Lecture 13

Spontaneous Breaking of a Local Symmetry

We have seen that the spontaneous breaking of a continuous symmetry results in the presence of massless states in the spectrum, the Nambu–Goldstone Bosons (NGB). We have seen this in particular for a $U(1)$ global symmetry where the potential was such that the ground state was not $U(1)$ invariant. In that case, the NGB corresponded to the degeneracy of the ground state, i.e. it was the fluctuation going around the degenerate minimum and as such it corresponded to a massless state. We will see later that this picuture generalizes for non-abelian global continuous symmetries so that the number of NGBs corresponds to the number of degenerate directions in group space, i.e. the number of broken generators.

Before we go into non-abelian symmetries, we will consider the situation when the $U(1)$ symmetry studied earlier is gauged. That is, is a local $U(1)$ symmetry such as for example in QED. As we will soon see, the consequences for the spectrum when the spontaneously broken symmetry is gauged are drastic. We start with the lagrangian of a scalar field charged under a gauged $U(1)$ symmetry just as QED. This is given by

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^* D^\mu\phi - V(\phi^*\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{13.1}$$

where the covariant derivative is defined by

$$D_\mu\phi = (\partial_\mu + ieA_\mu)\phi \ , \tag{13.2}$$

and the scalar and gauge field transformations under the $U(1)$ gauge symmetry are

$$
\begin{aligned}
\phi(x) &\rightarrow e^{i\alpha(x)}\phi(x) \\
A_\mu(x) &\rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \ .
\end{aligned}
\tag{13.3}
$$

Finally, the gauge field $A_\mu(x)$ has a kinetic term given by the square of the gauge invariant field strength as usual

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{13.4}
$$

With (13.2), (13.3) and (13.4) the lagrangian in (13.1) is clearly gauge invariant.
In order to implement spontaneous breaking we choose the potential as

$$
V(\phi^*\phi) = \frac{1}{2}\mu^2\phi^*\phi + \frac{\lambda}{4}\left(\phi^*\phi\right)^2 \ , \tag{13.5}
$$

which is the same form we used for the breaking fo the global $U(1)$ and corresponds to the only renormalizable terms allowed by the symmetry in four spacetime dimensions. What follows next pertaining the minimum of the potential is identical to what we saw for the global symmetry case. If $\mu^2 > 0$ the minimum of V in (13.5) is $\phi = 0$. However if $\mu^2 < 0$ then we rewrite the potential as

$$
V(\phi^*\phi) = -\frac{1}{2}m^2\phi^*\phi + \frac{\lambda}{4}\left(\phi^*\phi\right)^2 \ , \tag{13.6}
$$

where we have defined the positive constant $m^2 = -\mu^2$. As before, in this case the minimum is now given by the solution of

$$
-\frac{1}{2}m^2 + \frac{\lambda}{2}(\phi^*\phi)_0 = 0 \ , \tag{13.7}
$$

which results in

$$
(\phi^*\phi)_0 = \langle 0|\phi^*\phi|0\rangle = \frac{m^2}{\lambda} \equiv v^2 \ . \tag{13.8}
$$

Choosing the value of the field to be real at the minimum, we use the expansion

$$
\phi(x) = v + \eta(x) + i\xi(x) \ , \tag{13.9}
$$

such that the physical real fields satisfy

$$\langle 0|\eta(x)|0\rangle = \langle 0|\xi(x)|0\rangle = 0 \; . \tag{13.10}$$

Just as we expect, writing the potential in terms of $\eta(x)$ and $\xi(x)$

$$V(\phi^*\phi) = V((v^2 + \eta(x)^2) + \xi(x)^2) \; , \tag{13.11}$$

allows us to identify the spectrum which is given by

$$\begin{aligned} m_\eta &= \sqrt{2}m = \sqrt{2\lambda}v \\ m_\xi &= 0 \; . \end{aligned} \tag{13.12}$$

Thus, we identify $\xi(x)$ with the massless NGB. The difference with respect to the SSB of a global $U(1)$ comes in when we look at what happens in the scalar kinetic term. This is

$$\begin{aligned} \frac{1}{2}(D_\mu\phi)^* D^\mu\phi &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}\, e^2\, v^2\, A_\mu A^\mu \\ &\quad + e\, v\, A_\mu\, \partial^\mu\xi + \cdots \; , \end{aligned} \tag{13.13}$$

where we have explicitly written the terms quadratic in the fields, and the dots denote interactions terms that are cubic or quadratic in them. Besides the kinetic terms for $\eta(x)$ and $\xi(x)$ we notice two terms. The first one is an apparent gauge boson mass term. It implies that the gauge boson has acquired a mass given by

$$m_A = e\, v \; . \tag{13.14}$$

However, this does not mean that the gauge symmetry is not been respected. In fact, all we have done with respect to the (13.1) is to expand the theory around the ground state in terms of fields that have zero expectation values there. In other words, we just performed a change of variables. However, the fact the we are expanding the theory around a minimum that *does not* respect the symmetry is resulting in a mass for the gauge boson. This means that the gauge symmetry has been *spontaneously* broken. But since we have not added any terms that violated explicitly the $U(1)$ gauge symmetry, the symmetry *has not* been *explicitly* broken and therefore currents and charges must still be conserved. We will go into this poin in more detail later.

