Quantum Field Theory II

Homework 2

Due 23/10/2024

1. **Electron-Phonon Coupling (Altland-Simons pg.187)(3 pts.)**:

Work through the derivation of the attractive electron interaction resulting from the electron-phonon coupling. The phonon Hamiltonian is schematically written as

$$H_{\text{ph.}} = \sum_{\mathbf{q},j} \omega_q \, a^\dagger_{\mathbf{q},j} \, a_{\mathbf{q},j}$$

where $\mathbf{q}$ is the 3-momentum, the energies ω_q depend only on the momentum's abolute value, and $j = 1, 2, 3$ are spatial indices indicating that phonons can oscillate independently in all three directions. Phonons track the displacement of ions in the lattice. This displacement (in momentum space) is given by

$$\mathbf{u_q} = \frac{a_{\mathbf{q},\mathbf{j}} + a^\dagger_{-\mathbf{q},j}}{(2m\omega_q)^{1/2}} \, \hat{e}_j \ ,$$

where $\hat{e}_j$ is the unit vector in the j direction. These displacements induce a polarization $\mathbf{P} \sim \mathbf{u}$ resulting in an induced charge $\rho_{\text{ind.}} \sim \nabla \cdot \mathbf{P}$ that couples to electrons. Then we can write the electron-phonon coupling hamiltonian as

$$H_{\text{el.}-\text{ph.}} = \gamma \int d^d r \, \hat{n}(\mathbf{r}) \, \nabla \cdot \mathbf{u}(\mathbf{r})$$

where $\hat{n}(\mathbf{r}) = c^\dagger(\mathbf{r}) \, c(\mathbf{r})$ is the electronic density.

(a) Show that the electron-phonon interaction hamiltonian can be written as

$$H_{\text{el.}-\text{ph.}} = \gamma \sum_{\mathbf{q},j} \frac{iq_j}{(2m\omega_q)^{1/2}} \, \hat{n}_{\mathbf{q}} (a_{\mathbf{q},j} + a^\dagger_{-\mathbf{q},j})$$

with

$$\hat{n}_{\mathbf{q}} = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}}$$

the density of electrons in terms of creation and annihilation operators.

(b) Write the coherent-state partition function as

$$Z = \int \mathcal{D}(\bar{\psi}, \psi) \int \mathcal{D}(\bar{\phi}, \phi)\, e^{-S_{\text{el.}}[\bar{\psi},\psi]-S_{\text{ph.}}[\bar{\phi},\phi]-S_{\text{el.-ph.}}[\bar{\psi},\psi,\bar{\phi},\phi]}$$

in terms of the fermionic and bosonic coherent-state variables ψ and ϕ. Show that

$$S_{\text{ph.}}[\bar{\phi}, \phi] = \sum_{q,j} \bar{\phi}_{q,j}(-i\omega_n + \omega_q)\phi_{q,j}$$

and

$$S_{\text{el.-ph.}}[\bar{\psi}, \psi, \bar{\phi}, \phi] = \gamma \sum_{q,j} \frac{iq_j}{(2m\omega_q)^{1/2}}\, \rho_q \left(\phi_{q,j} + \bar{\phi}_{-q,j}\right)$$

where

$$\rho_q = \sum_k \bar{\psi}_{k+q}\psi_k \ .$$

(c) Integrate out the phonon fields and show that an attractive interaction between electrons is generated. (Remember that we need to shift the phonon fields to get rid of the electron-phonon interaction!)

<u>**Note**</u>: Show all your calculations in detail.

2. **<u>The Gap Equation in a Superconductor</u> (3 pts.)**:

The action for a superconductor in the coherent state representation can be written as

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \int d^d r \left\{ \bar{\psi}_\sigma \left[\partial_\tau + ie\phi + \frac{1}{2m} \left(-i\vec{\nabla} - e\mathbf{A} \right)^2 - \mu \right] \psi_\sigma - g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right\} ,$$

where $\sigma =\uparrow, \downarrow$ denotes spin, ϕ and $\mathbf{A}$ are the scalar and vector potentials and the last term is the BCS interaction with $g > 0$.

(a) Show that we can write the BCS interaction as

$$e^{g \int d\tau d^d r \, \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow} = \int \mathcal{D}(\bar{\Delta}, \Delta) e^{- \int d\tau d^d r \left\{ (1/g) |\Delta|^2 - \left(\bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow \right) \right\}} ,$$

where $\Delta(\mathbf{r}, \tau)$ is a complex scalar field obeying *periodic* boundary conditions, i.e. $\Delta(0) = +\Delta(\beta)$.

(b) Use the Nambu spinor notation

$$\bar{\Psi} = \begin{pmatrix} \bar{\psi}_\uparrow & \psi_\downarrow \end{pmatrix} , \qquad \Psi = \begin{pmatrix} \psi_\uparrow \\ \bar{\psi}_\downarrow \end{pmatrix}$$

to show that the partition function can be written as

$$Z = \int \mathcal{D}(\bar{\psi}, \psi) \mathcal{D}(\bar{\Delta}, \Delta) e^{- \int d\tau d^d r \left[(1/g) |\Delta|^2 - \bar{\Psi} \mathcal{O}^{-1} \Psi \right]} ,$$

and show the explicit form of the operator $\mathcal{O}^{-1}$.

(c) After integrating out the fermionic degrees of freedom, obtain the mean field equation satisfied by Δ. (Ignore the scalar and vector potentials, i.e. set $\phi = 0$ and $\mathbf{A} = 0$.) Show that, in the Matsubara representation, the mean field solution Δ_0 satisfies

$$\frac{1}{g} = \frac{T}{L^d} \sum_{\mathbf{p}, n} \frac{1}{\omega_n^2 + (\frac{p^2}{2m} - \mu)^2 + |\Delta_0|^2} .$$

This is the gap equation and, once the sum over the Matsubara frequencies is performed, it can be written as an integral over the energies. It contains the temperature dependence of Δ_0, the energy gap.

3. **Coleman-Weinberg with Fermions (4pts.)** :

Consider the following theory:

$$\mathcal{L} = \bar{\psi}(i \not{\partial} - m_f)\psi - g\phi \bar{\psi} \psi + \mathcal{L}(\phi) \ ,$$

where a fermion os mass m_f couples to a real scalar field. Here $\mathcal{L}(\phi)$ is the Lagrangean of the scalar field alone. We want to compute the contribution of the fermion to the effective potential of the scalar field.

Starting from the generating functional

$$Z[0] = \int D\phi \, D\bar{\psi} \, D\psi \, e^{i \int d^4x \mathcal{L}}$$

(a) Obtain an expression for the effective action integrating out the fermion.

(b) Compute the fermion contribution to the effective potential $V_{eff.}(\phi)$. Define a ϕ-dependent fermion mass $m(\phi) \equiv m_f + g\,\phi$. Then follow the same steps as for the case of the scalar contribution we did in class. Once you Fourier transform into momentum space you will still have a trace left: the one over the Dirac indices. It is useful to use (after proving) the following identity:

$$\mathrm{Tr}\,\mathrm{Ln}\left[(\not{p} - a)\right] = \frac{1}{2}\,\mathrm{Tr}\,\mathrm{Ln}\left[(-1)\,(p^2 - a^2)\right]$$