Lecture 12

The Effective Action as a Generating Functional

Here we will rederive the effective potential by making use of the fact that the effective action is a generating functional of the one-particle-irreducible (1PI) diagrams. To see this, we start from the definition of the generating functional of all Feynman diagrams as

$$Z[J] = e^{iW[J]} \,, \tag{12.1}$$

where, as we showed earlier, $W[J]$ is the generating functional of the *connected* Feynman diagrams. In other words, since the n-point function is generated from $Z[J]$ as

$$G^{(n)}(x_1, \ldots, x_n) = \frac{1}{Z[0]}(-i)^n \;\frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)} Z[J]\Big|_{J=0} \,, \tag{12.2}$$

we can write them as

$$Z[J] = Z[0]\sum_{n=0}^{\infty}\frac{i^n}{n!}\int dx_1 \ldots dx_n\, G^{(n)}(x_1, \ldots, x_n)\, J(x_1)\ldots J(x_n) \,. \tag{12.3}$$

Likewise, the connected generating functional $iW[J]$ can be expanded in terms of the connected n-point correlation functions $G_c^{(n)}(x_1, \ldots, x_n)$ as

$$iW[J] = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int dx_1 \ldots dx_n\, G_c^{(n)}(x_1, \ldots, x_n)\, J(x_1)\ldots J(x_n) \,. \tag{12.4}$$

On the other hand, we recall the definition of the effective action as

$$\Gamma[\phi_c] = W[J] - \int d^4x\, J(x)\phi_c(x)\ , \tag{12.5}$$

where $\phi_c(x) = \langle 0|\phi(x)|0\rangle$. As seen in the previous lecture, we have that

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J(x)\ , \qquad \frac{\delta W[J]}{\delta J(x)} = \phi_c(x)\ , \tag{12.6}$$

Furthermore, from the second equation in (12.6) we have

$$\frac{\delta^2 W[J]}{\delta J(y)\delta J(x)} = \frac{\delta\phi_c(x)}{\delta J(y)} = iD(x,y)\ , \tag{12.7}$$

where we used the fact that the second functional derivative of $W[J]$ must give us the *connected* two-point function $D(x,y)$. Now from the first equation in (12.6) we see that

$$\frac{\delta}{\delta J(y)}\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -\delta(x-y)\ , \tag{12.8}$$

from which we can write

$$\begin{aligned}\delta(x-y) &= -\int d^4z \frac{\delta\phi_c(z)}{\delta J(y)}\,\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(z)\delta\phi_c(x)} \\ &= -\int d^4z \frac{\delta^2 W[J]}{\delta J(y)\,\delta J(x)}\,\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(z)\delta\phi_c(x)}\ .\end{aligned} \tag{12.9}$$

The expression above means that the first factor is the functional inverse of the second, that is

$$-\frac{\delta^2 W[J]}{\delta J(y)\,\delta J(x)} = \left(\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(y)\,\delta\phi_c(x)}\right)^{-1}\ . \tag{12.10}$$

This tells us that the two-point connected correlation function is basically the functional inverse of the second derivative of the effective action. We now want to relate a higher order connected correlation function to the functional derivatives of the effective action. For this purpose, we first notice that

$$\begin{aligned}\frac{\delta}{\delta J(z)} &= \int d^4w \, \frac{\delta\phi_c(w)}{\delta J(z)} \, \frac{\delta}{\delta\phi_c(w)} \\ &= i\int d^4w D(z,w) \, \frac{\delta}{\delta\phi_c(w)} \;.\end{aligned} \tag{12.11}$$

Then, we can write

$$\begin{aligned}\frac{\delta^3 W[J]}{\delta J(x)\,\delta J(y)\,\delta J(z)} &= i\int d^4w \, D(z,w) \, \frac{\delta}{\delta\phi_c(w)} \frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)} \\ &= -i\int d^4w \, D(z,w) \, \frac{\delta}{\delta\phi_c(w)} \left(\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(x)\,\delta\phi_c(y)}\right)^{-1} \;.\end{aligned} \tag{12.12}$$

But the derivative of the functional inverse in the second line in (12.12) can be expressed in terms of the derivative of the functional by using that for a given M with inverse such that

$$M\,M^{-1} = \mathbb{1}, \tag{12.13}$$

then we have that

$$\frac{\delta M^{-1}}{\delta a} = -M^{-1}\,\frac{\delta M}{\delta a}\,M^{-1} \;. \tag{12.14}$$

Then we arrive at

$$\frac{\delta^3 W[J]}{\delta J(x)\,\delta J(y)\,\delta J(z)} = -i\int d^4w\,d^4u\,d^4v\,D(z,w)\,D(x,u)\,D(y,v)\,\frac{\delta^3\Gamma[\phi_c]}{\delta\phi_c(u)\,\delta\phi_c(v)\,\delta\phi_c(w)} \;. \tag{12.15}$$

