# Lecture 11

# The Effective Action

So far, in our treatment of the spontaneous breaking of a continuous symmetry, when we considered the the minimization of the potential of the field in question to obtain its ground state expectation value, we were working at the classical level. Quantum corrections will renormalize the parameters of the theory, including those of the potential. In fact, they can also add new field dependent terms and potentially change the position of the minimum of the potential. In order to include quantum corrections we will define a function in the full quantum theory such that its minimum is actually

$$\langle 0|\phi(x)|0\rangle \ , \tag{11.1}$$

where we define $|0\rangle$ as the state with no quanta of the field $\phi(x)$. To leading order, this function should coincide with the classical potential. We start with the generating functional in the presence of a linearly coupled source $J(x)$ given by

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\phi \, e^{iS[\phi]+i\int d^4x J(x)\phi(x)} \ . \tag{11.2}$$

We can then write

$$\frac{\delta W}{\delta J(x)} = -i\frac{\delta \ln Z}{\delta J(x)} = \frac{\int \mathcal{D}\phi \, \phi(x) \ e^{iS[\phi]+i\int d^4x J(x)\phi(x)}}{\int \mathcal{D}\phi \, e^{iS[\phi]+i\int d^4x J(x)\phi(x)}} \ , \tag{11.3}$$

which means that

$$\frac{\delta W}{\delta J(x)} = \langle 0|\phi(x)|0\rangle \equiv \phi_c(x) \ , \tag{11.4}$$

gives the expectation value of the operator $\phi(x)$. We now define the effective action as a functional of $\phi_c(x)$ as

$$\Gamma[\phi_c(x)] \equiv W[J] - \int d^4x J(x)\phi_c(x) \ . \tag{11.5}$$

From the definition above, we see that $\Gamma[\phi_c(x)]$ is the Legendre transform of $W[J]$, with $J(x)$ and $\phi_c(x)$ conjugate Legendre variables. We can then compute

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = \int d^4y \frac{\delta W[J]}{\delta J(y)} \frac{\delta J(y)}{\delta\phi_c(x)} - \int d^4y \frac{\delta J(y)}{\delta\phi_c(x)} \phi_c(y) - \int d^4y J(y)\delta^{(4)}(x-y) \ , \tag{11.6}$$

where in the last terms we used the defining property of the functional derivative. Using (11.4), we see that the first two terms in (11.6) cancel leaving us with

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J(x) \ . \tag{11.7}$$

In particular, in the absence of the external source (i.e. for $J(x) \to 0$) we have

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = 0 \ . \tag{11.8}$$

If we further assume that the solution of (11.8) is translationally invariant, then $\phi_c$ is a constant. Then, the effective action can be written as

$$\Gamma(\phi_c) = -\int d^4x \, V_{\text{eff.}}(\phi_c) = -VT \, V_{\text{eff.}}(\phi_c) \ , \tag{11.9}$$

where in the last equality we considered finite space-time box of volume $VT$. Then, (11.7) becomes

$$\frac{\partial V_{\text{eff.}}}{\partial\phi_c} = J(x) \ , \tag{11.10}$$

which in the $J \to 0$ limit turns into

$$\boxed{\frac{\partial V_{\text{eff.}}}{\partial\phi_c} = 0} \ . \tag{11.11}$$

The equation above says that the vacuum expectation value

$$\phi_c = \langle 0|\phi(x)|0\rangle \ , \tag{11.12}$$

is the solution that minimizes the effective potentail $V_{\text{eff.}}$. This formalizes what we had been saying before on a more sure footing. It also opens up the possibility of computing the contributions of quantum corrections.

## 11.1 Computing the Effective Action

In order to compute the effects of quantum corrections in the effective action, we first have to obtain the mean field or saddle point solution about which we will expand. Going back to the generating functional

$$Z[j] = e^{iS[\phi]+i\int d^4x\, J(x)\,\phi(x)} \ , \tag{11.13}$$

the saddle point configurations are solutions to

$$\frac{\delta\{S[\phi] + \int d^4x\, J(x)\,\phi(x)\}}{\delta\phi} = 0 \ . \tag{11.14}$$

