Lecture 6

Quantization of Fermion Fields

We will consider the spinor $\psi(\mathbf{x}, t)$ as a field and use to quantize the fermion field theory. For this we need to know its conjugate momentum. So it will be helpful to have the Dirac lagrangian. We will first insist in imposing *commutation* rules just as for the scalar field. But this will result in a disastrous hamiltonian. Fixing this problem will require a drastic modification of the commutation relations for the ladder operators.

6.1 The Dirac Lagrangian

Starting from the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \ , \tag{6.1}$$

we can obtain the conjugate equation

$$\bar{\psi}(x)\,(i\gamma^\mu\partial_\mu + m) = 0 \ , \tag{6.2}$$

where in this equation the derivatives act to their left on $\bar{\psi}(x)$. From these two equations for ψ and $\bar{\psi}$ is clear that the Dirac lagrangian must be

$$\mathcal{L} = \bar{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x) \ . \tag{6.3}$$

It is straightforward to check the the Eüler-Lagrange equations result in (6.1) and (6.2). For instance,

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = 0 \ . \tag{6.4}$$

But the second term above is zero since $\mathcal{L}$ does not depend (as written) on $\partial_\mu\bar{\psi}$. Thus, we obtain the Dirac equation (6.1) for ψ. Similarly, if we use ψ and $\partial_\mu\psi$ as the variables to put together the Eüler-Lagrange equations, we obtain (6.2).

6.2 Quantization of the Dirac Field

From the Dirac lagrangian we can obtain the conjugate momentum density defined by

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger \ . \tag{6.5}$$

This way, if we follow the quantization playbook we used for the scalar field, we should impose

$$[\psi_a(\mathbf{x},t), \pi_b(\mathbf{x}',t)] = [\psi_a(\mathbf{x},t), i\psi_b^\dagger(\mathbf{x}',t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta_{ab}\ , \tag{6.6}$$

or just

$$[\psi_a(\mathbf{x},t), \psi_b^\dagger(\mathbf{x}',t)] = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta_{ab}\ , \tag{6.7}$$

Following the same steps as in the case of the scalar field, we now expand $\psi(x)$ and $\psi^\dagger(x)$ in terms of solutions of the Dirac equation in momentum space. As we will see later, this will not work. But it is interesting to see why, because this will point directly to the correct quantization procedure. The most general expression for the fermion field in terms of the solutions of the Dirac equation in momentum space is

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(a_p^s\,u^s(\mathbf{p})\,e^{-iP\cdot x} + b_p^{s\dagger}\,v^s(\mathbf{p})\,e^{+iP\cdot x}\right)\ , \tag{6.8}$$

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(a_p^{s\dagger}\,u^{s\dagger}(\mathbf{p})\,e^{iP\cdot x} + b_p^{s}\,v^{s\dagger}(\mathbf{p})\,e^{+iP\cdot x}\right)\ , \tag{6.9}$$

The imposition of the quantization rule (6.7) on the field and its conjugate momentum in (6.8) and (6.9) would imply that the coefficients a_p^s, $a_p^{s\dagger}$, b_p^s and $b_p^{s\dagger}$ are ladder operators associated to the u-type and v-type "particles". But before we impose commutation rules on them we are going to compute the hamiltonian in terms of these operators.

Remember that the hamiltonian is defined by

$$
\begin{aligned}
H &= \int d^3x \,\{\pi(x)\,\partial_0\psi(x) - \mathcal{L}\}\ , \\
&= \int d^3x \,\left\{i\psi^\dagger(x)\,\partial_0\psi(x) - \bar{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x)\right\}\ ,
\end{aligned} \tag{6.10}
$$

which results in

$$
\boxed{H = \int d^3x\,\bar{\psi}(x)\,(-i\gamma\cdot\nabla + m)\,\psi(x)\ ,} \tag{6.11}
$$

