Lecture 8

Generating Functional Method

Here we will introduce a so-called external source into the formalism. This is an artifact in the calculation and for the purpose we will be using it here it has no physical significance. The propagator in the presence of an external source $j(t)$ is now [1]

$$\langle x_f, t_f | x_i, t_i \rangle_{j(t)} \equiv \int \mathcal{D}x(t)\; e^{\displaystyle i\int_{t_i}^{t_f} dt\,(L(t) + j(t)\,x(t))} \;. \tag{8.1}$$

Now the amplitude is a functional of the linearly coupled source $j(t)$. We define the functional derivative as the following operation

$$\frac{\delta j(t)}{\delta j(t')} = \delta(t - t') \;. \tag{8.2}$$

This definition is suggested by

$$j(t) = \int dt'\,\delta(t - t')\,j(t') \;. \tag{8.3}$$

As a result, we can write the matrix element of the time-ordered product of operators as

$$\boxed{\langle x_f, t_f | T(\mathbf{x}(t_1)\ldots\mathbf{x}(t_n)) | x_i, t_i \rangle \,=\, (-i)^n \,\frac{\delta^n}{\delta j(t_1)\ldots\delta j(t_n)}\,\langle x_f, t_f | x_i, t_i \rangle_{j(t)}\bigg|_{j(t)=0} \;.} \tag{8.4}$$

This should be obvious from (18.1), as for each functional derivative defined in (18.2) we

[1] From now one we restore $\hbar = 1$.

obtain one power of $x(t)$ but evaluated at the time of the source with respect to which we are deriving. That is

$$\frac{\delta}{\delta j(t_1)}\langle x_f, t_f|x_i, t_i\rangle_{j(t)} = \int \mathcal{D}x(t)\, ix(t_1)\, e^{iS[x(t)]+\int dt j(t)x(t)} , \tag{8.5}$$

so that

$$\begin{aligned}\frac{\delta^2}{\delta j(t_1)\delta j(t_2)}\langle x_f, t_f|x_i, t_i\rangle_{j(t)}\bigg|_{j=0} &= (i)^2 \int \mathcal{D}x(t)\, x(t_1)\, x(t_2)\, e^{iS[x(t)]} , \\ &= (i)^2 \langle x_f, t_f|T(\mathbf{x}(t_1)\mathbf{x}(t_2))|x_i, t_i\rangle , \end{aligned}\tag{8.6}$$

8.1 Selecting the Ground State

In most of our applications we will need to compute the expectation value of the time-ordered product of operators in the ground state (the "vacuum" in the case of quantum field theory). The ground state wave-function is defined by

$$\langle x, t|0\rangle = \psi_0(x)\, e^{-iE_0 t} . \tag{8.7}$$

In this way the desired matrix element can be written as

$$\begin{aligned}\langle 0|T(\mathbf{x}(t_1)\ldots\mathbf{x}(t_n))|0\rangle &= \int \langle 0|x_f, t_f\rangle\langle x_f, t_f|T(\mathbf{x}(t_1)\ldots\mathbf{x}(t_n))|x_i, t_i\rangle\langle x_i, t_i|0\rangle\, dx_f\, dx_i , \quad (8.8)\\ &= \int e^{iE_0(t_f-t_i)}\, \psi_0^*(x_f)\psi_0(x_i)\, \langle x_f, t_f|T(\mathbf{x}(t_1)\ldots\mathbf{x}(t_n))|x_i, t_i\rangle\, dx_f\, dx_i .\end{aligned}$$

However the above expression is not very useful as it involves the knowledge of all the matrix elements over all the states including the vacuum. In order to select the ground state matrix elements we will make use of a trick which called generically Wick rotation. In this incarnation it involves the analytic continuation of the correlation functions to imaginary time. We will use it later in field theory, in the energy instead of in the time variable, but the idea is the same. We start with

$$\langle x_f, t_f|x_i, t_i\rangle = \sum_n \psi_n(x_f)\, \psi_n^*(x_i)\, e^{-iE_n(t_f-t_i)} , \tag{8.9}$$

where we can see that if the energy had a small negative imaginary part, then when $(t_f - t_i) \to +\infty$ the largest contribution comes from the ground state E_0. We will use this method later. Alternatively, we can consider analytically continuing the amplitude to imaginary time in the following way:

$$t_f \to -i\tau_f \; , \qquad\qquad t_i \to +i\tau_i \; , \tag{8.10}$$

where τ_f and τ_i are real and positive , so that the ground state is selected in the limit $(\tau_f, \tau_i) \to \infty$

$$\lim_{(\tau_f,\tau_i)\to\infty} \langle x_f, t_f | x_i, t_i\rangle \simeq \psi_0(x_f)\psi_0^*(x_i)\, e^{-E_0(\tau_f+\tau_i)} \; . \tag{8.11}$$