As a typical example, let us consider a real scalar field with

$$\begin{aligned} S[\phi] &= \int d^4x \left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\} \\ &= \int d^4x \left\{-\frac{1}{2}\phi\,\partial^2\phi - V(\phi) + \frac{1}{2}\partial_\mu\left(\phi\partial^\mu\right)\right\} \ , \end{aligned} \tag{11.15}$$

from which we see that

$$\frac{\delta S[\phi]}{\delta\phi(x)} = -\left(\partial^2\phi(x) + V'(\phi)\right) \ , \tag{11.16}$$

where we defined

$$V'(\phi) \equiv \frac{\delta V[\phi]}{\delta\phi(x)} \ . \tag{11.17}$$

Then, the saddle point equation (11.14) becomes

$$\partial^2\phi(x) + V'(\phi) = J(x) \ . \tag{11.18}$$

Calling the solutions of the saddle point equation (11.18) $\phi_s(x)$, we will now expand around them. We then write

$$\phi(x) = \phi_s(x) + \eta(x) \ . \tag{11.19}$$

where we defined the fluctuation $\eta(x)$ around the saddle point solution $\phi_s(x)$. Thus, to expand the functional integral around the fixed saddle point solution, we must integrate over $\eta(x)$, i.e.

$$\mathcal{D}\phi \to \mathcal{D}\eta \ . \tag{11.20}$$

To prepare the functional integral we expand the action as

$$S[\phi] = \frac{1}{2}\partial_\mu\phi_s\partial^\mu\phi_s + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \partial_\mu\phi_s\partial^\mu\eta - V(\phi_s+\eta) \ , \tag{11.21}$$

where the potential can be expanded as

$$V(\phi_s+\eta) \simeq V(\phi_s) + \eta\left.\frac{\partial V}{\partial\phi}\right)_{\phi_s} + \frac{1}{2}\eta^2\left.\frac{\partial^2 V}{\partial\phi^2}\right)_{\phi_s} + \cdots \ , \tag{11.22}$$

It is straightforward to verify that the linear terms in $\eta$ in (11.21) will be cancelled once the saddle point condition (11.14) is imposed. Then, for small fluctuations $\eta(x)$ we can write the generating fuctional as

$$Z[J] = e^{iS[\phi_s]+i\int d^4x J(x)\phi_s(x)} \int \mathcal{D}\eta\, e^{i\int d^4x\left\{(1/2)\partial^\mu\eta\partial_\mu\eta-(1/2)\eta^2 V''(\phi_s)\right\}} \ . \tag{11.23}$$

The functional integral in $\eta$ in (11.23) is Gaussian so we perform it, resulting in

$$Z[J] = e^{iS[\phi_s]+i\int d^4x J(x)\phi_s(x)}\,[\det\mathcal{O}]^{-1/2} \ . \tag{11.24}$$

where we defined the operator

$$\mathcal{O} \equiv \partial^2 + V''(\phi_s) \ . \tag{11.25}$$

Using that

$$\det\mathcal{O} = e^{\mathrm{Tr}\ln\mathcal{O}} \,, \tag{11.26}$$

we arrive at

$$W[J] = S[\phi_s] + \int d^4y \, J(y)\, \phi_s(y) + \frac{i}{2}\mathrm{Tr}\ln\left(\partial^2 + \mathrm{V}''(\phi_\mathrm{s})\right) \,. \tag{11.27}$$

The last term in (11.27) encodes the quantum corrections coming from integrating out the fluctuations $\eta(x)$. In fact, if we compute the functional derivative of (11.27)

$$\frac{\delta W[J]}{\delta J} = \phi_s(x) + \frac{\delta\left[S[\phi_s] + \int d^4y J(y)\phi_s(y)\right]}{\delta\phi_s}\,\frac{\delta\phi_s}{\delta J} + \quad \text{quantum corrections} \,, \tag{11.28}$$

where the second term vanished due to the saddle point condition (11.14). Then, using (11.4) we obtain

$$\phi_c(x) = \phi_s(x) + \quad \text{quantum corrections} \,. \tag{11.29}$$

Thus, we conclude that the vacuum expectation value of the field, $\phi_c(x)$ is in fact the saddle point solution plus quantum corrections. We are now in a position to see how these corrections enter in the effective action.

Using the definition of the effective action in (11.5) and the exression (11.27) we arrive at

$$\boxed{\Gamma[\phi_s] = S[\phi_s] + \frac{i}{2}\mathrm{Tr}\ln\left(\partial^2 + \mathrm{V}''(\phi_\mathrm{s})\right)}\,, \tag{11.30}$$

which tells us that the effective action is an expansion around the saddle point action that includes the quantum fluctuations.