Inserting (6.8) and (6.9) into (6.11) we have

$$
\begin{aligned}
H \;=\; \int d^3x \Bigg\{ &\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_k}} \sum_r \left(a_k^{r\dagger}\,\bar{u}^{r\dagger}(\mathbf{k})\,e^{iK\cdot x} + b_k^r\,\bar{v}^r(\mathbf{k})\,e^{-iK\cdot x}\right) \qquad (6.12)\\
\times &\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s\,e^{-iP\cdot x}\,(\gamma\cdot\mathbf{p} + m)\,u^s(\mathbf{p}) + b_p^{s\dagger}\,e^{+iP\cdot x}\,(-\gamma\cdot\mathbf{p} + m)\,v^s(\mathbf{p})\right)\Bigg\}\ ,
\end{aligned}
$$

In the second line of (6.12) the hamiltonian operator was applied to the exponentials. Since $P\cdot x = Ex_0 - \mathbf{p}\cdot\mathbf{x}$, the $-i$ in the operator cancels with the $+i\mathbf{p}\cdot\mathbf{x}$ in when the derivative acts on the $-P\cdot x$ exponential. The opposite sign is picked up when acting on the $+P\cdot x$ exponential. Furthermore, since

$$
(\not{p} - m)\,u^s(\mathbf{p}) = 0 \quad\Longrightarrow\quad (E_p\gamma^0 - \gamma\cdot\mathbf{p} - m)\,u^s(\mathbf{p}) = 0\ , \tag{6.13}
$$

which results in

$$
\boxed{(\gamma\cdot\mathbf{p} + m)\,u^s(\mathbf{p}) = E_p\,\gamma^0\,u^s(\mathbf{p})\ .} \tag{6.14}
$$

Similarly, applying

$$
(\not{p} + m)\,v^s(\mathbf{p}) = 0 \quad\Longrightarrow\quad (E_p\gamma^0 - \gamma\cdot\mathbf{p} + m)\,u^s(\mathbf{p}) = 0\ , \tag{6.15}
$$

which gives us

$$
\boxed{(-\gamma\cdot\mathbf{p} + m)\,v^s(\mathbf{p}) = -E_p\,\gamma^0\,v^s(\mathbf{p})\ .} \tag{6.16}
$$

Using (6.14) and (6.16) and that

$$\int d^3x e^{\pm i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}} = (2\pi)^3\,\delta^{(3)}(\mathbf{k}-\mathbf{p})\ , \tag{6.17}$$

in (6.12) we can get rid of 2 of the 3 integrals. Then we have

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_{r,s}\left\{a_p^{r\dagger}\,a_p^s\,u^{r\dagger}(\mathbf{p})\,\gamma^0\,E_p\gamma^0\,u^s(\mathbf{p})\right. \\ &\qquad\qquad \left. -\,b_p^r\,b_p^{s\dagger}\,v^{r\dagger}(\mathbf{p})\,\gamma^0\,E_p\,\gamma^0\,v^s(\mathbf{p})\right\}\ , \end{aligned} \tag{6.18}$$

where we have also use the orthogonality of the $u^s(\mathbf{p})$ and $v^s(\mathbf{p})$ solutions. Finally, using the normalization os spinors

$$\begin{aligned} u^{r\dagger}(\mathbf{p})\,u^s(\mathbf{p}) &= 2E_p\,\delta^{rs}\ , \\ v^{r\dagger}(\mathbf{p})\,v^s(\mathbf{p}) &= 2E_p\,\delta^{rs}\ , \end{aligned} \tag{6.19}$$

we obtain

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s\left\{E_p\,a_p^{s\dagger}\,a_p^s - E_p\,b_p^s\,b_p^{s\dagger}\right\}\ . \tag{6.20}$$

In order to have a correct form of the hamiltonian, we must rearrange the second term in (6.20) into a number operator, such as the first term. For this purpose, we need to apply the commutation rules on b_p^s and $b_p^{s\dagger}$. If we were to impose the same commutation rules we used for scalar fields, and also in (6.7), we would have

$$[a_p^r, a_k^{s\dagger}] = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{k})\,\delta^{rs}, \qquad [b_p^r, b_k^{s\dagger}] = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{k})\,\delta^{rs}, \tag{6.21}$$

and zero otherwise. This would result in a hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3}\,E_p\sum_s\left\{a_p^{s\dagger}\,a_p^s - b_p^{s\dagger}\,b_p^s\right\} - \int E_p d^3p\,\delta^{(3)}(\mathbf{0})\ . \tag{6.22}$$

