polarization ϵ^-_μ of A^a_μ. So in sum, we conclude that the *forward* polarization of A^a_μ and the anti-ghosts must belong to $\mathcal{H}_1$ since they are not annihilated by Q; the *backward* polarization of A^a_μ and the ghost fields must belong to $\mathcal{H}_2$, since they are obtained by the application of Q to other states and therefore annihilated by it; and finally the *transverse* gauge bosons belong to $\mathcal{H}_0$ since they are always annihilated by Q. Although we derived this separation in the $g \to 0$ limit, we can extend this conclusions for the single-particle asymptotic states appearing in processes involving interactions. Thus, independent physical states are those that belong to $\mathcal{H}_0$ up to a term that belong to $\mathcal{H}_2$, since these states have zero inner product either with themselves or with states belonging to the kernel of Q.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 16.
- *The Quantum Theory of Fields, Vol. I*, by S. Weinberg, Chapter 15.