Lecture 14

Lie Algebras and Non Abelian Symmetries

Non abelian gauge theories are based on non abelian continuous groups. These are defined by the fact that they include elements that can be continuously deformed into the identity. For them then we have that

$$g \in G/ \tag{14.1}$$

the we can write

$$g(\alpha) = 1 + i\alpha^a t^a + \mathcal{O}(\alpha^2) \ , \tag{14.2}$$

where the α^a's are infinitesimally small real parameters, summation over the index a is understood and the t^a are called the generators of the group G. The definition (14.2) implies

$$g(0) = 1 \ . \tag{14.3}$$

If $g(\alpha)$ is *unitary* then the t^a must be a set of linearly independent *hermitian* operators. Groups defined by these properties are called Lie groups.

In order to obtain the defining property of Lie groups (its algebra) we start by defining the group's multiplication. The multiplication of two elements of the group results in another element of G:

$$g(\alpha)\, g(\beta) = g(\xi) \ , \tag{14.4}$$

where the real paramenters of the product element satisfy

$$\xi^a = f(\alpha^a, \beta^a) \ , \tag{14.5}$$

with f a continuously differentiable function of the α^a's and the β^a's. We can conclude various things about f. For instance,

$$f(\alpha^a, 0) = \alpha^a \ , \tag{14.6}$$

and similarly for $\alpha = 0$. On the other hand, if in (14.4) we have that

$$g(\beta) = g^{-1}(\alpha) \ , \tag{14.7}$$

then it must be that

$$f(\alpha, \beta) = 0 \ . \tag{14.8}$$

Armed with this knowledge we are going to compute the following quadruple multiplication:

$$g(\alpha)\, g(\beta)\, g^{-1}(\alpha)\, g^{-1}(\beta) = g(\xi) \ . \tag{14.9}$$

We will first focus on the left hand side of (14.9). This is given by

$$(1 + i\alpha^a t^a + \cdots)\,(1 + i\beta^b t^b + \cdots)\,(1 - i\alpha^c t^c + \cdots)\,(1 - i\beta^d t^d + \cdots) \ , \tag{14.10}$$

from which we can see that the terms linear in α and β cancel. Multiplying the first order parameters and keeping only up to second order products we have

$$1 - \alpha^a\beta^b t^a t^b + \alpha^a\alpha^c t^a t^c + \alpha^a\beta^d t^a t^d + \beta^d\alpha^c t^b t^c + \beta^b\beta^d t^b t^d - \alpha^c\beta^d t^c t^d + \cdots \ , \tag{14.11}$$

where the dots include the second order terms in the expansions of the g's and they will also contain second order products of α's and β's which are not explicitly written in (14.11). In fact, it is easy to see that the *third* and *sixth* terms in (14.11) actually are cancelled by them. Then the left hand side of (14.9) up to leading order in the infinitesimal paramenters α and β is given by

$$1 + \beta^b\alpha^c\,[t^b, t^c] + \cdots \ , \tag{14.12}$$

where $[t^b, t^c] = t^b t^c - t^c t^b$ is the commutator of the generators.
Now let us consider the right hand side of (14.9). We know that

$$\xi = f(\alpha, \beta) \ . \tag{14.13}$$

Then the most general expansion of ξ in terms of α and β is given by

$$\xi^e = A^e + B^{ef}\alpha^f + \tilde{B}^{ef}\beta^f + C^{efg}\alpha^f\beta^g + \tilde{C}^{efg}\alpha^f\alpha^g + \hat{C}^{efg}\beta^f\beta^g + \cdots \ , \tag{14.14}$$

where A^e, B^{ef}, $\tilde{B}^{ef}$, C^{efg}, $\tilde{C}^{efg}$ and $\hat{C}^{efg}$ are arbitrary real coefficients, and the dots correspond to terms with more than two infinitesimal parameters. However, since using (14.3), (14.7) and (14.8) we know that the function in (14.13) satisfies

$$f(\alpha, 0) = f(0, \beta) = 0 \ , \tag{14.15}$$

we immediately conclude that

$$A^e = B^{ef} = \tilde{B}^{ef} = \tilde{C}^{efg} = \hat{C}^{efg} = 0 \ . \tag{14.16}$$

Then we conclude that

$$\xi^e = C^{efg}\, \alpha^f \beta^g + \cdots \ , \tag{14.17}$$

and therefore

$$\begin{aligned} g(\xi) &= 1 + i\xi^e t^e + \cdots & (14.18) \\ &= 1 + iC^{efg}\, \alpha^f\, \beta^g\, t^e + \cdots \ . & (14.19) \end{aligned}$$

We can now equate this with our result for the left hand side (14.12). We then conclude that the commutator of the generators must satisfy

$$\boxed{[t^b, t^c] = i\, C^{bce}\, t^e} \ . \tag{14.20}$$

The expression above is the defining property of the group G and is called the algebra of the group. The set of constants C^{bce} are called structure constants and vary from one group to another.

