Quantum Field Theory II

Homework 3
Due 06/12/2024

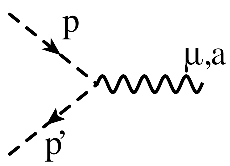
1. Consider a non-abelian gauge theory coupled to a fermion field (**2 pts.**):

   (a) Compute at one loop level the counter-term that regularizes the fermion field (δ_2).

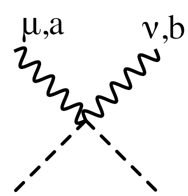
   (b) Also at one loop, compute the counter-term regularizing the vertex. (δ_1).

   (c) Using the result obtained in class for δ_3 compute the one-loop beta function $\beta(g)$.

2. **Non-Abelian Scalar Field (2pts.)**:
   Consider a non-abelian gauge theory with a gauge group G. Couple to it a complex scalar field in the representation r (instead of a fermion). The Feynman rules of the theory refering to the interaction are:



$= -ig\, t^a\, (p+p')_\mu$



$= ig^2\, (t^a t^b + t^b t^a)\, g_{\mu\nu}$

where the generators t^a correspond to the representation r. Compute the β function of the theory. Show that it is given by

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{1}{3}C(r)\right)$$

where $\mathrm{Tr}\left[t^a t^b\right] = C(r)\delta^{ab}$ and $C_2(G)$ is the quadratic Casimir operator of the adjoint representation, i.e. $f^{acd}f^{bcd} = C_2(G)\delta^{ab}$.

Hint: Use that $(\delta_1 - \delta_2)$ in a non-abelian gauge theory is universal, i.e. it does not depend on the matter content.

3. **Not the Standard Model (2pts.)** : Consider an $SU(2)$ gauge theory with lagrangian

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - V(\phi^\dagger \phi) \; - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu},$$

where the scalar field ϕ transforms in the *adjoint representation* of $SU(2)$. Thus, the covariant derivative is given by

$$(D_\mu \phi)^a = \left(\partial_\mu \phi^a + g\, \epsilon^{abc}\, A^b_\mu\, \phi^c\right) \,,$$

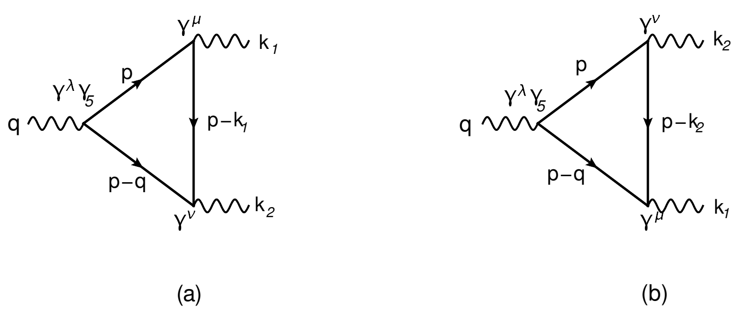
with g the $SU(2)$ coupling constant, $a, b, c = 1, 2, 3$ are the group indices and ϵ^{abc} is the Levi-Civitta tensor in three dimensions, i.e. the $SU(2)$ structure constants. Choose a vacuum with only one of the three components of $\phi^a(x)$ non zero, for instance

$$\langle \phi^3 \rangle = v, \qquad \langle \phi^1 \rangle = \langle \phi^2 \rangle = 0 \;.$$

Compute the resulting gauge boson masses. What is the symmetry breaking pattern ? I.e. Is there an remnant *unbroken* gauge symmetry ? What is it ?

Note: This is the Georgi-Glashow model. For a while it was a possible candidate for electroweak unification. Why was it discarded ?

4. **Anomalies** (2pts.)
The triangle diagrams in the figure



result in

$$\begin{aligned} i\,\Delta^{\lambda\mu\nu}(k_1,k_2) &= (-1)\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\Big[\gamma^\lambda\gamma_5\,\frac{i}{\not{p}-\not{q}}\,\gamma^\nu\,\frac{i}{\not{p}-\not{k}_1}\,\gamma^\mu\,\frac{i}{\not{p}}\Big] \\ &+ (-1)\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\Big[\gamma^\lambda\gamma_5\,\frac{i}{\not{p}-\not{q}}\,\gamma^\mu\,\frac{i}{\not{p}-\not{k}_2}\,\gamma^\nu\,\frac{i}{\not{p}}\Big] \end{aligned}$$

Prove that (show calculation in detail)

$$q_\lambda \, \Delta^{\lambda\mu\nu}(k_1, k_2) = -\frac{1}{4\pi^2} \, \epsilon^{\mu\nu\beta\alpha} \, k_{1\beta} \, k_{2\alpha} \; .$$

Explain how this is used, in general, to prove that the axial current is not conserved.

5. **Sine-Gordon Equation (2pts.)**:
Consider the Lagrangian in (1+1) dimensions:

$$\mathcal{L} = (\partial_\mu \phi) \, \partial^\mu \phi \; - V(\phi),$$

with

$$V(\phi) = \frac{a}{b} \, (1 - \cos(b\phi))$$

with a and b real parameters.

(a) Verify that the field configurations

$$\phi(t, x) = \frac{4}{b} \, \arctan \left\{ \exp \left[\pm\sqrt{ab} \, \frac{(x - vt)}{\sqrt{1 - v^2}} \right] \right\} \; ,$$

with v an arbitrary constant, are solutions of the equations of motion.

(b) Define a conserved current J^μ, such the associated charge

$$Q = \int_{-\infty}^{+\infty} dx \, J^0 \; ,$$

is the winding number of the solution.

(c) Show that the energy carried by these field configurations at $t = 0$ is

$$E = 8 \sqrt{\frac{a}{b^3}} \; .$$

(d) Expanding the Lagrangian in powers of ϕ, find the mass and the quartic coupling constant in terms of a and b. Express the energy in terms of the mass and coupling constants.