In order to go further in the computation, we need to make some assumptions that will allow us to treat the trace in (11.30). If we assume that the vacuum solution $\phi_s(x)$ is a constant, then $V''(\phi_s)$ is also a constant. Then the trace can be written as

$$\begin{aligned}
\mathrm{Tr}\ln\left(\partial^2 + \mathrm{V}''(\phi)\right) &= \int d^4x \langle x|\ln\left(\partial^2 + V''(\phi)\right)|x\rangle \\
&= \int d^4x \int d^4p \int d^4k \, \langle x|p\rangle \, \langle p|\ln\left(\partial^2 + V''(\phi)\right)|k\rangle \, \langle k|x\rangle \,,
\end{aligned} \tag{11.31}$$

where we have inserted the identity twice as integrals over the momenta $k$ and $p$, and we dropped the subscript s in the field. The action of the log operator on a momentum state is

$$\ln\left(\partial^2 + V''(\phi)\right)|k\rangle = \ln\left(-k^2 + V''(\phi)\right)|k\rangle \tag{11.32}$$

where the right-hand side is the eigenvalue times the state, plus using that $\langle p|k\rangle = \delta^{(4)}(p-k)$, we obtain

$$\mathrm{Tr}\ln\left(\partial^2 + \mathrm{V}''(\phi)\right) = \int \mathrm{d}^4\mathrm{x} \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4} \ln\left(-\mathrm{k}^2 + \mathrm{V}''(\phi)\right) , \tag{11.33}$$

where we have used the normalization of the space-time wave-function given by

$$|\langle x|k\rangle|^2 = \frac{1}{(2\pi)^4} . \tag{11.34}$$

Since $\phi$ is a constant, it is more relevant to compute the effective potential as defined in (11.9). We have

$$\boxed{V_{\text{eff.}}(\phi) = V(\phi) - \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \ln\left(\frac{-k^2 + V''(\phi)}{-k^2}\right)}, \tag{11.35}$$

where we inserted a $\phi$-independent term in order to have a dimensionless argument for the logarithm. This is the Coleman-Weinberg potential. It includes the first quantum corrections for the potential energy of the background (constant) field $\phi$. We will see below that this can be also obtained by a loop expansion, corresponding to the one loop effective potential. But before going into that, we will perform the momentum integral. As we will see below, it is divergent, which means that we need to define counterterms and impose renormalization conditions in order to obtain the renormalized effective potential.

## 11.2 Renormalization of the Effective Potential

The integral in (11.35) results in two divergencies: a quadratic and a logarithmic one. We include two counterterms as in

$$V_{\text{eff.}}(\phi) = V(\phi) - \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \ln\left(\frac{-k^2 + V''(\phi)}{-k^2}\right) + \delta\mu^2\phi^2 + \delta\lambda\phi^4 , \tag{11.36}$$

Performing a Wick rotation we have

$$\begin{aligned} V_{\text{eff.}}(\phi) &= V(\phi) + \frac{1}{2}\int^{\Lambda} \frac{dk_E d\Omega}{(2\pi)^4} k_E^3 \ln\left(1 + \frac{V''(\phi)}{k_E^2}\right) + \delta\mu^2\phi^2 + \delta\lambda\phi^4 \\ &= V(\phi) + \frac{2\pi^{d/2}}{(2\pi)^d\Gamma(d/2)}\int^{\Lambda} dk_E^2\, k_E^2 \left[\frac{V''(\phi)}{k_E^2} - \frac{1}{2}\left(\frac{V''(\phi)}{k_E^2}\right)^2 + \ldots\right] + \delta\mu^2\phi^2 + \delta\lambda\phi^4 \ , \end{aligned} \tag{11.37}$$

where we expanded the logartithm keeping only the UV-divergent terms. To perform the integrals in (11.37) we need to be careful with the IR limit since there is a logarithm. The lowest energy scale is $V''(\phi)$, so we obtain

$$\begin{aligned} V_{\text{eff.}}(\phi) &= V(\phi) + \frac{\Lambda^2}{32\pi^2}V''(\phi) - \frac{1}{64\pi^2}\left(V''(\phi)\right)^2 \ln\left(\frac{\Lambda^2}{V''(\phi)}\right) \\ &\quad +\delta\mu^2\phi^2 + \delta\lambda\phi^4 \ . \end{aligned} \tag{11.38}$$