Similarly, we can write

$$\lim_{(\tau_f,\tau_i)\to\infty} \langle x_f, t_f | T(\mathbf{x}(t_1)\ldots\mathbf{x}(t_n))| x_i, t_i\rangle \simeq \psi_0(x_f)\psi_0^*(x_i)\, e^{-E_0(\tau_f+\tau_i)}\, \langle 0|T(\mathbf{x}(t_1)\ldots\mathbf{x}(t_n))|0\rangle \; , \tag{8.12}$$

from which we can extract the ground state expectation value of the time-ordered product of operators as

$$\langle 0|T(\mathbf{x}(t_1)\ldots\mathbf{x}(t_n))|0\rangle = \lim_{t_f\to -i\infty, t_i\to +i\infty} \frac{e^{iE_0(t_f-t_i)}}{\psi_0(x_f)\psi_0^*(x_i)}\, \langle x_f, t_f | T(\mathbf{x}(t_1)\ldots\mathbf{x}(t_n))| x_i, t_i\rangle \; . \tag{8.13}$$

This way of selecting the ground state will allow us to make use of the generating functional method as we sketch as follows.

8.2 The Generating Functional

We define the generating functional in the presence of the source $j(t)$ as

$$Z[j] \equiv \lim_{t_f, ti\to \mp i\infty} \langle x_f, t_f | x_i, t_i\rangle_{j(t)} \; . \tag{8.14}$$

Then its representation in terms of the path integral is

$$Z[j] = \lim_{t_f, ti\to \mp i\infty} \int \mathcal{D}x(t) e^{i\int_{t_i}^{t_f} dt(L(t)+x(t)j(t))} \; . \tag{8.15}$$

Since

$$Z[0] = \lim_{t_f,ti\to\mp i\infty} \langle x_f, t_f | x_i, t_i\rangle = \psi_0(x_f)\psi_0^*(x_i)e^{-E_0(\tau_f+\tau_i)} \;, \tag{8.16}$$

we arrive, using (18.13), at

$$\boxed{\langle 0|T(\mathbf{x}(t_1)\ldots\mathbf{x}(t_n))|0\rangle = (-i)^n\,\frac{1}{Z[0]}\,\frac{\delta^n}{\delta j(t_1)\ldots\delta j(t_n)}Z[j]\Big|_{j=0}} \;. \tag{8.17}$$

The above expression is the master formula to obtain the correlation functions of the theory and it will have an analogous expression in the quantum field theory formulation. The procedure defined above to select the ground state relies on using imaginary time and the limits $t\to\pm i\,\infty$, wich amounts to using Euclidean four dimensional space. However, in most applications we will need to work in Minkowski space, which requires we use real time limits $t\to\pm\infty$. To see that the results above derived in imaginary time still hold we have to define an analytic continuation back to real time. We will do this below by making use of a $i\epsilon$ prescription that will pick up the correct singularities of the Green functions. As we will see, the result will be equivalent to a rotation from imaginary to real time.

8.3 Example: The Harmonic Oscillator

Before we start with the application of the path integral to quantum field theory it will be very instructive to work out an example in some detail. In particular, the harmonic oscillator in quantum mechanics is simple enough but contains many of the elements we will be using later. We start with the action in the presence of an external source defined by

$$S^j[x(t)] \equiv \int_{-\infty}^{+\infty} dt\,\left(\frac{m}{2}\dot{x}^2(t) - \frac{m\omega^2}{2}x^2(t) + x(t)j(t)\right) \;, \tag{8.18}$$

where ω is the oscillator's frequency and m the mass. It will be convenient to work on the Fourier transformed variables. We then define $\tilde{x}(E)$ by

$$x(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\,e^{-iEt}\,\tilde{x}(E) \;, \tag{8.19}$$

as well as the Fourier transform of the external source $j(t)$ by

$$j(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, e^{-iEt}\, \tilde{\jmath}(E) \ . \tag{8.20}$$