$$= i\,e\,v\,(-i\,q_\mu) = m_A\,q_\mu$$

Figure 13.1: Feynman rule for the non-diagonal contribution to the two-point function in (13.13).

The second notable aspect in (13.13) is the term mixing the gauge boson with the $\xi(x)$ field, the would-be NGB. Having a term like this, i.e. non-diagonal two-point function, implies that we have to include a Feynman diagram as the one in Figure 13.1. Although in principle there is no problem with having a non-diagonal Feynman rule such as this as long as we always remember to include it, it is interesting to see how to diagonalize it and what are the consequences of doing that. The idea is to choose a gauge for $A_\mu(x)$ such that we can cancel this term once we go to the new gauge. The theory has to be physically equivalent to the one with (13.13). Choosing a specific gauge corresponds to choosing a scalar function $\alpha(x)$ in the gauge tranformations (13.3). In particular, if we choose

$$\alpha(x) = -\frac{1}{v}\,\xi(x)\ , \tag{13.15}$$

we then have the gauge transformation

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{ev}\xi(x)\ . \tag{13.16}$$

Replacing $A_\mu(x)$ in terms of $A'_\mu(x)$ and $\xi(x)$ in (13.13) we have

$$\begin{aligned}\frac{1}{2}(D_\mu\phi)^* D^\mu\phi \;&=\; \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}e^2\,v^2\left(A'_\mu - \frac{1}{ev}\partial_\mu\xi\right)\left(A'^\mu - \frac{1}{ev}\partial^\mu\xi\right) \\ &\quad + e\,v\left(A'_\mu - \frac{1}{ev}\partial_\mu\xi\right)\partial^\mu\xi + \cdots\ ,,\end{aligned} \tag{13.17}$$

Carefully collecting all the terms in (13.17) we arrive at the surprisingly simple expression for the scalar kinetic term:

$$\frac{1}{2}(D_\mu\phi)^* D^\mu\phi = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\,e^2\,v^2\,A'_\mu A'^\mu + \cdots\ . \tag{13.18}$$

We see that the gauge boson mass term is still the same as before. However, the $\xi(x)$ field, the massless field that we thought would be the NGB is now gone. Its kinetic term is gone and, as we will see later, no term with $\xi(x)$ remains in the lagrangian after this gauge transformation. So the would-be NGB is not! When a degree of freedom disappears from the theory just by performing a gauge transformation, we say that this is not a physical degree of freedom. This particular gauge without the NGB $\xi(x)$ is called the *unitary gauge*, since it exposes the actual degrees of freedom of the theory: a real scalar field $\eta(x)$ with mass $m_\eta = \sqrt{2}m$ and a gauge boson with mass $m_A = ev$. In fact if we count degrees of freedom before and after we expanded around the non-trivial ground state, we see that before we had *two real scalar fields*, and *two degrees of freedom* corresponding to the two helicities of a massless gauge boson, for a total of *four degrees of freedom*. But after we expanded around the ground state, we have *one real scalar field*, plus *three polarizations* for the now massive gauge boson, again a total of *four degrees of freedom*. It is in this sense that sometimes we say that when a gauge symmetry is spontaneously broken, the NGB is *"eaten"* by the gauge boson to become its longitudinal polarization. This statement can be made more precise through the *equivalence theorem*, which says that in processes at energies much larger than v (so that it does not matter that the expectation value of the field is not zero in the ground state) computing any observable by using the theory with a massive gauge boson should yield the same result as using the theory with a massless gauge boson and a massless NGB, up to corrections that go like v^2/E^2, where E is the characteristic energy scale of the process in question. We will come back to the equivalence theorem later on when we consider the spontaneous breaking of non-abelian gauge symmetries.