The last term in (6.22) is an infinite constant. It corresponds to the sum over all the zero-point energies of the infinite harmonic oscillators each with a "frequency" E_p. This will always be present in quantum field theory (just as the zero-point energy is present in the harmonic oscillator!) and we will deal with it throughout the course. However, since it is a constant, we can always shift the origin of the energy in order to cancel it[1]. So this is not what is wrong with this hamiltonian. The problem is in the first term, particularly the negative term. The presence of this negative term tells us that we can lower the energy by producing additional v-type particles. For instance, the state $|\bar{1}_p\rangle$ with one such particle would have an energy

$$\langle\bar{1}_p|H|\bar{1}_p\rangle = -E_p < \langle 0|H|0\rangle \ , \tag{6.23}$$

smaller than the vacuum. This means that we have a runaway hamiltonian, i.e. its ground state corresponds to the state with infinite such particles. This is of course non-sense. The problem comes from the use of the commutation relations (6.21). On the other hand if we used anti-commutation relations such as

$$\{a_p^r, a_k^{s\dagger}\} = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{k})\,\delta^{rs}, \qquad \{b_p^r, b_k^{s\dagger}\} = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{k})\,\delta^{rs}, \tag{6.24}$$

together with

$$\{a_p^r, a_k^s\} = 0 = \{a_p^{r\dagger}, a_k^{s\dagger}\} \ , \qquad \{b_p^r, b_k^s\} = 0 = \{b_p^{r\dagger}, b_k^{s\dagger}\} \ , \tag{6.25}$$

and we go back to (6.20), using (6.25) instead of (6.21) we obtain

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3}\,E_p \sum_s \left\{a_p^{s\dagger}\,a_p^s + b_p^{s\dagger}\,b_p^s\right\} + \text{constant} \ .} \tag{6.26}$$

This is now a well behaved hamiltonian, where for each fixed value of the momentum we have a contribution to the energy of $a_p^{s\dagger}\,a_p^s$ number of particles of type u, and $b_p^{s\dagger}\,b_p^s$ number of particles of type v. This is the expected form of the hamiltonian, and we arrived at it by using the anti-commutation relations (6.24) and (6.25) for the ladder operators. It is straightforward to show that they imply anti-commutation rules also for the fermion field and its conjugate momentum. That is

$$\boxed{\{\psi_a(\mathbf{x},t), \psi_b^\dagger(\mathbf{x}',t)\} = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta_{ab} \ ,} \tag{6.27}$$

and zero otherwise, instead of (6.7).

[1]The fact that this constant is negative will remain true and is an important fact. For instant, for scalar fields is positive.

6.3 Charge Operator and Fermion Number

In order to better understand the meaning of the u and v solutions it is useful to build another operator other than the hamiltonian. We start with the Dirac current. We know that it is given by

$$j^\mu = \bar{\psi}\gamma^\mu\psi \ , \tag{6.28}$$

satisfying current conservation

$$\partial_\mu \, j^\mu = 0 \ . \tag{6.29}$$

Noether's theorem tells us that the conserved current is associated with a conserved charge defined by

$$Q = \int d^3x \, j^0(x) = \int d^3x \, \bar{\psi}(x)\gamma^0\psi(x) = \int d^3x \, \psi^\dagger(x) \, \psi(x) \ . \tag{6.30}$$

We have seen this before: it is the probability density obeying a continuity equation (6.29). The fact that the charge Q is time independent is a direct consequence of (6.29). We build this operator in terms of ladder operators in momentum space just as we did for the hamiltonian. Using (6.8) and (6.9) and following the same steps that lead to (6.12) we obtain

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left\{ a_p^{s\dagger} \, a_p^s + b_p^s \, b_p^{s\dagger} \right\} \ , \tag{6.31}$$

Using the anti-commutation relations (6.25) on the second term we arrive at

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left\{ a_p^{s\dagger} \, a_p^s - b_p^{s\dagger} \, b_p^s \right\} \ , \tag{6.32}$$

where we have omitted the a and b-independent, infinite constant. We see clearly that each u-type particle contributes to Q with $+1$, whereas each v-type particle contributes with -1. The continuous symmetry associated with the current j^μ is just the global fermion number. That is the lagrangian is invariant under

$$\begin{aligned} \psi(x) &\longrightarrow e^{i\alpha} \, \psi(x) \ , \\ \psi^\dagger(x) &\longrightarrow e^{-i\alpha} \, \psi^\dagger(x) \ . \end{aligned} \tag{6.33}$$

with α a real constant. This just says that the Dirac lagrangian conserves fermion number, meaning that there are fermions with charge $+1$ and anti-fermions (the v-type states) with charge -1. To summarize:

- a_p^s: annihilates fermions
- $b_p^{s\dagger}$ creates anti-fermions
- $a_p^{s\dagger}$ creates fermions
- b_p^s annihilates anti-fermions

Or, in other words

- $\psi(x)$ annihilates fermions or creates anti-fermions
- $\psi^\dagger(x)$ creates fermions or annihilates anti-fermions

6.4 Pauli Exclusion Principle and Statistics

One of the most important consequences of having anti-commutation rules for the ladder and field operators is that fermions obey Fermi-Dirac statistics and the Pauli exclusion principle. To see this, consider a two fermion state. It is built out of creation operators as

$$|1_p^s\, 1_k^r\rangle = a_p^{s\dagger}\, a_k^{r\dagger}|0\rangle \ . \tag{6.34}$$

The anti-commutation rules (6.25) imply

$$a_p^{s\dagger}\, a_k^{r\dagger} = -\, a_k^{r\dagger}\, a_p^{s\dagger} \ . \tag{6.35}$$

which means that the state is odd under the exchange of two particles (for instance switching positions), or

$$|1_p^s\, 1_k^r\rangle = -|1_k^r\, 1_p^s\rangle \tag{6.36}$$

In particular if both fermions have the same exact quantum numbers, here in our example the helicity s and the momentum $\mathbf{p}$, we have

$$|1_p^s\, 1_p^s\rangle = -|1_p^s\, 1_p^s\rangle = 0 \ , \tag{6.37}$$

which means that this state is forbidden. As a result, we cannot put two fermions (or two anti-fermions) with the exact same quantum numbers in the same state. So the occupation numbers in states made of fermions are either 0 or 1 for a given set of quantum numbers. This is what is called Fermi-Dirac statistics. Equation (6.37) is an expression of the Pauli exclusion principle.

6.5 Spin

To complete our acquaintance with fermions, we go back to its properties under Lorentz transformations. We had seen that under a Lorentz transformation defined by

$$x'^{\mu} = \Lambda^{\mu}_{\nu}\, x^{\nu} \simeq (\delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}) \ , \tag{6.38}$$

the spinors respond as

$$\psi'(x') = S(\Lambda)\,\psi(x) \simeq (\mathbf{1} - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu})\,\psi(x) \ , \tag{6.39}$$

where $\omega^{\mu\nu}$ parametrizes an infinitesimal Lorentz transformation and, as obtained previously, we have

$$\sigma_{\mu\nu} = \frac{i}{2}\,[\gamma_{\mu}, \gamma_{\nu}] \ . \tag{6.40}$$

We would like to consider a particular Lorentz transformation: an infinitesimal rotation about the $\hat{z} = \hat{3}$ axis. This is

$$\Lambda^{\mu}_{\nu} \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{6.41}$$

which is a rotation by an infinitesimal angle θ. This results in $\omega^{1}_{2} = -\theta$, which means $\omega^{12} = \theta$. On the other hand, we have

$$\sigma_{12} = \frac{i}{2}\,[\gamma_1, \gamma_2] = \frac{i}{2}\left[\begin{pmatrix} \mathbf{0} & \sigma_1 \\ -\sigma_1 & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \sigma_2 \\ -\sigma_2 & \mathbf{0} \end{pmatrix}\right] \ . \tag{6.42}$$

Knowing that the Pauli matrices satisfy

$$[\sigma_1, \sigma_2] = 2i\, \epsilon_{123}\, \sigma_3 \ , \tag{6.43}$$

we arrive at

$$\sigma_{12} = \begin{pmatrix} \sigma_3 & \mathbf{0} \\ \mathbf{0} & \sigma_3 \end{pmatrix} \equiv \Sigma_3 \ . \tag{6.44}$$