Baseed on these definitions we can now write the action (18.18) in the E variable. To do this we must first compute

$$\dot{x}(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, (-iE)\, e^{-iEt}\, \tilde{x}(E) \ , \tag{8.21}$$

which results in

$$\int_{-\infty}^{+\infty} dt\, \dot{x}^2(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \int_{-\infty}^{+\infty} \frac{dE'}{2\pi} (-iE)(-iE')\tilde{x}(E)\tilde{x}(E') \int_{-\infty}^{+\infty} dt e^{-i(E+E')t} \ . \tag{8.22}$$

The last integral in the expression above results in a factor of $2\pi\delta(E+E')$, which gives us

$$\int_{-\infty}^{+\infty} dt\, \dot{x}^2(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, (+E^2)\, \tilde{x}(E)\, \tilde{x}(-E) \ . \tag{8.23}$$

Following similar steps we obtain

$$\int_{-\infty}^{+\infty} dt\, x^2(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, \tilde{x}(E)\, \tilde{x}(-E) \ . \tag{8.24}$$

and

$$\int_{-\infty}^{+\infty} dt\, x(t)\, j(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, \tilde{x}(E)\, \tilde{\jmath}(-E) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, \tilde{x}(-E)\, \tilde{\jmath}(E) \ . \tag{8.25}$$

Replacing the expression above into the action (18.18) we have

$$S^j[x(t)] = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \left(\frac{m}{2}(E^2-\omega^2)\tilde{x}(E)\tilde{x}(-E) + \tilde{x}(E)\tilde{\jmath}(-E) \right) \ . \tag{8.26}$$

Now we would like to make a change of our functional variable of integration in the path integral, here $x(t)$ or rather its Fourier transform $\tilde{x}(E)$, such that the new variable is decoupled from the external source. In other words, we want a change of variables that cancels the last term in (18.26). The desired new variable is

$$\tilde{x}'(E) \equiv \tilde{x}(E) + \frac{\tilde{\jmath}(E)}{m(E^2-\omega^2)} \; . \tag{8.27}$$

When replacing $\tilde{x}(E)$ from (18.27) in (18.26) we get

$$\begin{aligned} S^j[x(t)] \;\; = \;\; & \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \left\{ \frac{m}{2}(E^2-\omega^2)\tilde{x}'(E)\tilde{x}'(-E) - \frac{1}{2}\tilde{x}'(E)\tilde{\jmath}(-E) - \frac{1}{2}\tilde{x}'(-E)\tilde{\jmath}(E) \right. \\ & + \frac{1}{2}\tilde{\jmath}(E)\frac{1}{m(E^2-\omega^2)}\tilde{\jmath}(-E) + \frac{1}{2}\tilde{x}'(E)\tilde{\jmath}(-E) + \frac{1}{2}\tilde{x}'(-E)\tilde{\jmath}(E) \\ & \left. - \frac{1}{m}\,\tilde{\jmath}(E)\,\frac{1}{(E^2-\omega^2)}\,\tilde{\jmath}(-E) \right\} \end{aligned} \tag{8.28}$$

which results in

$$S^j[x(t)] = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \left\{ \frac{m}{2}(E^2-\omega^2)\tilde{x}'(E)\tilde{x}'(-E) - \frac{1}{2m}\,\tilde{\jmath}(E)\,\frac{1}{(E^2-\omega^2)}\,\tilde{\jmath}(-E) \right\} \; . \tag{8.29}$$

Finally, we use the inverse Fourier transforms

$$\tilde{x}'(E) = \int_{-\infty}^{+\infty} dt\, e^{iEt}\, x'(t) \; , \qquad\qquad \tilde{\jmath}(E) = \int_{-\infty}^{+\infty} dt\, e^{iEt}\, j(t) \; , \tag{8.30}$$

to obtain the action back in the time variable

$$\boxed{S^j[x(t)] = \int_{-\infty}^{+\infty} dt \left\{ \frac{m}{2}\dot{x}'^2(t) - \frac{m\omega^2}{2}x'^2(t) \right\} - \frac{1}{2m}\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt'\, j(t)\, D(t-t') j(t')} \; , \tag{8.31}$$

where we have defined

$$D(t-t') \equiv \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, e^{-iE(t-t')}\, \frac{1}{E^2-\omega^2} \; . \tag{8.32}$$

It is interesting to point out that $D(t-t')$ is the Green function associated to the equation of motion, which for the harmonic oscillator is

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2 \right) x(t) = 0 \; , \tag{8.33}$$

It is straightforward to check that $D(t-t')$ as written in (18.32) satisfies

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) D(t-t') = -\delta(t-t') \ . \tag{8.34}$$

Coming back to (18.31), we can see that the action is now separated into two pieces: one that is source-independent and one that contains all the dependence on the external source. The first term in (18.31) will give rise to $Z[0]$, which means that all the dynamical information on the correlation functions defined by (18.17) is in the second term. Thus, we need to concentrate in understanding the function $D(t-t')$. We will do this by performing the integral in (18.32).

