Quantum Field Theory II

Homework 2

Due 25/10/2023

1. **Electron-Phonon Coupling (Altland-Simons pg.187)(4 pts.)**:

Work through the derivation of the attractive electron interaction resulting from the electron-phonon coupling. The phonon Hamiltonian is schematically written as

$$H_{\text{ph.}} = \sum_{\mathbf{q},j} \omega_q \, a^\dagger_{\mathbf{q},j} \, a_{\mathbf{q},j}$$

where $\mathbf{q}$ is the 3-momentum, the energies ω_q depend only on the momentum's abolute value, and $j = 1, 2, 3$ are spatial indices indicating that phonons can oscillate independently in all three directions. Phonons track the displacement of ions in the lattice. This displacement (in momentum space) is given by

$$\mathbf{u}_\mathbf{q} = \frac{a_{\mathbf{q},\mathbf{j}} + a^\dagger_{-\mathbf{q},j}}{(2m\omega_q)^{1/2}} \, \hat{e}_j \ ,$$

where $\hat{e}_j$ is the unit vector in the j direction. These displacements induce a polarization $\mathbf{P} \sim \mathbf{u}$ resulting in an induced charge $\rho_{\text{ind.}} \sim \nabla \cdot \mathbf{P}$ that couples to electrons. Then we can write the electron-phonon coupling hamiltonian as

$$H_{\text{el.}-\text{ph.}} = \gamma \int d^d r \, \hat{n}(\mathbf{r}) \, \nabla \cdot \mathbf{u}(\mathbf{r})$$

where $\hat{n}(\mathbf{r}) = c^\dagger(\mathbf{r}) \, c(\mathbf{r})$ is the electronic density.

(a) Show that the electron-phonon interaction hamiltonian can be written as

$$H_{\text{el.}-\text{ph.}} = \gamma \sum_{\mathbf{q},j} \frac{i q_j}{(2m\omega_q)^{1/2}} \, \hat{n}_\mathbf{q} (a_{\mathbf{q},j} + a^\dagger_{-\mathbf{q},j})$$

with

$$\hat{n}_\mathbf{q} = \sum_\mathbf{k} c^\dagger_{\mathbf{k}+\mathbf{q}} c_\mathbf{k}$$

the density of electrons in terms of creation and annihilation operators.

(b) Write the coherent-state partition function as

$$Z = \int \mathcal{D}(\bar{\psi}, \psi) \int \mathcal{D}(\bar{\phi}, \phi)\, e^{-S_{\mathrm{el.}}[\bar{\psi},\psi] - S_{\mathrm{ph.}}[\bar{\phi},\phi] - S_{\mathrm{el.-ph.}}[\bar{\psi},\psi,\bar{\phi},\phi]}$$

in terms of the fermionic and bosonic coherent-state variables ψ and ϕ. Show that

$$S_{\mathrm{ph.}}[\bar{\phi}, \phi] = \sum_{q,j} \bar{\phi}_{q,j}(-i\omega_n + \omega_q)\phi_{q,j}$$

and

$$S_{\mathrm{el.-ph.}}[\bar{\psi}, \psi, \bar{\phi}, \phi] = \gamma \sum_{q,j} \frac{iq_j}{(2m\omega_q)^{1/2}}\, \rho_q \left(\phi_{q,j} + \bar{\phi}_{-q,j}\right)$$

where

$$\rho_q = \sum_k \bar{\psi}_{k+q}\psi_k \ .$$

(c) Integrate out the phonon fields and show that an attractive interaction between electrons is generated. (Remember that we need to shift the phonon fields to get rid of the electron-phonon interaction!)

<u>**Note**</u>: Show all your calculations in detail.

2. **Spontaneous Breaking of Global $U(1)$ Symmetry in a Superfluid (3 pts.)**:

Starting from the action for an interacting Bose gas in coherent state representation

$$S[\bar{\phi}, \phi] = \int_0^\beta d\tau \int d^d r \left\{ \bar{\phi}(\mathbf{r}, \tau) \left(\partial_\tau + \frac{\left(-i\vec{\nabla}\right)^2}{2m} - \mu \right) \phi(\mathbf{r}, \tau) + \frac{g}{2} \left(\bar{\phi}(\mathbf{r}, \tau) \phi(\mathbf{r}, \tau) \right)^2 \right\} ,$$

use the change of variables

$$\phi(\mathbf{r}, \tau) = \sqrt{\rho(\mathbf{r}, \tau)}\, e^{i\theta(\mathbf{r}, \tau)} ,$$

to expand around the ground state given by the mean field solution. That is using

$$\rho(\mathbf{r}, \tau) = \rho_0 + h(\mathbf{r}, \tau) ,$$

where ρ_0 is the density of the ground state condensate, and $h(\mathbf{r}, \tau)$ is a density fluctuation around it, obtain the action for $h(\mathbf{r}, \tau)$ and $\theta(\mathbf{r}, \tau)$, up to leading order in the fluctuations. Show how and why at low energies only the $\theta(\mathbf{r}, \tau)$ mode survives and its effective action is given by

$$S_{\text{eff.}}[\theta] = \int d\tau \int d^d r \left\{ \frac{1}{2g} \left(\partial_\tau \theta\right)^2 + \frac{\rho_0}{2m} \left(\vec{\nabla}\theta\right)^2 \right\} .$$

Comment on the dispersion relation (relation between energy and momentum) for $\theta(\mathbf{r}, \tau)$.

3. **Spontaneous Breaking of Chiral Symmetry (3 pts.)**:

Consider a Dirac fermion $\psi(x)$ coupled to a complex scalar field $\Phi(x)$ in a theory described by the Lagrangian

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi - g \left(\bar{\psi}_L \psi_R \Phi + \bar{\psi}_R \psi_L \Phi^* \right) + \mathcal{L}_\Phi \ ,$$

where

$$\mathcal{L}_\Phi = \partial_\mu \Phi^* \partial^\mu \Phi - V(\Phi^* \Phi) \ .$$

(a) Show that this theory is invariant under the global $U(1)_L \times U(1)_R$ chiral symmetry defined by the transformations

$$\psi_L \to e^{i\theta_L} \psi_L, \qquad \psi_R \to e^{i\theta_R} \psi_R, \qquad \Phi \to e^{i(\theta_L - \theta_R)} \Phi$$

(b) Consider now the potential given by

$$V(\Phi^* \Phi) = -m^2 |\Phi|^2 + \frac{\lambda}{2} (|\Phi|^2)^2 \ ,$$

where $\lambda > 0$. Show that the vacuum state (ground state), is non trivial, i.e. $\langle \Phi \rangle \neq 0$. What is the value of $\langle \Phi \rangle \equiv v$ in terms of the parameters of the theory ? Discuss.

(c) We can parameterize the fluctuations around the vacuum by writing the complex scalar field as

$$\Phi(x) = (v + h(x)) \ e^{i\pi(x)/f} \ ,$$

where we introduced the scale f to make the argument of the exponent dimensionless. Write the Lagrangian in terms of the fields $h(x)$ and $\pi(x)$. What are their masses, m_h and m_π ? What is the value of the scale f ?

(d) From the Lagrangian in the previous point identify the fermion mass m_ψ and its couplings to $\pi(x)$ and $h(x)$. Discuss in detail the fact that the fermion mass term $\bar{\psi}_L \psi_R$+h.c. appears here, despite originally being forbidden by the global chiral symmetry. Is the symmetry explicitly broken ?