Lecture 16

Feynman Rules in Non Abelian Gauge Theories

Here we press on with non-abelian gauge theories by deriving their Feynman rules. However, before we can safely apply them to compute scattering amplitudes in perturbation theory and, specially before we can study the renormalization of these gauge theories, we will see at the end of this lecture that there is something missing. In order to solve this problem, we will have to be carefull in quantizing non-abelian gauge theories, as we will do in the next lecture.

16.1 Derivation of the Feynman Rules

We start by considering a generic a theory of a fermion that transforms as

$$\psi(x) \to g(x)\,\psi(x) = e^{i\alpha^a(x)t^a}\,\psi(x)\ , \tag{16.1}$$

under a generic non abelian gauge symmetry. The lagrangian of the theory is then

$$\mathcal{L} = \bar{\psi}\,(i\,\not{D} - m)\,\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\ , \tag{16.2}$$

where the covariant is given by

$$D_\mu\psi(x) = (\partial_\mu - ig\,A^a_\mu(x)\,t^a)\,\psi(x)\ , \tag{16.3}$$

and the t^a are the generators of the gauge group G written in the appropriate representation. The non abelian field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \ . \tag{16.4}$$

As we saw earlier, this means that there will be interactions terms in the gauge field "kinetic term", the last one in (16.2). Thus, for the purpose of deriving all the Feynman rules it is convenient to split the lagrangian in (16.2) into a truly free lagrangian and interacting terms. We define

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int.}} \tag{16.5}$$

where the free lagrangian is now

$$\mathcal{L}_0 \equiv \bar{\psi}(i \not{\partial} - m)\psi - \frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right) \ . \tag{16.6}$$

On the other hand, the interaction part of the lagrangian defined in (16.5) can be itself separated into three terms given by

$$\mathcal{L}_{\text{int.}} = \mathcal{L}^f_{\text{int.}} + \mathcal{L}^{3G}_{\text{int.}} + \mathcal{L}^{4G}_{\text{int.}} \ , \tag{16.7}$$

denoting the interactions of gauge bosons with fermions,

$$\mathcal{L}^f_{\text{int.}} = g A^a_\mu \bar{\psi}\gamma^\mu t^a \psi \ , \tag{16.8}$$

the triple gauge boson interaction

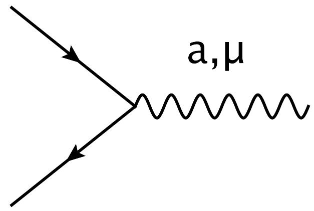
$$\mathcal{L}^{3G}_{\text{int.}} = -g f^{abc} \partial^\mu A^{a\nu} A^b_\mu A^c_\nu \ , \tag{16.9}$$

and the quartic one

$$\mathcal{L}^{4G}_{\text{int.}} = -\frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \ , \tag{16.10}$$

respectively. It is now straightforward to derive the Feynman rules from (16.8), (16.9) and (16.10).

We start with the fermion interaction. The Feynman rule is very similar to that of QED, but with the addition of the gauge group generator. This is shown in the figure below:



$$= \quad i\, g\gamma^\mu\, t^a$$

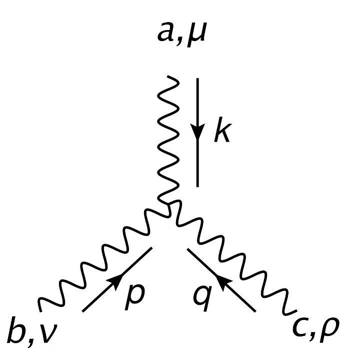
Next, we consider the triple gauge boson interaction in (16.9). Here we have to be more careful with the momentum flow since it involves a derivative on one of the gauge fields. To obtain the Feynman rule from $i\mathcal{L}^{3G}_{\rm int.}$ we need to contract it with all possible combinations of the state

$$|k, \epsilon(k);\, p, \epsilon(p);\, q, \epsilon(q)\rangle\ . \tag{16.11}$$

There are 3! such contractions. For instance, if we contract the gauge boson of momentum k with $\partial^\mu A^{a\nu}$, the one with momentum p with A^b_μ and the one with momentum q with A^c_ν, we obtain the following contribution to the Feynman rule