We can now replace all this into (6.39) to obtain

$$\psi(x') + \delta\psi(x') \simeq (\mathbf{1} - \frac{i}{4}\, 2\, \theta\, \Sigma_3)\, \psi(\Lambda^{-1}x') \ , \tag{6.45}$$

where the 2 multiplying Σ_3 comes from adding the ω^{12} and the ω^{21} contributions, and in the last factor we evaluate the argument in x but write it in terms of x' in order to have an expression in terms of the new spatial coordinate only. Dropping the prime for notational simplicity, we have that the change in the spinor field in (6.45) is

$$\delta\psi(x) \simeq (\mathbf{1} - \frac{i}{2}\, \theta\, \Sigma_3)\, \psi(\Lambda^{-1}x) - \psi(x) \ . \tag{6.46}$$

We need to Taylor expand $\psi(\Lambda^{-1}x)$ for small values of θ. From (6.41) we have

$$\Lambda^{-1}\, x \simeq= (t, x + \theta y, y - \theta x, z) \ , \tag{6.47}$$

which results in

$$\psi(t, x + \theta y, y - \theta x, z) \simeq \psi(t, x, y, z) + \theta\, y\, \frac{\partial\psi}{\partial x} - \theta\, x\, \frac{\partial\psi}{\partial y} \ . \tag{6.48}$$

Replacing this in the last factor of (6.46) we obtain, in linear approximation in θ

$$\delta\psi(x) \simeq -\theta \left(x\partial_y - y\partial_x + \frac{i}{2}\, \Sigma_3 \right)\, \psi(x) \equiv \theta\, \Delta\psi(x) \ . \tag{6.49}$$

So now we have the complete response of the spinor to the rotation. There is the response to the change of coordinates in the argument of $\psi(\Lambda^{-1}x)$, plus there is the genuine spinor rotation itself associated with Σ_3. In terms Noether currents, if we take the time component of the conserved current, its volume integral is a conserved quantity. In this case it would be the $\hat{z}$ component of the angular momentum. We have

$$j^0_{(z)} = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\,\Delta\psi \;. \tag{6.50}$$

From the Dirac lagrangian, we obtain

$$\frac{\partial \mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 \;, \tag{6.51}$$

which together with the definition for $\Delta\psi$ in (6.49) gives

$$\begin{aligned} J_z &= \int d^3x\, j^0_{(z)} = \int d^3x\,(-i)\,\psi^\dagger(x\partial_y - y\partial_x + \frac{i}{2}\,\Sigma_3)\psi \\ &= \int d^3x\,\psi^\dagger([\mathbf{r}\times(-i\nabla)]_z + \frac{1}{2}\Sigma_3)\psi \;, \end{aligned} \tag{6.52}$$

which clearly can be generalized by adding the rotations about $\hat{x}$ and $\hat{y}$ to obtain

$$\mathbf{J} = \int d^3x\,\psi^\dagger(\mathbf{r}\times(-i\nabla) + \frac{1}{2}\boldsymbol{\Sigma})\psi. \tag{6.53}$$

The above expression is the total angular momentum operator. Clearly the second term refers to the response of the internal degrees of freedom of the spinor to space rotations. In general this is not Lorentz invariant, but in the non-relativistic limit we can identify the first term as the orbital angular momentum and the second as the spin operator. So even at rest ($\mathbf{p} = 0$) this last contribution is present. It is customary to define the spin operator as

$$\mathbf{S} \equiv \frac{1}{2}\,\boldsymbol{\Sigma} \;, \tag{6.54}$$

which has eigenvalues $\pm 1/2$.

6.6 Discrete Transformations

Lorentz transformations that can be obtained by continuously deforming from the identity are called proper and orthochronous. On the other hand, there is a group of Lorentz transformations that cannot be obtained in this way. These are the discrete transformations called Parity and Time-reversal. Unlike proper Lorentz transformations, which have $\det(\Lambda) = +1$, they have $\det(\Lambda) = -1$. But they still preserve the interval. We briefly review below these, as well as the associated transformation called Charge Conjugation, corresponding to the exchange of particle and anti-particle.