Finally, the structure constants in (14.20) satisfy an identity that is derived from the following cyclic property of commutators:

$$[t^a,[t^b,t^c]] + [t^b,[t^c,t^a]] + [t^c,[t^a,t^b]] = 0\ . \tag{14.21}$$

Using (14.20) and the equation above we arrive at

$$\boxed{C^{ade}\,C^{bcd} + C^{bde}\,C^{cad} + C^{cde}\,C^{abd} = 0}\ , \tag{14.22}$$

which is the Jacobi identity for the structure constants.

14.1 Classification of Lie Algebras

For the applications we are manly interested in here, we focus on unitary transformations on a finite number of fields. These can be represented by a finite number of hermitian operators. When the number of of generators is finite we say that the group is *compact*. If one of the generators commutes with all others, then it generates a $U(1)$ subgroup. If the algebra does not contain such a $U(1)$ factor is called *semi-simple*. Furthermore, if it does not contain at least two sets of generators whose members commute with the ones from the other set, then the algebra is called *simple*. The most general Lie algebra can be expressed as a direct sum of simple algebras plust $U(1)$ abelian factors.

The restriction that the algebra be compact and simple results in the three so called classical groups, plus five exceptional groups. Here we will not talk about the exceptional groups (G_2, F_4, E_6, E_7 and E_8) although some of them have found applications, for instance in attempts to build model of the unification of all fundamental interactions. In fact we will mostly concentrate on $SU(N)$, which is relevant in many applications such as, for instance, the description of gauge theories in the standard model of particle physics. The other classical groups, $SO(N)$ and $Sp(N)$ have been also used in many applications.

$\underline{SU(N)}$: Unitary transformations of N-dimensional vectors.
If u and v are N-dimensional vectors, a linear transformation on them defined by

$$u \to U\,u\ , \qquad\qquad v \to U\,v\ , \tag{14.23}$$

is a unitary transformation if it preserves the product

$$u^\dagger\,v\ . \tag{14.24}$$

This is satisfied if

$$U^\dagger = U^{-1} \ . \tag{14.25}$$

These transformations defined in this way also include the multiplication by an overall phase:

$$u \to e^{i\alpha}\, u \ . \tag{14.26}$$

But the transformation above corresponds to an example of a $U(1)$ factor. If we want our algebra to be *simple*, we should remove it. We do this by requiring that

$$\det U = 1 \ . \tag{14.27}$$

This requirement removes the phase transformation in (14.26) since we have

$$U = e^{iH} \ , \tag{14.28}$$

where H must be hermitian due to (14.25). The unit determinant constraint (14.27) means that

$$\mathrm{Tr}\left[\mathrm{H}\right] = 0 \ , \tag{14.29}$$

excluding the $U(1)$ transformation in (14.26). Without this exclusion we would have $U(N) = SU(N) \times U(1)$. The generetors of $SU(N)$ are represented by $N^2 - 1$ $N \times N$ traceless matrices. Of these, $N-1$ are diagonal, which define the rank of the group. As mentioned earlier, $SU(N)$ gauged groups figure prominently in the standard model of particle physics, where the interactions are described by the gauge group $SU(3) \times SU(2) \times U(1)$, where the first factor refers to strong interactions and the last two to the electroweak ones.

$\underline{SO(N)}$: Orthogonal transformations on N-dimensional vectors.
It is defined as the unitary transformations that preserve the scalar product of any two N dimensional vectors

$$u \cdot v = u_a\, \delta_{ab}\, v_b \ . \tag{14.30}$$

This is just the group of rotations in N dimensions, but we need to exclude the reflection so that (14.27) is satisfied. Otherwise we would have $O(N)$, which is not a simple group. The number of generators is

$$\frac{N(N-1)}{2} \,, \tag{14.31}$$

which is the number of independent angles in N dimensions.

$SO(N)$ gauge theories have been used in extensions of the standard model, such as for example $SO(10)$ grand unification models. The are also often used as spontaneously broken global symmetries in models where the Higgs boson is composite.

$\underline{Sp(N)}$: Symplectic transformations on N-dimensional vectors.
These transformations preserve the anti-symmetric product of N dimensional vectors

$$u \cdot v = u_a \, \epsilon_{ab} \, v_b \,, \tag{14.32}$$

with

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,. \tag{14.33}$$

The groups has

$$\frac{N(N+1)}{2} \,, \tag{14.34}$$

generators, that means that it is represented by this number of $N \times N$ unitary matrices.

14.2 Representations

A representation is a realization of the multiplication of group elements by using matrices. That is

$$a\,b = c \qquad \rightarrow \qquad M(a)\,M(b) = M(c) \,, \tag{14.35}$$

where $M(a)$, $M(b)$ and $M(c)$ are matrices. A representation is said to be *reducible* if it can be written in diagonal block form, that is as

$$M(a) = \begin{pmatrix} M_1(a) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & M_2(a) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_3(a) \end{pmatrix} \,. \tag{14.36}$$

A reducible representation is the direct sum of irreducible representations (irreps).