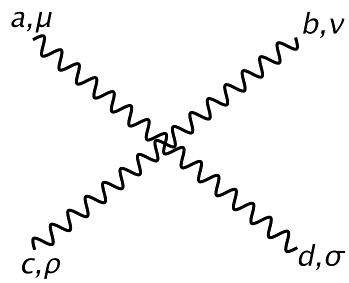
$$-igf^{abc}(-ik^\nu)g^{\mu\rho}\ . \tag{16.12}$$

This corresponds to the last term in the Feynman rule shown in the figure below. All possible 6 contractions result in the Feynman rule shown there.



$$= \quad g\, f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$$

Finally, we derive the Feynman rule for the quartic interaction from (16.10). coming from the product of the last term in $G^a_{\mu\nu}$ with the similar term in $G^{a\mu\nu}$. This is given by



$$
\begin{aligned}
= \qquad & -ig^2\,\big[f^{abe}f^{cde}\,(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\
& + \; f^{ace}f^{bde}\,(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\
& + \; f^{ade}f^{bce}\,(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\big]
\end{aligned}
$$

Notice that, although this last Feynman rule starts at order g^2, it cannot be considered of a higher order in perturbation theory than the other two. What matter is the computation of the amplitude o a given process to the desired order in g. For instance, if we wish to compute the leading order contributions to the scattering of two gauge bosons going to two gauge bosons, we see that the second Feynman rule can be used to form contributions with two vertices and one gauge boson propagator. These are of order g^2. On the other hand, the last Feynman rule is a contribution to the amplitude in and on itself. So all the leading order contributions to this process are of the same order, g^2.

16.2 Ward Identity and the Missing Link

We have seen that in QED the Ward identity being satisfied is equivalent to gauge invariance. Specifically, when we consider amplitude with an external gauge boson, such as the one depicted in Figure 16.1, (taken from Part I, lecture 18).

Generically, we can write the amplitude of such process as

$$\mathcal{A} = \epsilon_\mu(k)\,\mathcal{M}^\mu \ , \tag{16.13}$$

Assuming all external particles are on-shell[1], the Ward identity states that

$$k_\mu\,\mathcal{M}^\mu = 0 \ . \tag{16.14}$$

We would like to check the generalization of the validity of (16.14) for non-abelian gauge theories. Naively, we would expect that gauge invariance would impose it. However, as

[1]The contact terms vanish when particles are on-shell.

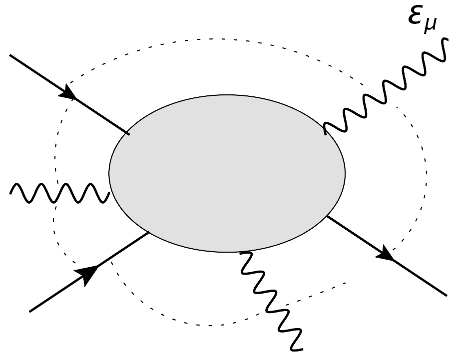


Figure 16.1: Process with an external gauge boson. Dotted lines denote possible additional external particles. All shown particles are on-shell.

we will see below, this is not the case. Or at least it is not at this stage. We will see that the problem is not that gauge invariance is not satisfied, but that we have not properly quantized the theory.

To see that we have a problem, we will compute the amplitude for a process involving external gauge bosons in a non-abelian gauge theory. In particular we will consider the pair production of gauge bosons in the scattering of fermions: $f\,\bar{f} \to V_1 V_2$. The corresponding Feynman diagrams are shown in Figure 16.2.

