Lecture 15

Non Abelian Gauge Symmetries

We consider here the generalization of the concept of gauge invariance when the gauge group G is non abelian. Below, we will see what this means by presenting the basics of non abelian group theory. We will also study the physical consequences of non abelian gague invariance. But before we do all that, we will take another look at abeliang gauge theory, i.e. when $G = U(1)$, by thinking about gauge invariance in a geometric way.

15.1 Gauge Invariance and Geometry

We consider the action of a $U(1)$ local symmetry transformation on a fermion field $\psi(x)$. It is given by

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\,\psi(x)\ . \tag{15.1}$$

As we well know, terms in the lagrangian that do no contain derivatives are trivially invariant under (15.1). For instance, the fermion mass term transforms as

$$m\bar{\psi}\psi \to m\bar{\psi}'\psi' = m\bar{\psi}\psi\ . \tag{15.2}$$

However, terms containing derivatives are not invariant. Let us study in detail how the problem arises. We write the derivative by using a direction in spacetime defined by a four-vector n_μ, such that

$$n^\mu\partial_\mu\psi(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,[\psi(x+\epsilon n) - \psi(x)]\ , \tag{15.3}$$

where the argument of the first term on the left hand side must be understood as

$$x_\mu + \epsilon n_\mu = x_\mu + \Delta x_\mu \ . \tag{15.4}$$

But the fields $\psi(x+\epsilon n)$ and $\psi(x)$ have *different* gauge transformations as clearly seen from (15.1). The fact that they are evaluated in different spacetime points means that the gauge parameters of their transformations are different, i.e. $\alpha(x+\epsilon n)$ and $\alpha(x)$. This translates in $\partial_\mu\psi(x)$ not having a well defined gauge transformation.

The situation is similar to what happens in general relativity when we want to compare two objects with non-trivial transformation properties, e.g. vectors or spinors, at two different positions in spacetime. For instance, if the objects being compared are two vectors, then part of the variation comes from the fact that the curvature will change the orientation of a vector as we move it from one point to another. But we are interested in the *intrinsic* variation due to some dynamical effect. For this purpose we define a *parallel transport*. Our case is no different.

We define the scalar function

$$U(y,x) \ , \tag{15.5}$$

depending on two spacetime points x and y in such as way that it transforms under the $U(1)$ gauge symmetry as

$$U(y,x) \to e^{i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)} \ . \tag{15.6}$$

We call $U(y,x)$ a comparator. This clearly means that $U(y,y)=1$. Also, it means that

$$U(y,x)\,\psi(x) \to e^{i\alpha(y)}\, U(y,x)\,\psi(x) \ . \tag{15.7}$$

Thus, the product of the comparator times the field in x, transforms as an object located in y. We can use this to define a new derivative as

$$n^\mu D_\mu \equiv \lim_{\epsilon\to 0} \frac{1}{\epsilon}\,\left[\psi(x+\epsilon n) - U(x+\epsilon n, x)\,\psi(x)\right] \ , \tag{15.8}$$

so that the two terms being subtracted transform in the same way under the gauge symmetry. This is the case given that under a $U(1)$ gauge transformation

$$\begin{aligned} \psi(x+\epsilon n) \quad &\to \quad e^{i\alpha(x+\epsilon n)}\,\psi(x+\epsilon n) \\ U(x+\epsilon n, x)\,\psi(x) \quad &\to \quad e^{i\alpha(x+\epsilon n)}\, U(x+\epsilon n, x)\,\psi(x) \ . \end{aligned} \tag{15.9}$$

Based on the definition of the covariant derivative in (15.8) we can recover the familiar form of $D_\mu\psi(x)$. For this purpose, we first expand the comparator at leading order in ϵ as

$$U(x+\epsilon n, x) = 1 - i\epsilon n^\mu\, A_\mu(x) + \mathcal{O}(\epsilon^2)\ , \tag{15.10}$$

where the linear term in the expansion must depend also on the direction n^μ, but then this Lorentz index must be contracted with a four-vector that generally depends on x, which we call $A_\mu(x)$. Implicit in the form of the expansion we used in (15.10) is the assumption that the comparator can be written as a phase, since the normalization can always be absorbed in redefinitions of the fields, here $\psi(x)$. Replacing (15.10) in (15.8) we have