The expression above gives us a relationship between the connected three-point correlation function, on the left hand side and the third funcional derivative of the effective action. Just by inspection of the right hand side we see that the triple functional derivative must be the *amputated* three-point correlation function since it corresponds to the connected one on the left hand side but stripped of the two-point functions of the three external legs, here represented by $D(z,w)$, $D(x,u)$ and $D(y,v)$. Then, noticing that the connected three-piont function is given by

$$(-i)^3 \frac{\delta^3 iW[J]}{\delta J(x)\,\delta J(y)\,\delta J(z)} = -\frac{\delta^3 W[J]}{\delta J(x)\,\delta J(y)\,\delta J(z)} \;, \tag{12.16}$$

we have that

$$\frac{\delta^3 i\Gamma[\phi_c]}{\delta\phi_c(x)\,\delta\phi_c(y)\,\delta\phi_c(z)} = G^{(3)}_{1PI}(x,y,z) \;, \tag{12.17}$$

where on the right hand side is the amputated or one-particle irreducible (1PI) three-point function. This can be generalized to higher order correlation functions so that

$$\boxed{\frac{\delta^n i\Gamma[\phi_c]}{\delta\phi_c(x_1)\,\ldots\,\delta\phi_c(x_n)} = G^{(n)}_{1PI}(x_1,\ldots,x_n) \equiv \Gamma^{(n)}(x_1,\ldots,x_n)} \;, \tag{12.18}$$

where we renamed the 1PI n-point correlation functions for simplicity. The expression above means that the effective action is the generating functional of the 1PI diagrams. Then, we can expand it as

$$i\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int d^4x_1 \ldots d^4x_n\, \Gamma^{(n)}(x_1,\ldots,x_n)\, \phi_c(x_1)\ldots\phi_c(x_n) \;. \tag{12.19}$$

which is equivalent to (12.18). In order to compute the effective potential and make contact with our previous results for it, we start by writing the 1PI correlation functions in momentum space. We define them by the Fourier transform

$$\Gamma^{(n)}(x_1,\ldots,x_n) = \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}\, e^{i(k_1\cdot x_1+\cdots+k_n\cdot x_n)}\, \Gamma^{(n)}(k_1,\ldots,k_n)\,(2\pi)^4\,\delta^{(4)}(k_1+\cdots+k_n) \;. \tag{12.20}$$

Then we can write the effective action expansion in (12.19) as

$$\begin{aligned} i\Gamma[\phi_c] \;&=\; \sum_n \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}\,(2\pi)^2\delta^{(4)}(k_1+\cdots+k_n) \\ &\quad \times e^{(k_1\cdot x_1+\cdots+k_n\cdot x_n)}\,\Gamma^{(n)}(k_1,\ldots,k_n)\phi_c(x_1)\ldots\phi_c(x_n) \;, \end{aligned} \tag{12.21}$$

On the other hand, we may write the effective action as a derivative expansion. This is

$$\Gamma[\phi_c] = \int d^4x \left\{ -V_{\rm eff.}(\phi_c) + \frac{1}{2} Z(\phi_c)\partial_\mu\phi_c\partial^\mu\phi_c + \dots \right\} , \qquad (12.22)$$

where the effective potential is the zero-derivative term, the kinetic term (including field renormalization) is the two-derivative term and the dots denote terms with a higher number of derivatives. The derivative expansion in (12.22) can be matched with a momentum expansion in (12.21). Expanding about zero momenta (12.21) can be written as

$$\begin{aligned} i\Gamma[\phi_c] &= \sum_n \frac{1}{n!} \int d^4x_1 \dots d^4x_n \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \int d^4x\, e^{i(k_1+\dots+k_n)\cdot x} \\ &\quad \times e^{(k_1\cdot x_1 + \dots + k_n \cdot x_n)} \left\{ \Gamma^{(n)}(0,\dots,0)\phi_c(x_1)\dots\phi_c(x_n) + \dots \right\} , \end{aligned} \qquad (12.23)$$

where we have replaced the momentum conservation delta function by its integral representation and we have only written explicitly the first term in the momentum expansion of $\Gamma^{(n)}(k_1,\dots,k_n)$ about zero momenta, i.e. $\Gamma^{(n)}(0,\dots,0)$. Noticing that

$$\int \frac{d^4k_i}{(2\pi)^4} e^{ik_i\cdot(x_i+x)} = \delta^{(4)}(x_i + x) , \qquad (12.24)$$

we obtain

$$i\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int d^4x \left\{ \Gamma^{(n)}(0,\dots,0)\phi_c^n(x) + \dots \right\} . \qquad (12.25)$$

