Quantum Field Theory II

Homework 1

Due 18/09/2024

1. **Beta Functions of Yukawa Theory**: (5 pts.)
   Consider the point 1. of Homework 5 of the first semester, the Yukawa lagrangian with pseudoscalar coupling:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\,\partial\!\!\!/ - M)\psi - ig\,\bar{\psi}\gamma_5\psi\,\phi,$$

   where ϕ is a real scalar field and ψ is a Dirac fermion. The Lagrangean is invariant under parity transformations defined by

$$\psi(t,\vec{x}) \to \gamma^0\,\psi(t,-\vec{x})$$

$$\phi(x,\vec{x}) \to -\phi(t,-\vec{x})\ ,$$

   which implies that ϕ is odd (pseudoscalar). You have computed the counterterms δ_ψ, δ_ϕ, δ_g and δ_λ. They are (up to finite and μ-independent pieces):

$$\begin{aligned}
\delta_\psi &= -\frac{g^2}{32\pi^2}\left(\frac{2}{\epsilon} - \ln\mu^2\right)\\
\delta_\phi &= -\frac{g^2}{8\pi^2}\left(\frac{2}{\epsilon} - \ln\mu^2\right)\\
\delta_g &= \frac{g^3}{16\pi^2}\left(\frac{2}{\epsilon} - \ln\mu^2\right)\\
\delta_\lambda &= \frac{3\lambda^2 - 48g^4}{32\pi^2}\left(\frac{2}{\epsilon} - \ln\mu^2\right)
\end{aligned}$$

   Using these, compute the beta functions $\beta(g)$ and $\beta(\lambda)$ to first order in the couplings, assuming λ and g^2 are of the same order.

2. **RG Description of the Ferromagnetic Transition**: (5 pts.)
   The Landau-Ginzburg description of a ferromagnet corresponds to d dimensional Euclidean ϕ^4 theory, with the action

$$S[\phi] = \int d^dx\,\left\{\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\,\phi^2 + \frac{\lambda}{4!}\phi^4\right\}\ ,$$

   (a) Obtain the leading order shifts for λ and m^2 when rescaling dow to lower energies or larger distance scales. That is, the scale shift $x \to x/b$ plus integrating the high energy modes from $\Lambda/b \to \Lambda$, where Λ is the UV cutoff.

(b) Obtain the renormalization group equation for this shift in length scale, i.e. obtain

$$\frac{d\lambda}{d\ln b} \qquad \text{and} \qquad \frac{dm^2}{d\ln b}$$

to leading order in the ϵ expansion, where $\epsilon = 4 - d$. Show that this results in an IR-stable fixed point for

$$\lambda_* = \frac{16\pi^2}{3}\,\epsilon \qquad m_*^2 = -\frac{\epsilon}{3}$$

What is the behavior in the vicinity of the Gaussian fixed point

$$\lambda_* = 0 \qquad m_*^2 = 0 \ ?$$

Hint: To eliminate the dependence on the cutoff Λ, rescale all momenta in terms of Λ as $k \to k' = k/\Lambda$ so as to make the momenta dimensionless and a fraction of the cutoff. Similarly, with length scales, $x \to x' = x\Lambda$. See lecture 4 for details.