$$\begin{aligned} n^\mu\, D_\mu\, \psi(x) &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\, \left[\psi(x+\epsilon n) - \psi(x) + i\epsilon n^\mu A_\mu(x)\right] \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\, \left[\psi(x+\epsilon n) - \psi(x)\right] + i n^\mu A_\mu(x)\ , \end{aligned} \tag{15.11}$$

where we neglected terms of higher order in ϵ since they do not contribute when taking the limit $\epsilon \to 0$. The first term above is just the normal derivative as defined in (15.3), so we obtain

$$D_\mu\psi(x) = \partial_\mu\psi(x) + iA_\mu(x)\psi(x)\ , \tag{15.12}$$

which is of course the usual definition of the covariant derivative. The vector field $A_\mu(x)$ will also transform under the gauge symmetry. To extract its transformation law, we need to look at the expansion of the transformation of the comparator $U(y,x)$ which defines $A_\mu(x)$. This is,

$$\begin{aligned} U(x+\epsilon n, x) &\to e^{i\alpha(x+\epsilon n)}\, U(x+\epsilon n, x)\, e^{-i\alpha(x)} \\ 1 - i\epsilon\, n^\mu A_\mu(x) + \cdots &\to (1 + i\alpha(x+\epsilon n) + \cdots)\,(1 - i\epsilon\, n^\mu A_\mu(x) + \cdots)\,(1 - i\alpha(x) + \cdots) \\ &\to 1 + i(\alpha(x+\epsilon n) - \alpha(x)) - i\epsilon\, n^\mu A_\mu(x) + \cdots\ , \end{aligned} \tag{15.13}$$

where the dots indicate both higher orders in ϵ and in the α's. We point out that we are not using an infinitesimal $\alpha(x)$, but that the higher orders terms in α actually identically cancel. Dividing both sides of (15.13) by ϵ and taking the limit for $\epsilon \to 0$ we obtain

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha(x) \;, \tag{15.14}$$

as expected. Combining (15.14) and (15.12) one can easily verify that the covariant derivative transforms as

$$D_\mu \psi(x) \to e^{i\alpha(x)}\, D_\mu \psi(x) \;, \tag{15.15}$$

which guarantees that all terms in the lagrangian are now gauge invariant if the covariant derivative replaces the normal derivative. That is, the first term in

$$\mathcal{L} = \bar{\psi} i \gamma^\mu D_\mu \psi - m \bar{\psi}\psi \;, \tag{15.16}$$

is $U(1)$ gauge invariant since the covariant derivative of $\psi(x)$ transforms as the field $\psi(x)$. A final question is the definition of a kinetic term for the *connection* field $A_\mu(x)$. Here, we will make use of a method that, although appears too complicated for the abelian case, it will be very useful when applied to non abelian gauge theories later. What we are after is a term that depends quadratically on derivatives of $A_\mu(x)$. What we will start with is the following differential operator applied to the fermion field:

$$[D_\mu, D_\nu]\, \psi(x) \;. \tag{15.17}$$

This is the commutator of the covariant derivatives applied $\psi(x)$. Using (15.15) is easy to verify that (15.17) transforms like the field, that is

$$[D_\mu, D_\nu]\, \psi(x) \to e^{i\alpha(x)}\, [D_\mu, D_\nu]\, \psi(x) \;. \tag{15.18}$$

This can be interpreted as a transformation rule for the commutator:

$$[D_\mu, D_\nu] \to e^{i\alpha(x)}\, [D_\mu, D_\nu]\, e^{-i\alpha(x)} \;. \tag{15.19}$$

On the other hand, we can explicitly compute the commutator by using (15.12). This is

$$\begin{aligned} [D_\mu, D_\nu]\, \psi(x) &= [\partial_\mu + iA_\mu, \partial_\nu + iA_\nu]\, \psi(x) \\ &= i\, (\partial_\mu A_\nu - \partial_\nu A_\mu)\; \psi(x) \;, \end{aligned} \tag{15.20}$$

which reveals that the commutator of the covariant derivatives is itself *not* a differential operator.