Comparing with the first term in (12.22) we obtain that the effective potential is given by

$$\boxed{V_{\rm eff.}(\phi_c) = \sum_n \frac{1}{n!}\, i\,\Gamma^{(n)}(0,\dots,0)\, \phi_c^n(x)} . \qquad (12.26)$$

This expression states that the effective potential is an expansion in the vacuum expectation value of the field, $\phi_c(x)$, where the coefficients are fixed by the 1PI correlation functions at zero external momentum. For instance, for a theory of a real scalar, at tree level we have

$$i\Gamma^{(2)}(0,0) = \mu^2 , \qquad i\Gamma^{(4)}(0,\dots,0) = \lambda . \qquad (12.27)$$

Since at leading order only these two amplitudes are non-zero, the effective potential is very simple and coincides with the classical potential. But if we want to go beyond leading

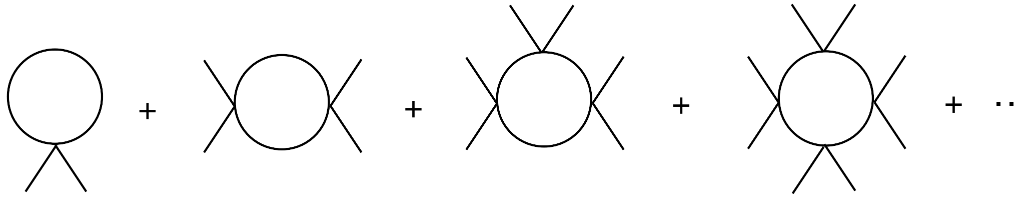

Figure 12.1: One loop contributions to the effective potential $V_{\text{eff.}}$.

order in computing $V_{\text{eff.}}$ we need to consider loop corrections to the $\Gamma^{(n)}(0,\ldots,0)$. Thus, we need to consider loop contributions to the zero-momentum amplitudes. And, as we will see below, all powers of $\phi_c(x)$ will contribute starting at one loop.

12.1 The One Loop Effective Potential

We consider the usual real scalar theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 \ . \tag{12.28}$$

To go beyond leading order in the calculation of $V_{\text{eff.}}$ we need to compute the one loop contributions to the $\Gamma^{(n)}(0,\ldots,0)$. These are shown in Figure 12.1, where the ϕ^4 interaction is inserted. Since the theory respects $\phi \to -\phi$, only diagrams with an even number of external legs contribute. The infinite sum corresponds to adding all the one-loop contributions to the zero external momentum amplitudes, i.e.

$$\Gamma^{(2)}(0,0) + \Gamma^{(4)}(0,\ldots,0) + \Gamma^{(6)}(0,\ldots,0) + \Gamma^{(8)}(0,\ldots,0) + \ldots \ . \tag{12.29}$$

The expression for the one loop contribution to the zero-momentum 2n-point amplitude is

$$\Gamma^{(2n)}(0,\ldots,0) = \frac{(2n)!}{2^n\,2n}\int\frac{d^4k}{(2\pi)^4}\left[(-i\lambda)\frac{i}{k^2-\mu^2}\right]^n \ . \tag{12.30}$$

In the expression above, there are n insertions of $(-i\lambda)$ since there are n vertices. There are also n propagators. Finally, the combinatoric factor outside the integral deserves a more detailed explanation. First, the factor of $(2n)!$ in the numerator corresponds to the number of ways to distribute $2n$ particles in $2n$ external lines. However, some of these are redundant. The factor of 2^n in the denominator corresponds to the fact that exchanging the two external lines at each of the n vertices does not result in a new contribution.

Finally, the factor of $2n$ also in the denominator accounts for the indistinguishable overall rotations of the vertices, which do not generate distinct diagrams. We can now use (12.30) in the expression (12.26) for the effective potential, noticing that only terms even contribute in it (i.e. the factorial dividing in (12.26) is $(2n)!$ now). We then obtain

$$V_{\text{eff.}}(\phi_c) = V(\phi_c) + \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{(\lambda/2)\,\phi_c^2}{k^2-\mu^2}\right]^n \ . \tag{12.31}$$

It is straightforward to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{(\lambda/2)\,\phi_c^2}{k^2-\mu^2}\right]^n = -\ln\left(1 - \frac{(\lambda/2)\,\phi_c^2}{k^2-\mu^2}\right) \ , \tag{12.32}$$

which results in

$$V_{\text{eff.}}(\phi_c) = V(\phi_c) - \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \ln\left(1 - \frac{(\lambda/2)\,\phi_c^2}{k^2-\mu^2}\right) \ . \tag{12.33}$$

The result above for the one-loop effective potential is very similar to the one we obtained before, with the only difference that now we have a nonzero mass μ^2. The method we used here makes it clear that the corrections to the potential are coming first from the sum of *all* the one loop contributions to the zero external momentum amplitudes.