There is another, perhaps more direct, way to see that the NGB can be *gauged away*, i.e. it disappears from the theory by performing a gauge transformation. For this purpose, it is advantageous to parametrize the scalar field not in terms of real and imaginary parts, but of modulus and phase. We write

$$\phi(x) = e^{i\pi(x)/f}\ (v + \sigma(x))\ , \tag{13.19}$$

where we see that this automatically satisfies (13.8). We have two real scalar fields, just as before. One is the modulus field $\sigma(x)$ and the other one is the phase field $\pi(x)$. The scale f is defined so that the argument of the exponent is dimensionless. To fix f we demand that the $\pi(x)$ field has a canonically normalized kinetic term, i.e. we impose it be

$$\frac{1}{2}\partial_\mu\pi\partial^\mu\pi\ . \tag{13.20}$$

This fixes

$$f = v\ , \tag{13.21}$$

so that we have

$$\phi(x) = e^{i\pi(x)/v}\ (v + \sigma(x))\ , \tag{13.22}$$

instead of (13.9). From the form above, it is immediately clear that $\pi(x)$ will not appear in the potential. In fact, this is given by

$$V(\phi^*\phi) = -\frac{m^2}{2}\,[v + \sigma(x)]^2 + \frac{\lambda}{4}\,[v + \sigma(x)]^4\ . \tag{13.23}$$

From this form above we see that $\sigma(x)$ is the massive real scalar field with

$$m_\sigma = \sqrt{2\lambda}\,v\ , \tag{13.24}$$

just as before. This also means that $\pi(x)$ cannot get a mass, i.e.

$$m_\pi = 0\ , \tag{13.25}$$

and therefore is the NGB. In fact, it will only appear in the lagrangian in derivative form since it is the only way it will come down from the exponentials before these annihilate in the kinetic scalar term.

From the parametrization (13.22) it is also obvious how to remove $\pi(x)$ by means of a gauge transformation. Clearly, choosing the gauge transformation

$$\phi(x) \to \phi'(x) = e^{-i\pi(x)/v}\,\phi(x)\ , \tag{13.26}$$

results in

$$\phi'(x) = [v + \sigma(x)]\ . \tag{13.27}$$

Of course, the gauge transformation (13.26) is the same we introduced earlier in (13.15) only substituting $\pi(x)$ for $\xi(x)$, and it therefore results in the same transformation for the gauge fields as in (13.16). Therefore, our conclusions are exactly the same as the ones we derived by using (13.9) as the field expansion: there is a massive gauge boson field with mass $m_A = e\,v$ and a massive reals scalar with mass given by (13.24).

We finally comment on the meaning of spontaneously breaking a gauge symmetry. Specifically, we want to address the point that although the gauge boson has acquired a mass, the gauge symmetry is still present. To show this, let us go back to the gauge where we have both the gauge boson and the NGB. We want to compute the gauge boson two-point function at tree level. In particular we want to consider the effect of spontaneous symmetry breaking. We will need to use the Feynman rule illustrated in Figure 13.1.

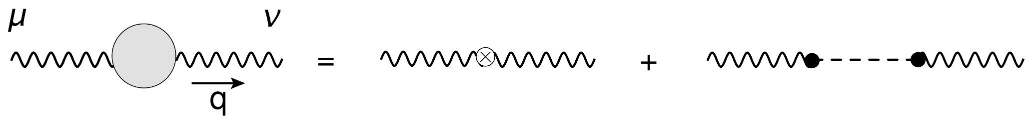

Figure 13.2: New contributions to the gauge boson two-point function at tree level in the presence of spontaneous symmetry breaking. The first diagram is the gauge boson mass term insertion. The second one corresponds to the massless NGB contribution.

The calculation is illustrated in Figure 13.2. In addition to the tree-level gauge boson propagator, there are two new terms contributing: the gauge boson mass insertion and the massless NGB pole. They are

$$\begin{aligned} i\delta\Pi_{\mu\nu} &= im_A^2 g_{\mu\nu} + m_A q_\mu \frac{i}{q^2} m_A(-q_\nu) \\ &= im_A^2 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) . \end{aligned} \tag{13.28}$$

In the first line in (13.28) we used the gauge boson–NGB mixing Feynman rule of Figure 13.1. The result is that the new additions to the two-point function result to be actually transverse. That is, we have that

$$q^\mu \delta\Pi_{\mu\nu} = 0 \ , \tag{13.29}$$

so that the two-point function remains transverse, therefore respecting the Ward identities. Since the Ward identities are equivalent to current conservation, we conclude that the gauge symmetry is still preserved, even in the presence of the gauge boson mass term. We can see that this required the presence of the NGB pole. Just having the gauge boson mass term would have resulted in a non-transverse contribution to the two-point function, and an explicit violation of the gauge symmetry. So having a gauge boson mass is compatible with gauge invariance as long as it is the result of spontaneous symmetry breaking.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 20.1.
- *Quantum Field Theory* , by M. Srednicki, Chapter 85.
- *Quantum Field Theory in a Nutshell*, by A. Zee. Chapter 4.6.