We then define

$$[D_\mu, D_\nu] \equiv i\, F_{\mu\nu} \ , \tag{15.21}$$

which is clearly gauge invariant, since the commutator transformation rule (15.19) implies

$$F_{\mu\nu} \to e^{i\alpha(x)}\, F_{\mu\nu}\, e^{-i\alpha(x)} = F_{\mu\nu} \ . \tag{15.22}$$

This can be alternatively seen from (15.18) in combination with the field transformation (15.1), since the commutator is not a differential operator. Then, two powers of $F_{\mu\nu}$ would give us what we want for a gauge field kinetic term.

This concludes our rederivation of the $U(1)$ gauge invariant lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cdots \ , \tag{15.23}$$

where the factor of $-1/4$ is necessary to recover the electromagnetic strength tensor in the classical limit, and the dots denote possible gauge invariant higher dimensional (non-renormalizable) terms.

15.2 Non Abelian Gauge Groups

We will now follow the same geometric procedure we applied for a $U(1)$ gauge theory for the case of non abelian groups. We first consider the case of $G = SU(2)$ and later generalize our results for arbitrary non abelian groups. $SU(2)$ is isomorphic with $SO(3)$ the group of rotations in 3 dimensions, so it should be familiar from the study of angular momentum in quantum mechanics. The elements of $SU(2)$ are unitary matrices which we write as

$$g(x) = e^{i\alpha^a(x)t^a} \ , \tag{15.24}$$

where t^a are the generators (three of them from $2^2 - 1$), which are given in terms of the Pauli matrices by

$$t^a = \frac{\sigma^a}{2} \ , \tag{15.25}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{15.26}$$

As we see from (15.24), there are 3 coefficient functions of x, $\alpha^1(x)$, $\alpha^2(x)$ and $\alpha^3(x)$, so that the exponent is the most general x dependent expansion of the generators. Let us consider, just as in the previous section for the $U(1)$ case, the transformation of a fermion field under a $SU(2)$ gauge group. This is given by

$$\psi(x) \to \psi'(x) = e^{i\alpha^a(x)t^a}\,\psi(x) = g(x)\,\psi(x) . \tag{15.27}$$

If a fermion field does transform as in (15.27) this implies that it has an $SU(2)$ internal index. Depending on the representation under which they transform they will be different *multiplets*. The *fundamental* representation correspond to using (15.25) and implies that the fermion field is an $SU(2)$ *doublet*

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \\ \psi_2(x) \end{pmatrix} , \tag{15.28}$$

which means there are two fermions. The local transformation (15.27) mixes these two components.

We are now in a position to define the covariant derivative. Just as before, we define the comparator $U(y, x)$ with the gauge transformation property

$$U(y, x) \to g(y)\,U(y, x)\,g^\dagger(x) . \tag{15.29}$$

and just as for the $U(1)$ case before, the covariant derivative has the geometric definition

$$n^\mu D_\mu \psi(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\,\left[\psi(x+\epsilon n) - U(x+\epsilon n, x)\,\psi(x)\right] . \tag{15.30}$$

Noticing that

$$U(y, y) = \mathbb{1} , \tag{15.31}$$

the identity in 2×2 matrices, we can expand $U(y, x)$ around this considering infinitesimal gauge transformations $\alpha^a(x) \sim \mathcal{O}(\epsilon)$. The most general expansion to leading order is

$$U(x+\epsilon n, x) = \mathbb{1} + ig\epsilon n^\mu A^a_\mu(x)\,t^a + \mathcal{O}(\epsilon^2) , \tag{15.32}$$

where we included a factor g, the coupling, and the Lorentz index in n^μ is contracted by the fields $A^a_\mu(x)$, where the index a contracts with the one in the generator. This reflects the fact that the most general expansion is a linear combination of the 3 Pauli matrices, meaning that now we will have 3 gauge fields, $A^1_\mu(x)$, $A^2_\mu(x)$ and $A^3_\mu(x)$. Then, we have

$$\begin{aligned} n^\mu D_\mu \psi(x) &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\,\left[\psi(x+\epsilon n) - U(x+\epsilon n, x)\,\psi(x)\right] \\ &= n^\mu \partial_\mu \psi(x) - ign^\mu\, A^a_\mu(x)\, t^a \psi(x)\ , \end{aligned} \tag{15.33}$$

which results in the covariant derivative

$$D_\mu \psi(x) = \left(\partial_\mu - ig\, A^a_\mu(x)\, t^a\right)\psi(x)\ . \tag{15.34}$$

For the case at hand, i.e. $G = SU(2)$ the generators in (15.34) are one half of the Pauli matrices. This is the covariant derivative acting on a fermion ψ that trasnforms under the $SU(2)$ gauge group as in (15.27). As we will see below, this determines the interactions of fermions with the $SU(2)$ gauge bosons $A^a_\mu(x)$.