To compute $D(t-t')$ we will analytically continue E to the complex plane. In particular we will write

$$D(t-t') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, e^{-iE(t-t')}\, \frac{1}{E^2-\omega^2+i\epsilon} \ , \tag{8.35}$$

where ϵ is a real, positive and infinitesimal constant. For instance, the singularities in E can be shifted to complex values by shifting the oscillator frequency from ω^2 to $\omega^2 - i\epsilon$. Now the poles are shifted: from $\pm\omega$ to $\omega - i\epsilon$ and $-\omega + i\epsilon$ (see Figure 18.1). We can already see that allowing for an imaginary part for the energy could be equivalent to having an imaginary time, as long as the signs of these imaginary times are appropriately chosen. Thus, we will see that choosing the shift $i\epsilon$ in the energy singularities is taken the place of the imaginary time prescription.

Now to perform the integral we have to decide how to close the contour in order to use Cauchy's theorem. The choice of contour will depend on the time ordering.

$\underline{(t-t')>0}$: We need to close the contour with $Im[E] < 0$, so that when $Im[E]^2 + Re[E]^2 \to \infty$ the semi-circle does not contribute.[2] In this case then only the pole at $\omega - i\epsilon$ contributes and according to Cauchy's theorem we have

$$\begin{aligned} D(t-t') &= (-2\pi i)\,\frac{1}{2\pi}\, e^{-i\omega(t-t')}\,\frac{1}{2\omega} \ , \\ &= -i\,\frac{e^{-i\omega(t-t')}}{2\omega} \ . \end{aligned} \tag{8.36}$$

$\underline{(t-t')<0}$: For this time ordering we need to close the contour for $Im[E] > 0$ so that the semi-circle does not contribute.[3] This results in

[2] We see that this is equivalent to $t \to -i\tau$ and $t' \to +i\tau'$.

[3] Conversely, this is equivalent to $t \to +i\tau$ and $t' \to -i\tau'$.

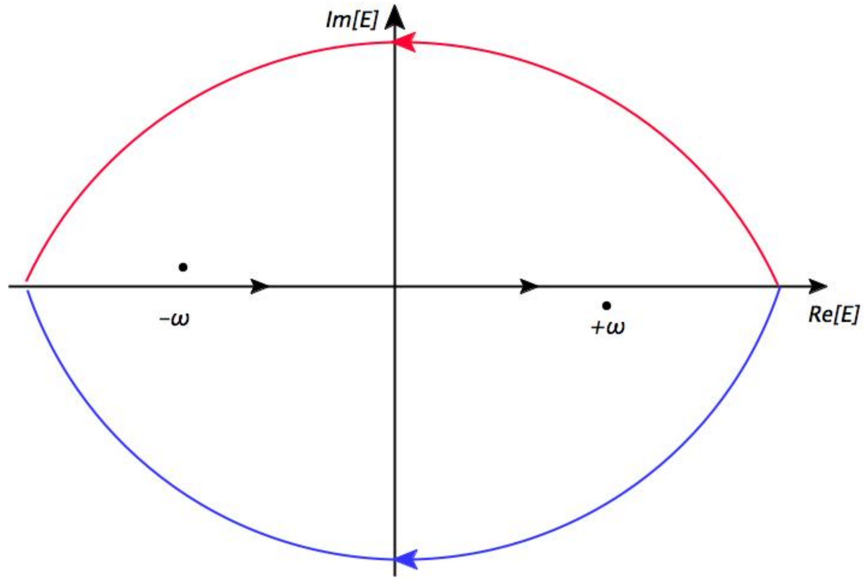

Figure 8.1: Contours in the complex E plane. For $(t-t')>0$ we need to close the (blue) contour for $Im[E]<0$. For $(t-t')<0$ we can use the (red) contour above, i.e. for $Im[E]>0$.

$$\begin{aligned} D(t-t') &= (+2\pi i)\,\frac{1}{2\pi}\,e^{+i\omega(t-t')}\,\frac{1}{-2\omega}\ , \\ &= -i\,\frac{e^{+i\omega(t-t')}}{2\omega}\ . \end{aligned} \tag{8.37}$$