Parity:
It is defined as

$$x^\mu \to x'^\mu = (x_0, -\mathbf{x}) \ , \tag{6.55}$$

and it corresponds to a reflection. This defines the coordinate Lorentz transformation, i.e. $\Lambda_P = \mathrm{diag}(1,-1,-1,-1)$. Now we would like to know how fermions transform under parity. For this we consider the Dirac equation

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x) = 0 \ . \tag{6.56}$$

We want (6.56) to be invariant under (6.55). For this we need to define the fermion parity transformation

$$\psi'(x') = S(\Lambda_P)\, \psi(x) \ , \tag{6.57}$$

so that it satisfies

$$\left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m \right) \psi'(x') = 0 \ . \tag{6.58}$$

But the action of parity on the derivatives results in

$$\begin{aligned} \frac{\partial}{\partial x_0} &\to +\frac{\partial}{\partial x'_0} \\ \frac{\partial}{\partial x_i} &\to -\frac{\partial}{\partial x'_i} \end{aligned} \tag{6.59}$$

Given that

$$(\gamma^0)^2 = 1, \qquad \text{and} \qquad \gamma^0\gamma^i = -\gamma^i\gamma^0 \ , \tag{6.60}$$

we can see that choosing

$$S(\Lambda_P) = \eta\,\gamma^0 \ , \tag{6.61}$$

with η just an arbitrary phase (i.e. $|\eta|^2 = 1$) does the job.

Charge Conjugation:

In principle, we can introduce the charge-conjugation operation without talking about interactions. However, it will be instructive to consider the case of a fermion coupled a gauge field. We know that the lagrangian in the presence of this interaction is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + eA_\mu\bar{\psi}\gamma^\mu\psi \ . \tag{6.62}$$

Here e is the fermion charge, and in the last term the fermion current is coupled with the vector potential in a way that we already know is gauge invariant.

The equation of motion corresponding to (6.62) is

$$\left[i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right]\psi(x) = 0 \ . \tag{6.63}$$

The *charge conjugate* field ψ_C has charge $-e$ and therefore it must satisfy

$$\left[i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right]\psi_C(x) = 0 \ . \tag{6.64}$$

We want to know how to obtain ψ_C from ψ. As a first step we can write the complex conjugate of the original equation of motion (6.63):

$$\left[-i\gamma^{*\mu}\left(\partial_\mu + ieA_\mu\right) - m\right]\psi^*(x) = 0 \ . \tag{6.65}$$

We now define the charge conjugation operation as

$$\psi_C = C\gamma^0\psi^* \ , \tag{6.66}$$

where C is an operator to be defined and the γ^0 factor is there just for convenience. With this definition we will obtain (6.64) from (6.65) if we impose that C satisfies

$$C\gamma^0\gamma^{*\mu} = -\gamma^\mu\, C\gamma^0 \ . \tag{6.67}$$

To see that this is the case, we multiply (6.65) from the left by $C\gamma^0$ obtaining

$$C\gamma^0 \left[-i\gamma^{*\mu}\left(\partial_\mu + ieA_\mu\right) - m\right]\psi^*(x) = 0 \ ,$$

$$\left[i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right] C\gamma^0\psi^* = 0 \ , \tag{6.68}$$

where in the last line we made use of (6.67), and we see that using the definition of ψ_C, this is (6.64). A possible choice for C that satisfies (6.67) is

$$C = i\gamma^2\,\gamma^0 \ , \tag{6.69}$$

since γ^2 is the only imaginary one. In this way the conjugate field is

$$\psi_C = i\gamma^2\psi^* \ . \tag{6.70}$$

Time Reversal:

It simply corresponds to the transformation $t \to -t$. As a result all velocities, and therefore all momenta, change sign. Since the positions $\mathbf{x}$ stay the same the angular momentum

$$\mathbf{L} \to -\mathbf{L} \ . \tag{6.71}$$

The same is true for spin. That is

$$\mathbf{S} \to -\mathbf{S} \ . \tag{6.72}$$

Let us first look at the effect of time reversal in the Schrödinger/Dirac equation (depending on the H used) . Initially we have

$$i\,\frac{\partial\psi(t)}{\partial t} = H\psi(t) \ . \tag{6.73}$$

Under time reversal $t \to t' = -t$. We want $\psi'(t')$ to also satisfy the Schrödinger/Dirac equation, so

$$i\,\frac{\partial\psi'(t')}{\partial t'} = H\psi'(t') \ . \tag{6.74}$$

Defining the time reversal operation on the wave function as

$$\psi'(t') = T\,\psi(t) \ , \tag{6.75}$$

and inserting (6.75) in (6.74) we have

$$i\,\frac{\partial T\psi(t)}{\partial(-t)} = HT\psi(t) \ . \tag{6.76}$$

Multiplying (6.76) by $T^{-1} = T^{\dagger}$ on the left we have

$$T^{-1}(-i)\frac{\partial T\psi(t)}{\partial t} = T^{-1}HT\psi(t) \ . \tag{6.77}$$