The next step is to obtain the gauge transformations for the gauge fields. Once again, to do this we consider the infinitesimal gauge transformation of the comparator. Using (15.29) this is given by

$$\begin{aligned} U(x+\epsilon n, x) &\to g(x+\epsilon n)\, U(x+\epsilon n, x)\, g^\dagger(x) \\ \mathbb{1} + ig\,\epsilon n^\mu A^a_\mu(x)\, t^a &\to g(x+\epsilon n)\,\left(\mathbb{1} + ig\epsilon n^\mu A^a_\mu(x)\, t^a\right) g^\dagger(x)\ , \end{aligned} \tag{15.35}$$

where in the second line we use the expansion in (15.32). We notice that

$$\begin{aligned} g(x+\epsilon n)\, g(x) &= \left[\left(\mathbb{1} + \epsilon n^\mu \frac{\partial}{\partial x^\mu} + \mathcal{O}(\epsilon^2)\right) g(x)\right]\, g^\dagger(x) \\ &= \mathbb{1} + \epsilon\, n^\mu \partial_\mu(g(x))\, g^\dagger\ . \end{aligned} \tag{15.36}$$

Replacing the equation above in (15.35), we have that

$$A^a_\mu(x)\, t^a \to g(x)\,\left(A^a_\mu(x)\, t^a\right)\, g^\dagger(x) - \frac{i}{g}\,(\partial_\mu g(x))\, g^\dagger(x)\ . \tag{15.37}$$

If we now define the gauge field matrix

$$A_\mu(x) \equiv A_\mu^a(x)\, t^a \ , \tag{15.38}$$

we can rewrite (15.37) as

$$\boxed{A_\mu(a) \to g(x)\left(A_\mu(x) + \frac{i}{g}\,\partial_\mu\right)\, g^\dagger(x)} \ , \tag{15.39}$$

where we have used the fact that $g^\dagger g = g g^\dagger = \mathbb{1}$ in order to make the replacement

$$\partial_\mu(g(x))\, g^\dagger(x) = -g(x)\partial_\mu g^\dagger(x) \ . \tag{15.40}$$

The gauge transformation of the matrix gauge field (15.39) is actually valid for any non abelian gauge group, not just $SU(2)$, as long as $g(x)$ is a group element expressed in terms of the generators t^a as in (15.24). We can also recover the *abelian* gauge field transformation (15.14) if we replace t^a by the identity and $\alpha^a(x)$ is just $\alpha(x)$. However this is deceiving since there are new contributions that appear exclusively in the non abelian case. To see this in the gauge field transformation, we consider an infinitesimal gauge transformation with

$$g(x) = \mathbb{1} + i\,\alpha^a(x)\, t^a + \cdots \tag{15.41}$$

where the dots denote terms higher in powers of $\alpha^a(x)$. Replacing (15.41) in (15.39) we arrive at

$$A_\mu^a(x)\, t^a \to A_\mu^a(x)\, t^a + \frac{1}{g}\,\partial_\mu\alpha^a(x)\, t^a + i\,\left[\alpha^a(x)\, t^a, A_\mu^b(x)\, t^b\right] + \cdots \ . \tag{15.42}$$

The first two terms in (15.42) are analogous to what we find in the abelian case. But the third term is only present in non abelian gauge groups since it is proportional to the commutator of two generators. We will see below that this non commutativity has important physical consequences.

With the definition of the covariant derivative in (15.34) and the gauge field tranformation (15.39) we can prove that the fermion kinetic term given by

$$\bar{\psi}\gamma^\mu D_\mu\psi \ , \tag{15.43}$$

is invariant under the gauge transformations (15.27). This means that under these gauge transformations

$$D_\mu\psi(x) \to g(x)\, D_\mu\psi(x) \ , \tag{15.44}$$

must be satisfied. This can be explicitly verified just by substitution.