The first two diagrams, (a) and (b) are similar to those present in the abelian case, i.e. in QED. On the other hand, diagram (c) is a new element: it involves the interactions among three gauge bosons and it marks the non-abelian character of the gauge interaction. We then consider first the abelian part of the amplitude given by the sum of diagrams (a) and (b). This is given by

$$\begin{aligned} i\mathcal{A}^{(a+b)} &= i\mathcal{M}^{\mu\nu}_{(a+b)}\,\epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2) \\ &= (ig)^2\,\bar{v}(p_1)\gamma^\mu t^a \frac{i}{\not{p}_2 - \not{k}_2 - m}\gamma^\nu t^b u(p_2)\,\epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2) \\ &+ (ig)^2\,\bar{v}(p_1)\gamma^\nu t^b \frac{i}{\not{k}_2 - \not{p}_1 - m}\gamma^\mu t^a u(p_2)\,\epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2)\ , \end{aligned} \tag{16.15}$$

In order to test the Ward identity, we will first compute the contraction of the contributions to $\mathcal{M}^{\mu\nu}_{(a+b)}$ with one of the external momenta, say k_2, replacing the associated polarization, $\epsilon^*_\nu(k_2)$. Then we have

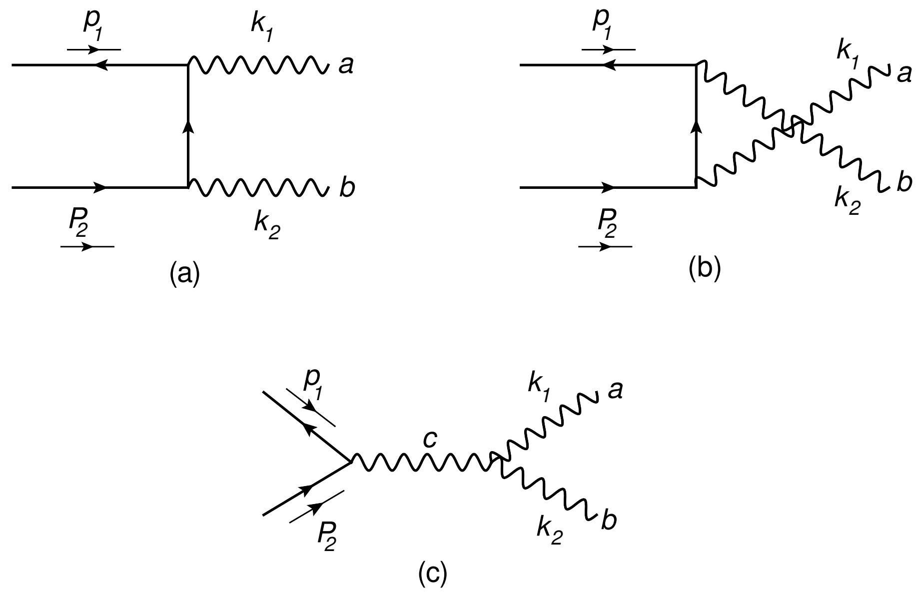


Figure 16.2: Feynman diagrams for the process $f\bar{f} \to V_1V_2$ in a non-abelian gauge theory. The fermions are both incoming with momenta (p_1 and p_2, and the gauge bosons outgoing with k_1 and k_2. The arrows in the fermion lines indicate a fermion or an anti-fermion. The indices a, b and c in the gauge bosons refer to the associated generators, t^a, t^b and t^c.

$$\begin{aligned} i\mathcal{M}^{\mu\nu}_{(a+b)}\epsilon^*_\mu(k_1)\,k_{2\nu} &= (ig)^2\bar{v}(p_1)\left\{\gamma^\mu t^a\,\frac{i}{\not{p}_2 - \not{k}_2 - m}\;\not{k}_2 t^b\right. \\ &\quad \left. + \not{k}_2 t^b\,\frac{i}{\not{k}_2 - \not{p}_1 - m}\,\gamma^\mu t^a\right\} u(p_2)\,\epsilon^*_\mu(k_1)\;. \end{aligned} \tag{16.16}$$