Then, we can combine both results independently of the time ordering in the following expression

$$\boxed{D(t-t') = -\frac{i}{2\omega}\,e^{-i\omega|t-t'|}}. \tag{8.38}$$

With the answer in (18.38) we will be able to compute correlation functions using the generating functional

$$Z[j] = Z[0]\ e^{-\dfrac{i}{2m}\displaystyle\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt'\, j(t)\, D(t-t')\, j(t')}\ , \tag{8.39}$$

where we used

$$Z[0] = \int \mathcal{D}x'(t) e^{iS[x'(t)]} , \tag{8.40}$$

and

$$S[x'(t)] = \int_{-\infty}^{+\infty} dt \left\{ \frac{m}{2} \dot{x}'^2(t) - \frac{m\omega^2}{2} x'^2(t) \right\} . \tag{8.41}$$

The assumption we are making in (18.40) is that the measure of the path integral is invariant under the transformation

$$x(t) \longrightarrow x'(t) = x(t) + \frac{1}{m} \int_{-\infty}^{+\infty} dt' D(t-t') j(t') , \tag{8.42}$$

that is that $\mathcal{D}x'(t) = \mathcal{D}x(t)$. This is a good assumption in the example we are considering. In general, the failure of the invariance of the measure under (18.42) would signal the existence of a so-called anomaly: an apparent symmetry of the classical theory (in this case translation) that is not respected by the quantization procedure, in this case represented by the path integral and in particular its measure. We will see examples of anomalies later in the course. But for now we do not have to worry about them.

We are now in a position to compute correlation functions using (18.17). We start with the product of two position operators

$$\langle 0 | T(\mathbf{x}(t_1)\mathbf{x}(t_2)) | 0 \rangle = \frac{(-i)^2}{Z[0]} \frac{\delta^2}{\delta j(t_1)\, \delta j(t_2)} Z[j] \bigg|_{j=0} . \tag{8.43}$$

If we expand the exponential in (18.39) as

$$e^{-\frac{i}{2m}\int dt \int dt'\, j(t)\, D(t-t')\, j(t')} = 1 - \frac{i}{2m} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt'\, j(t)\, D(t-t')\, j(t') + \ldots , \tag{8.44}$$

it is clear that for the correlation function in (18.43) we only need the first term in the expansion, since all the higher order terms in $j(t)$ will vanish when making $j(t) = 0$ after the functional derivative. In this way we obtain

$$\langle 0 | T(\mathbf{x}(t_1)\mathbf{x}(t_2)) | 0 \rangle = \frac{i}{m} D(t_1 - t_2) = \frac{1}{2m\omega} e^{-i\omega|t_1 - t_2|} . \tag{8.45}$$

Then we can take the $t_1 \to t_2$ limit to obtain

$$\langle 0|x^2|0\rangle = \lim_{t_1\to t_2} \langle 0|T(\mathbf{x}(t_1)\mathbf{x}(t_2))|0\rangle = \frac{1}{2m\omega} \;, \tag{8.46}$$

which is the well known result from the usual treatment of the harmonic oscillator.

For applications to matrix elements between occupied states we need to introduce the definition of states with n particles as

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n\,|0\rangle \tag{8.47}$$

where the creation operator is defined in terms of position and momentum in the usual way

$$a^\dagger = \sqrt{\frac{m\omega}{2}}\left(x - \frac{i}{m\omega}\,p\right) = \sqrt{\frac{m\omega}{2}}\left(1 - \frac{i}{\omega}\frac{\partial}{\partial t}\right) x(t) \;, \tag{8.48}$$

and where in the last equality we replaced the momentum by $p = \dot{x}/m$. With these definitions now we can compute, for instance

$$\begin{aligned}
\langle 0|x|1\rangle &= \lim_{t_1\to t_2} \sqrt{\frac{m\omega}{2}}\left(1 - \frac{i}{\omega}\frac{\partial}{\partial t_2}\right) \langle 0|\mathbf{x}(t_1)\,\mathbf{x}(t_2)|0\rangle \\
&= \lim_{t_1\to t_2} \sqrt{\frac{m\omega}{2}}\,(-i)^2\left(1 - \frac{i}{\omega}\frac{\partial}{\partial t_2}\right) \frac{1}{Z[0]}\,\frac{\delta^2 Z[j]}{\delta j(t_1)\,\delta j(t_2)} \\
&= \lim_{t_1\to t_2} \sqrt{\frac{m\omega}{2}}\left(1 - \frac{i}{\omega}\frac{\partial}{\partial t_2}\right) \frac{i}{m}\,D(t_1 - t_2) \;\; = \frac{1}{\sqrt{2m\omega}} \;,
\end{aligned} \tag{8.49}$$