The final step, just as in the abelian case considered earlier, is to obtain the kinetic term for the gauge fields. Following the steps taken there, we need to compute

$$[D_\mu, D_\nu]\psi(x) \ . \tag{15.45}$$

Using the matrix notation (15.38) and replacing the explicit form of the covariant derivative (15.34) in (15.45) we obtain

$$[D_\mu, D_\nu]\psi(x) = -ig\ (\partial_\mu A_\mu - \partial_\nu A_\mu)\,\psi(x) - g^2\ [A_\mu, A_\nu]\,\psi(x) \ . \tag{15.46}$$

Once again, just as for the abelian case, we see that the commutator in (15.45) is not a differential operator. But unlike for the abelian case, there is a new term proportional to the commutator

$$[A_\mu, A_\nu] = A_\mu^a A_\nu^b\,[t^a, t^b] \ . \tag{15.47}$$

Defining the gauge field strength (matrix) by

$$[D_\mu, D_\nu]\psi(x) \equiv -ig\, F_{\mu\nu}\,\psi(x) \ , \tag{15.48}$$

we have that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu] \ , \tag{15.49}$$

which can be expressed in gauge field components using (15.38) to give

$$F_{\mu\nu} = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\ t^a - ig\, A_\mu^a A_\nu^b\,[t^a, t^b] \ . \tag{15.50}$$

Defining the gauge field strength $F_{\mu\nu}^a$ by

$$F_{\mu\nu} \equiv F_{\mu\nu}^a\, t^a \ , \tag{15.51}$$

and writing the commutator out in terms of the structure constants we arrive at

$$\boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu} \ , \tag{15.52}$$

which is the non abelian gauge field strength in all generality. For instance for and $SU(2)$ gauge theory we have

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu \ , \tag{15.53}$$

since the structure constants are given by the epsilon tensor ϵ^{abc}.

Now, given (15.44), we know that the commutator acting on the fermion field transforms as

$$[D_\mu, D_\nu]\psi(x) \to g(x)\,[D_\mu, D_\nu]\psi(x), \tag{15.54}$$

which results in the gauge transformation for the commutator

$$[D_\mu, D_\nu] \to g(x)\,[D_\mu, D_\nu]\,g^\dagger(x) \ . \tag{15.55}$$

Then, using (15.48), we obtain the gauge transformation for the matrix $F_{\mu\nu}$:

$$F_{\mu\nu} \to g(x)\,F_{\mu\nu}\,g^\dagger(x) \ . \tag{15.56}$$

We can use this information to guess the form of the gauge invariant kinetic term. From (15.56), we see that $F_{\mu\nu}$ is not gauge invariant, unlike what happens in the abelian case. Then, although

$$F_{\mu\nu}F^{\mu\nu} \to g(x)\,F_{\mu\nu}F^{\mu\nu}\,g^\dagger(x) \ , \tag{15.57}$$

is not gauge invariant, its trace actually is. Then, we have

$$\begin{aligned} \mathrm{Tr}[\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}] &= F^a_{\mu\nu}\,F^{b\mu\nu}\,\mathrm{Tr}[\mathrm{t^a t^b}] \\ &= F^a_{\mu\nu}\,F^{b\mu\nu}\,\frac{\delta^{ab}}{2} \ , \end{aligned} \tag{15.58}$$

so the form of the kinetic term that corresponds to the abelian normalization is

$$\boxed{-\frac{1}{2}\mathrm{Tr}[\mathrm{F}_{\mu\nu}\,\mathrm{F}^{\mu\nu}] = -\frac{1}{4}\mathrm{F}^{\mathrm{a}}_{\mu\nu}\,\mathrm{F}^{\mathrm{a}\mu\nu}}\ . \tag{15.59}$$

Although at first the form of the gauge kinetic term above looks just like a simple sum of the kinetic terms of the individual gauge bosons (for $a = 1, \ldots, N^2 - 1$), this is deceiving. When plugging in the explicit form of $F^a_{\mu\nu}$ from (15.52) we see that (15.59) not only leads to terms quadratic in the derivatives of each of the fields, but also to interactions among the gauge fields: there will be a triple interaction and a quartic one. This is a crucial feature of non abelian gauge theories: the gauge bosons interact with each other, whereas this is not the case for the gauge bosons of the abelian $U(1)$, e.g. the photons. This will have very important consequences, from the behavior of scattering amplitudes to the renormalization group flow.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 15.
- *Quantum Field Theory* , by M. Srednicki, Chapter 69.