As usual, we need to regulate the momentum integral and fix the relevant counterterms by imposing renormalization conditions on them. After a Wick rotation with $k_0 = ik_4$ we have

$$V_{\text{eff.}}(\phi_c) = V(\phi_c) + \frac{1}{16\pi^2}\int_0^{\Lambda} dk\, k^3 \,\ln\left(1 + \frac{(\lambda/2)\phi_c^2}{k^2+\mu^2}\right) + \frac{1}{2}\delta\mu^2\phi_c^2 + \frac{\delta\lambda}{4!}\phi_c^4 \ , \tag{12.34}$$

where we are already using the euclidean momentum and we added the only two necessary counterterms since the integral has quadratic and logarithmic divergences coming from the first two terms in the expansion of the logarithm. We impose the renormalization conditions

$$\left.\frac{d^2V_{\text{eff.}}(\phi_c)}{d\phi_c^2}\right|_{\phi_c=0} = \mu^2 \ , \tag{12.35}$$

that fixes the physical mass, and

$$\left.\frac{d^4 V_{\text{eff.}}(\phi_c)}{d\phi_c^4}\right|_{\phi_c=0} = \lambda \;, \tag{12.36}$$

for fixing the physical coupling. Performing the euclidean momentum integration with a cutoff Λ, we have

$$\begin{aligned} V_{\text{eff.}}(\phi_c) = V(\phi_c) \quad + \quad & \frac{1}{64\pi^2}\left\{\frac{\lambda}{2}\phi_c^2\Lambda^2 + \left(\frac{\lambda}{2}\phi_c^2+\mu^2\right)^2 \ln\left(\frac{(\lambda/2)\phi_c^2+\mu^2}{\Lambda^2}\right) + \mu^2 \ln\left(\frac{\mu^2}{\Lambda^2}\right)\right. \\ & \left. +\frac{1}{2}\delta\mu^2\phi_c^2 + \frac{\delta\lambda}{4!}\phi_c^4\right\} \;. \end{aligned} \tag{12.37}$$

Imposing (12.35) and (12.36) in (12.37) fixes the counterterms to be

$$\delta\mu^2 = -\frac{1}{64\pi^2}\left\{\lambda\Lambda^2 + \lambda\mu^2 + 2\lambda\mu^2\ln\left(\frac{\mu^2}{\Lambda^2}\right)\right\} \;, \tag{12.38}$$

and

$$\delta\lambda = -\frac{1}{64\pi^2}\left\{9\lambda^2 + 6\lambda^2\ln\left(\frac{\mu^2}{\Lambda^2}\right)\right\} \;. \tag{12.39}$$

Replacing these back in (12.37) we obtain the renormalized effective potential to one loop accuracy,

$$V_{\text{eff.}}(\phi_c) = V(\phi_c) + \frac{1}{64\pi^2}\left\{\left(\frac{\lambda}{2}\phi_c^2+\mu^2\right)^2 \ln\left(\frac{(\lambda/2)\phi_c^2+\mu^2}{\mu^2}\right) - \frac{\lambda\mu^2}{2}\phi_c^2 - \frac{3}{8}\lambda^2\phi_c^4\right\} \;. \tag{12.40}$$

This expression is the same we obtained before but for nonzero μ^2. Once again then we conclude that the loop corrections to the potential not only shift the parameters, in this case μ^2 and λ, but also change the functional form of $V_{\text{eff.}}(\phi_c)$.

The result above is independent of the sign of μ^2. Thus, if $\mu^2 < 0$ we have spontaneous symmetry breaking and the vacuum expectation value (VEV) of ϕ is nonzero. Then μ^2 is not the physical mass of the field ϕ. However, it is customary to define the VEV-dependent mass as

$$m^2(\phi_c) \equiv \mu^2 + \frac{\lambda}{2}\phi_c^2 \;, \tag{12.41}$$

which then enters in the expression (12.40) for $V_{\rm eff.}$. This results in

$$V_{\rm eff.}(\phi_c) = V(\phi_c) + \frac{1}{64\pi^2} m^4(\phi_c)\,\ln\left(\frac{m^2(\phi_c)}{\mu^2}\right) + \cdots\ , \tag{12.42}$$

where the dots denote the one-loop shifts of the ϕ_c powers already existing at leading order. This form will be very handy when computing the effective potential of theories with spontaneous symmetry breaking.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 11.
- *Quantum Field Theory in a Nutshell*, by A. Zee. Chapter 4.3.
- *Gauge Theory of Elementary Particle Physics*, T.-P. Cheng and L.-F. Li, Section 6.4.