which is the correct answer. Even more interesting is computing

$$\langle 1|x^2|1\rangle = \frac{m\omega}{2} \lim_{t_1\to t^+,t_2\to t^-} \left(1 + \frac{i}{\omega}\frac{\partial}{\partial t_1}\right)\left(1 - \frac{i}{\omega}\frac{\partial}{\partial t_2}\right) \langle 0|\mathbf{x}(t_1)\,\mathbf{x}^2(t)\,\mathbf{x}(t_2)|0\rangle \;, \tag{8.50}$$

We will see that this is similar to computing a four-point correlation function in quantum field theory. We obtain

$$\begin{aligned}\langle 1|x^2|1\rangle &= \frac{(-i)^4}{Z[0]} \lim_{t_1\to t^+, t_2\to t^-} \left(\frac{m\omega}{2}\right) \left(1+\frac{i}{\omega}\frac{\partial}{\partial t_1}\right) \left(1-\frac{i}{\omega}\frac{\partial}{\partial t_2}\right) \frac{\delta^4}{\delta j(t_1)\,\delta j(t_2)\,\delta j(t)\,\delta j(t')} Z[j]\Big|_{j=0} \\ &= \frac{m\omega}{2}\left(\frac{i}{m}\right)^2 \lim_{t_1\to t^+, t_2\to t^-} \left(1+\frac{i}{\omega}\frac{\partial}{\partial t_1}\right)\left(1-\frac{i}{\omega}\frac{\partial}{\partial t_2}\right) \\ &\qquad \times \{D(t_1-t_2)\,D(0) + D(t_1-t)\,D(t-t_2)\} = \frac{3}{2m\omega}\ ,\end{aligned} \tag{8.51}$$

which again is the right answer for the harmonic oscillator. The equality before last says that the four-point function can be obtained as the sum over products of two-point functions, the D's. We will obtain a similar result in quantum field theory, where it will have the name of Wick's theorem.

8.4 The Path Integral and Statistical Mechanics

We end this lecture with a very important connection of the path integral formulation to statistical mechanics. In general we can write the partition function of a system as

$$Z = Tr\left[e^{-\beta H}\right]\ , \tag{8.52}$$

where the trace is understood to run over all possible indices and

$$\beta = \frac{1}{k_B\, T} \tag{8.53}$$

For instance, in terms of eigenstates of the Hamiltonian we can write

$$Z = \sum_n \langle n|e^{-\beta H}|n\rangle\ , \tag{8.54}$$

with $|n\rangle$ energy eigenstates. Inserting position eigenstates we get

$$\begin{aligned} Z &= \int dx \sum_n \langle n|x\rangle\langle x|e^{-\beta H}|n\rangle \\ &= \int dx \sum_n \langle x|e^{-\beta H}|n\rangle\langle n|x\rangle\ , \end{aligned} \tag{8.55}$$

where in the last equality we swapped the factor with impunity assuming we are dealing with bosons, and we would have picked up a sign if we were dealing with fermions. We then have

$$Z = \int dx \, \langle x | e^{-\beta H} | x \rangle \ , \tag{8.56}$$

Clearly, the partition function in (18.56) can be understood as a path integral with $x_i = x_f = x$ and we make the identifications

$$\beta = \frac{i}{\hbar} t \ . \tag{8.57}$$

In particular, we can write the partition function as

$$Z = \int \mathcal{D}x(\tau) \; e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \left\{ \frac{1}{2} m \left(\frac{\partial x}{\partial \tau} \right)^2 + V(x) \right\}} \ , \tag{8.58}$$

where we have defined $\tau \equiv it$. Clearly, we can see that going to back to real time we would obtain the action. However in this imaginary time τ, which is actually $\tau = \hbar\beta$ the inverse temperature, we should remember the periodicity condition

$$x(0) = x(\hbar\beta) \ . \tag{8.59}$$

The path integral in (18.58) is said to be in terms of the Euclidean action, but this is nothing but the hamiltonian. The expression in (18.58) is the starting point for the path integral formulation of statistical systems. We will see that many such systems then can be treated in this way using the methods of quantum field theory we will derive in the functional integral formulation in future lectures.

Additional suggested readings

- *Modern Quantum Mechanics*, by J. J. Sakurai, Chapter 2.5.
- *Quantum Field Theory* , by M. Srednicki, Chapter 7.