The dimension of representation r, $d(r)$, is the dimension of the vector space in which the matrices $M(a)$ act. Irreps can be used to have matrices representing the generators of the group, t^a. We denote these matrices as t^a_r. To fix their normalization we define the trace of the product as

$$\mathrm{Tr}[\mathrm{t^a_r\,t^b_r}] \equiv \mathrm{D^{ab}} \;, \tag{14.37}$$

which satisfies $D^{ab} > 0$ if the t^a_r are hermitian. We can always choose a basis for the matrices t^a_r such that

$$D^{ab} \propto \delta^{ab} \;, \tag{14.38}$$

meaning that

$$\mathrm{Tr}[\mathrm{t^a_r\,t^b_r}] = \mathrm{C(r)}\,\delta^{\mathrm{ab}} \;, \tag{14.39}$$

with $C(r)$ a constant that depends on the particular representation r.

Expressing the generators by the t^a_r, we may write the algebra of the Lie group as

$$[t^a_r, t^b_r] = i\,f^{abc}\,t^c_r \;, \tag{14.40}$$

where the f^{abc} are the structure constants (which we called C^{abc} before). Making use of (14.39) and (14.40) we can write the structure constants as

$$f^{abc} = \frac{-i}{C(r)}\,\mathrm{Tr}[[\mathrm{t^a_r, t^b_r}]\,\mathrm{t^c_r}] \;. \tag{14.41}$$

Expanding the commutator and the trace it is straightforward to show that (14.41) implies that f^{abc} is totally anti-symmetric under the exchange of the group indices a, b and c.

Complex Conjugate Representation

For each irrep r we can define a *complex conjugate* representation $\bar{r}$. For instance, if we have a field ϕ undergoing an infinitesimal transformation we write

$$\phi \to (1 + i\alpha^a\,t^a_r)\,\phi \;. \tag{14.42}$$

Then, the complex conjugate of the field transforms as

$$\phi^* \to (1 - i\alpha^a\,(t_r^a)^*)\,\phi^* \ . \tag{14.43}$$

Then, the generators of the complex conjugate representation are defined as

$$t_{\bar{r}}^a = -(t_r^a)^* = -(t^a)^T \ , \tag{14.44}$$

where the last equality is a consequence of t_r^a being hermitian. There are cases when the complex conjugate representation $\bar{r}$ is equivalent with r. This is the case if a unitary transformation U exists such that

$$t_{\bar{r}}^a = U\,t_r^a\,U^\dagger \ . \tag{14.45}$$

Then we say that the representation r is *real*.

Adjoint Representation

The generators of the adjoint representation G are defined by the structure constants f^{abc} by

$$\left(t_G^b\right)_{ac} \equiv i\,f^{abc} \ . \tag{14.46}$$

It is straightforward to verify that they satisfy the algebra, that is that

$$[t_G^b, t_G^c]_{ae} = i f^{bcd}\,\left(t_G^d\right)_{ae} \ , \tag{14.47}$$

which is in fact the Jacobi identity (14.22). Since the structure constants f^{abc} are real, we can see that the generators of the adjoint representation satisfy

$$t_G^a = -(t_G^a)^* \ , \tag{14.48}$$

which means that he adjoint representation is real. The dimension of the adjoint representations, $d(G)$ is given by the number of generators of the group, e.g. $N^2 - 1$ for $SU(N)$, etc.

Casimir Operator

The operator defined by

$$T^2 \equiv t^a\,t^a \ , \tag{14.49}$$

is called the Casimir operator and it has the property that it commutes with all the generators of the group. That is,

$$[T^2, t^a] = 0 \ . \tag{14.50}$$

The most well known example is the operator for the total angular momentum squared, J^2, which commutes with all the components of $\vec{J}$. In a given irrep r the Casimir is given by a constant:

$$t_r^a \, t_r^a = C_2(r) \, \mathbb{1} \ , \tag{14.51}$$

where $\mathbb{1}$ is the identity in $d(r) \times d(r)$ dimensions. Here we defined $C_2(r)$, the quadratic Casimir operator of the representation r. For the particular case of the adjoint representation, we have

$$(t^c)_{ad} \, (t^c)_{bd} = f^{acd} \, f^{bcd} = C_2(G) \, \delta^{ab} \ . \tag{14.52}$$

For a given representation r it is possible to relate the Casimir $C_2(r)$ with $C(r)$. To see this we start from (14.39). We have that if we multiply it by δ^{ab} on each side we arrive at

$$\delta^{ab} \, \mathrm{Tr}[\mathrm{t_r^a \, t_r^b}] = \mathrm{C(r) \, \delta^{ab} \, \delta^{ab}} \tag{14.53}$$

The product of the two deltas in the right hand side above gives the number of generators, which we can write as $d(G)$, the dimension of the adjoint representation G. But inserting the facor of δ^{ab} on the right hand side of (14.53) inside the trace, we obtain the trace of (14.51). Noticing that $\mathrm{Tr}[\mathbb{1}] = \mathrm{d(r)}$ we arrive at the useful relation

$$\boxed{d(r) \, C_2(r) = d(G) \, C(r)} \ . \tag{14.54}$$

We are now ready to tackle gauge symmetries based on non abelian groups.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 15.
- *Gauge Theory of Elementary Particle Physics*, T.-P. Cheng and L.-F. Li, Chapter 4.