In order to put the expression above in a more useful form, we will make use of the Dirac equation for the spinors. In particular, using that

$$\begin{aligned} (\not{p}_2 - m)\;u(p_2) &= 0 \\ \bar{v}(p_1)\,(\not{p}_1 + m) &= 0\;, \end{aligned} \tag{16.17}$$

we rewrite (16.16) as

$$
\begin{aligned}
i\mathcal{M}^{\mu\nu}_{(a+b)}\epsilon^*_\mu(k_1)\,k_{2\nu} &= (ig)^2\bar{v}(p_1)\left\{\gamma^\mu t^a\,\frac{i}{\not{p}_2-\not{k}_2-m}\,(\not{k}_2-\not{p}_2+m)t^b\right.\\
&\qquad \left.+(\not{k}_2-\not{p}_1-m)t^b\,\frac{i}{\not{k}_2-\not{p}_1-m}\,\gamma^\mu t^a\right\}\,u(p_2)\,\epsilon^*_\mu(k_1)\ ,\\
&= (ig)^2\,\bar{v}(p_1)(-i)\gamma^\mu\left[t^a,t^b\right]u(p_2)\,\epsilon^*_\mu(k_1)\ ,
\end{aligned}
\tag{16.18}
$$

where in the second line we simply collected the two terms after cancelling the propagators. Using the algebra of the group

$$
[t^a,t^b] = i\,f^{abc}\,t^c\ , \tag{16.19}
$$

we obtain

$$
i\mathcal{M}^{\mu\nu}_{(a+b)}\,\epsilon^*_\mu(k_1)\,k_{2\nu} = (ig)^2\,f^{abc}\,\bar{v}(p_1)\,\gamma^\mu\,t^c\,u(p_2)\,\epsilon^*_\mu(k_1)\ . \tag{16.20}
$$

The expression above is clearly not equal to zero. However, its form being proportional to the structure constants f^{abc} suggests that it might in fact be cancelled by the contribution of diagram (c) in Figure 16.2, which according to the Feynman rules derived for the triple gauge boson interaction in the previous section, contains such dependence. To check if this is the case, we first write the amplitude for diagram (c):

$$
\begin{aligned}
iA^{(c)} &= i\mathcal{M}^{\mu\nu}_{(c)}\,\epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2)\\
&= ig\,\bar{v}(p_1)\,\gamma_\rho\,t^c\,u(p_2)\,\frac{(-i)}{k_3^2}\,g\,f^{abc}\\
&\times\ \left\{g^{\mu\nu}(k_2-k_1)^\rho+g^{\nu\rho}(k_2-k_3)^\mu+g^{\rho\mu}(k_1-k_2)^\nu\right\}\epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2)\ .
\end{aligned}
\tag{16.21}
$$

We now replace one of the external polarizations, say $\epsilon^*_\nu(k_2)$, by the associated momentum $k_{2\nu}$. We obtain

$$
\begin{aligned}
i\mathcal{M}^{\mu\nu}_{(c)}\,\epsilon^*_\mu(k_1)\,k_{2\nu} &= ig^2\,\bar{v}(p_1)\,\gamma_\rho\,t^c\,u(p_2)\,\frac{(-i)}{k_3^2}\,\epsilon^*_\mu(k_1)\,f^{abc}\\
&\times\ \left\{(k_2^\mu(k_2-k_1)^\rho+k_2^\rho(k_2-k_3)^\mu+g^{\rho\mu}(k_1-k_2)\cdot k_2\right\}\ .
\end{aligned}
\tag{16.22}
$$

The expression above can be greatly simplified by using momentum conservation at the vertices. For instance, using

$$k_2 = -k_1 - k_3 \ , \tag{16.23}$$

and substituting k_2 in (16.22), we obtain

$$\begin{aligned} i\mathcal{M}^{\mu\nu}_{(c)}\,\epsilon^*_\mu(k_1)\,k_{2\nu} &= ig^2\,\bar{v}(p_1)\,\gamma_\rho\,t^c\,u(p_2)\,\frac{(-i)}{k_3^2}\,\epsilon^*_\mu(k_1)\,f^{abc} \\ &\times \left\{g^{\rho\mu}k_3^2 - k_3^\rho\,k_3^\mu - g^{\rho\mu}\,k_1^2 + k_1^\rho\,k_1^\mu\right\} . \end{aligned} \tag{16.24}$$

We can already verify that the first term in the last line in the brackets in (16.24) results in a total cancellation with $i\mathcal{M}^{\mu\nu}_{(a+b)}\,\epsilon^*_\mu(k_1)\,k_{2\nu}$ in (16.20). Thus, if we can argue that the remaining three contributions vanish, we would prove the Ward identity.