Then, if the hamiltonian H is invariant under time reversal, i.e. if

$$T^{-1}HT = H \ , \tag{6.78}$$

which is equivalent to $[H, T] = 0$, we arrive at

$$T^{-1}(-i)T\frac{\partial\psi(t)}{\partial t} = H\psi(t) \ , \tag{6.79}$$

which appears to be wrong by a sign. The solution is to make T an anti-unitary operation, instead of unitary. What this means is that it does not only act on the states but also on c-numbers. In particular, it satisfies

$$T^{-1}(-i)T = +i \ , \tag{6.80}$$

which is what is needed to turn (6.79) into the desired equation. To see how this works, let us write T as

$$T = UK \ , \tag{6.81}$$

where U is a normal unitary operation and K means complex-conjugate whatever is on the right of K. With this definition K satisfies

$$K^{-1} = K^{\dagger} = K \ , \tag{6.82}$$

as it can be easily checked by explicitly noticing that

$$K^{-1}K\phi = K^{-1}\phi^* = \phi\ . \tag{6.83}$$

The definition (6.81) means that

$$T^{-1}(-i)T = KU^{-1}(-i)UK = K(-i)K = +iKK = +i\ , \tag{6.84}$$

as advertised.

We now want to specify for the case of the Dirac equation. For this case we have,

$$H = -i\gamma^0\,\vec{\gamma}\cdot\vec{\nabla} + \gamma^0 m\ , \tag{6.85}$$

We would like to define T in terms of γ matrices so that it satisfies (6.78). That is

$$\begin{aligned} T^{-1}HT &= H \\ KU^{-1}HUK &= H\ . \end{aligned} \tag{6.86}$$

Using (6.85) we arrive at the following conditions defining U:

$$KU^{-1}\gamma^0 UK = \gamma^0 \tag{6.87}$$

$$KU^{-1}(-i\gamma^0\gamma^j)UK = -i\gamma^0\gamma^j\ . \tag{6.88}$$

Multiplying both equations above by K on the left and on the right we arrive at

$$U^{-1}\gamma^0 U = \gamma^{0*} \tag{6.89}$$

$$U^{-1}(i\gamma^0\gamma^j)U = -i\gamma^{0*}\gamma^{j*}\ . \tag{6.90}$$

Since $\gamma^{0*} = \gamma^0$, these conditions result in

$$U^{-1}\gamma^j U = -\gamma^{j*}\ . \tag{6.91}$$

Since γ^2 is the only imaginary γ matrix, these conditions are satisfied by choosing

$$U = \eta\gamma^1\gamma^3 \ , \tag{6.92}$$

where again η is an arbitrary phase.

An interesting consequence is that for fermions, if we have a state ψ, then if we apply T twice

$$T^2\psi = \eta\gamma^1\gamma^3\eta\gamma^1\gamma^3\psi = (-1)\psi \ . \tag{6.93}$$

The fact that $T^2 = -1$ results in the so called Kramers degeneracy. For instance, let us assume that ψ and $T\psi$, which are states of the same energy, are actually the same state. This means that they differ at most by a phase, that is that

$$T\psi = e^{i\alpha}\psi \ , \tag{6.94}$$

for some arbitrary α. But then this means that

$$T^2\psi = T(T\psi) = T(e^{i\alpha}\psi) = e^{-i\alpha}T\psi = \psi \neq -\psi \ , \tag{6.95}$$

Therefore, the assumption that ψ and $T\psi$ are the same state only differing by a phase must have been incorrect. The result is then that there are two distinct states, ψ and $T\psi$ with the same energy. The Kramers degeneracy is present as long as we have one fermion or an odd number of them.