We see that if $V(\phi)$ is a quartic polinomial in $\phi$, then $V''(\phi)$ is quadratic, and $(V''(\phi))^2$ is quartic, resulting in divergences matched by the counterterms we wrote to begin with. On the other hand, if $V(\phi)$ contains higher powers of $\phi$, we will require additional counterterms, resulting in non-renormalizable terms. E.g. if $V(\phi)$ goes like $\phi^6$, then $V''(\phi)$ is quartic already, and $(V''(\phi))^2$ goes as $\phi^8$, requiring new higher dimensonal counterterms.

As an example, let us consider

$$V(\phi) = \frac{\lambda}{4!}\phi^4 \ , \tag{11.39}$$

which means that we choose $m = 0$. Then we have

$$\begin{aligned} V_{\text{eff.}}(\phi) &= \frac{\lambda}{4!}\phi^4 + \frac{\Lambda^2}{64\pi^2}\lambda\phi^2 - \frac{\lambda^2}{256\pi^2}\phi^4 \ln\left(\frac{\Lambda^2}{\lambda\phi^2/2}\right) \\ &\quad +\delta\mu^2\phi^2 + \delta\lambda\phi^4 \ . \end{aligned} \tag{11.40}$$

To fix the counterterms we impose the following renormalization conditions. First, since $m = 0$ we impose

$$\frac{d^2 V_{\text{eff.}}(\phi)}{d\phi^2}|_{\phi=0} = 0 \ , \tag{11.41}$$

resulting in

$$\boxed{\delta\mu^2 = -\lambda\frac{\Lambda^2}{64\pi^2}}\,. \tag{11.42}$$

For the quartic counterterm, we cannot impose the renormalization condition at $\phi = 0$ because, once again, we have a logarithm. We then choose $\phi = \mu$, an arbitrary nonzero scale. Then we impose

$$\frac{d^4 V_{\text{eff.}}(\phi)}{d\phi^4}|_{\phi=\mu} = \lambda(\mu) \ . \tag{11.43}$$

which results in

$$\delta\lambda = \frac{\lambda^2}{256\pi^2}\left(\ln\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{25}{6}\right) \ . \tag{11.44}$$

Replacing in $V_{\text{eff.}}(\phi)$ we arrive at

$$\boxed{V_{\text{eff.}}(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2}{256\pi^2}\left(\ln\left(\frac{\phi^2}{\mu^2}\right) - \frac{25}{6}\right)\phi^4} \ , \tag{11.45}$$

where it is understood that $\lambda = \lambda(\mu)$ at a given order. We see from the expression above that the quantum corrections to the effective potential are not just shifts of the coupling constant $\lambda$, but there is a new $\phi$-dependent term in the form of logarithm, giving the potential a different shape that the mere $\phi^4$ it had a lowest level.

The fact that quantum corrections modify the shape of the potential posses the question of whether we can have spontaneous symmetry breaking triggered just from them. In the example above, imposing

$$\frac{\partial V_{\text{eff.}}}{\partial\phi} = 0 \ ,$$

results in the condition

$$\lambda \ln\left(\frac{\phi_0^2}{\mu^2}\right) = -\frac{32\pi^2}{3} + \mathcal{O}(\lambda) \ , \tag{11.46}$$

where $\phi_0$ is the value of the field at the minimum. However, the expression in (11.46) appears incompatible with perturbation theory. This does not mean that spontaneous symmetry breaking cannot be triggerd by quamtum corrections. But we will need to include at least one more parameter, for instance a nonzero value for $m$. Then, it is possible to start with a theory with a minimum at $\phi = 0$ at tree level, but that it acquires a nonzero ground state expectation value $\phi_0$ due to quantum corrections. This mechanism is widely used, for instance, in theories trying to explain the origin of the scalar (Higgs) sector of the standard model of particle physics, where it is referred to as radiative spontaneous symmetry breaking.

## Additional suggested readings

- *Aspects of Symmetry*, S. Coleman, Chapter 5.3.
- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 11.
- *Quantum Field Theory in a Nutshell*, by A. Zee. Chapter 4.3.
- *Gauge Theory of Elementary Particle Physics*, T.-P. Cheng and L.-F. Li, Section 6.4.