First, we consider the obvious: since the external gauge bosons are set on shell, we have $k_1^2 = 0$, which makes the third term in the brackets trivially zero.

Let us now consider the second term in the brackets in (16.24): $-k_3^\rho k_3^\mu$. The first four vector is contracted with the gamma matrix resulting in a factor of

$$\begin{aligned} \bar{v}(p_1)\ \not{k}_3\,t^c\,u(p_2) &= -\bar{v}(p_1)\ (\not{p}_1 + \not{p}_2)\ u(p_2) \\ &= -\bar{v}(p_1)\ (-m+m)\ u(p_2) = 0\ , \end{aligned}$$

where we used $p_1 + p_2 = -k_3$, momentum conservation in the left vertex in diagram (c) of Figure 16.2. So this contribution is also zero.

Finally, the last term in the brackets gets contracted with the polarization, resulting in a factor

$$\epsilon^*_\mu(k_1)\,k_1^\mu, k^\rho\ . \tag{16.25}$$

But, if we impose that this external gauge boson is *transverse*, then it should be satisfied that

$$\epsilon^*(k_1)\cdot k_1 = 0\ , \tag{16.26}$$

which means that this term also has a vanishing contribution. This would complete our test of the Ward identity, i.e. it would prove that

$$\boxed{i\mathcal{M}^{\mu\nu}_{(a+b+c)}\,\epsilon^*_\mu(k_1)\,k_{2\nu} = 0}\ . \tag{16.27}$$

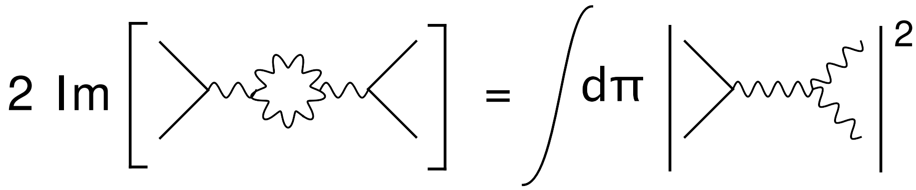


Figure 16.3: The optical theorem and non abelian gauge theory pair production of gauge bosons.

Unfortunately, this proof is incorrect. The culprit is the very last step, assuming that external gauge bosons are transverse and therefore satisfy (16.26). In QED, when we test the Ward identity, the fact that the longitudinal and time-like polarizations do not contribute is automatic and does not need to be imposed by hand. It is a consequence of gauge invariance. Here, however, we seem to be forced to impose (16.26) in order to satisfy the Ward identity. This should not be necessary. On the other hand, not imposing transversality we seem to be concluding that the polarizations that should be non-physical not only violate the Ward identity, but also seem to be contributing to physical observables.

One could that we could just ignore the contributions of the unphysical polarizations to the process at hand: the pair production of two gauge bosons. However, this would be inconsistent with the optical theorem. According to it, the cross section in question (in leading order in perturbation theory) should be related to the imaginary part of the one loop amplitude of $f\bar{f} \to f\bar{f}$ scattering, as shown in Figure 16.3.

In the loop diagram we need to include all gauge boson polarizations, which are contained in each factor of $g_{\mu\nu}$ gauge boson propagators. This means that we cannot simply choose to ignore the unphysical contributions in the pair production amplitude in the right of the Figure. The optical theorem forces us to keep these contributions. The meaning of their presence will be revealed when we realized that we have not properly quantized the non abelian gauge theory. We will do this in the next lecture.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 15.

- *Quantum Field Theory* , by M. Srednicki, Chapter 69.