Lecture 4

Fermions

In the previous lectures we have seen that attempts to describe the quantum mechanical evolution by aplying the Klein-Gordon equation to the wave-function leads to several problems. In particular the zero component of the Klein-Gordon current cannot be interpreted as a probability density, as it is in the case of the Schrödinger current, since is not definite positive. Here we will follow Dirac's path in obtaining a relativistic description of the wave-function that relies on one time derivative as well as one spatial derivative. Eventually, the technology developed for the Dirac equation, will be used once we consider the quantization of fermionic fields obeying it.

4.1 The Dirac Equation

The aim is to write an evolution equation for the wave function that is first order in both time and space derivatives in order for it to have a chance of being Lorentz invariant. The starting point is to write the following

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t} = \aleph\,\psi(\mathbf{x},t)\ , \tag{4.1}$$

where the operator $\aleph$ needs to be at most linear in spatial derivatives, and such that the "squared" of this equation

$$-\frac{\partial^2\psi(\mathbf{x},t)}{\partial t^2} = \aleph^2\psi(\mathbf{x},t)\ , \tag{4.2}$$

is the Klein-Gordon equation. The most general form of the operator $\aleph$ is

$$\aleph = -i\vec{\alpha}\cdot\vec{\nabla} + \alpha_4 m\ , \tag{4.3}$$

where the α_i's with $i = 1, 2, 3$ and α_4 are arbitrary coefficients.The last term, containing α_4 is defined proportional to the mass m for reasons that will become clear next. Constraints on these coefficients result from imposing that (4.2) is the Klein-Gordon equation. That is, we need

$$\begin{aligned} \aleph^2 &= -\alpha_i\alpha_j \frac{\partial^2\psi}{\partial x_i \partial x_j} - im(\alpha_j\alpha_4 + \alpha_4\alpha_j)\frac{\partial\psi}{\partial x_j} + \alpha_4^2 m^2\psi \\ &= -\frac{\partial^2\psi}{\partial x_j \partial x_j} + m^2\psi\ , \end{aligned} \tag{4.4}$$

where in the last equality we are enforcing the Klein-Gordon equation, the j index is summed, and in the first line we did not assume the α_i's or α_4 commute. In fact in order for the last equality in (4.4) to be valid the α's must satisfy the following constraints:

- For $i \neq j$, $\boxed{\alpha_i\alpha_j + \alpha_j\alpha_i = 0}$
- For all i's $\boxed{\alpha_i\alpha_4 + \alpha_4\alpha_i = 0}$
- For $i = j$, $\boxed{\alpha_i^2 = 1}$
- $\boxed{\alpha_4^2 = 1}$

Clearly this means that these are not just numbers since they do not commute. Then they must be matrices. The above properties tell us quite a bit about them.

Traceless: We can see that all the α_i matrices must be traceless. For instance,

$$\begin{aligned} &\alpha_i\alpha_4 + \alpha_4\alpha_i = 0 \\ &\alpha_i = -\alpha_4\alpha_i\alpha_4 \\ &\mathrm{Tr}[\alpha_\mathrm{i}] = -\mathrm{Tr}[\alpha_4\alpha_\mathrm{i}\alpha_4] \\ &\mathrm{Tr}[\alpha_\mathrm{i}] = -\mathrm{Tr}[\alpha_\mathrm{i}] = 0 \end{aligned} \tag{4.5}$$

The same can be done for α_4 to show that $\mathrm{Tr}[\alpha_4] = 0$.

Squared Matrices: Since the operator $\aleph$ must be hermitian the matrices must be $N \times N$ squared matrices.

Even dimension: Given that $\alpha_i^2 = 1 = \alpha_4^2$, the eigenvalues must be ± 1. But for them to be traceless we will need for N to be even.

So we learned that the Dirac matrices in $\aleph$ must be even dimensional, squared and traceless matrices. For $N = 2$ there are *three* such matrices: Pauli's σ_1, σ_2 and σ_3. But we need *four*. If the particle were massless, we would not need α_4, and we could use the Pauli matrices to write the Dirac equation. We will see this at work later. But for now we want to consider the most general case when $m \neq 0$. This leaves us with $N = 4$, so the Dirac matrices are 4×4. We can rewrite the Dirac equation in increasingly compact ways. Starting as

$$i\left(\alpha_4\frac{\partial}{\partial t} + \alpha_4\alpha_j\,\frac{\partial}{\partial x_j}\right)\psi(\mathbf{x},t) - m\psi(\mathbf{x},t) = 0\ , \tag{4.6}$$

we can define the following matrices suggesting a covariant notation:

$$\begin{aligned} \gamma^0 &\equiv \alpha_4\ , \\ \vec{\gamma} &\equiv \alpha_4\,\vec{\alpha}\ , \end{aligned} \tag{4.7}$$

which then results in

$$i\left(\gamma^0\frac{\partial}{\partial x_0} + \gamma_j\,\frac{\partial}{\partial x_j}\right)\psi(\mathbf{x},t) - m\psi(\mathbf{x},t) = 0\ . \tag{4.8}$$

We may use the more compact covariant notation as in

$$(i\gamma^\mu\partial_\mu - m)\,\psi(\mathbf{x},t) = 0\ , \tag{4.9}$$

where we should remember that

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t},\vec{\nabla})\ , \tag{4.10}$$

so that is clear that γ^μ contracts as a contravariant four-vector (although it does not transform like one!), which means $\gamma_\mu = (\gamma^0, -\vec{\gamma})$. Finally, we can introduce the following notation for the contraction with gamma matrices:

$$\gamma^\mu\,a_\mu = \not{a}\ , \tag{4.11}$$

where a_μ is some object with a Lorentz index such as a four-vector or a four-gradient. Then we can rewrite (4.9) as

$$\boxed{(i\not{\partial} - m)\psi(\mathbf{x},t) = 0}\ . \tag{4.12}$$

To obtain the anti-commutators of the γ^μ matrices, we rewrite the one of the α_j's with $j = 1, 2, 3$ as

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \ . \tag{4.13}$$

Using (4.7) we obtain

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ . \tag{4.14}$$

where $g^{\mu\nu}$ is Minkowski's metric, i.e. $g^{\mu\nu} = \text{diag.}(1, -1, -1, -1)$. The relation in (4.14) defines the algebra satisfied by the gamma matrices, called the Clifford algebra.

An example of a representation of the gamma matrices is

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} \mathbf{0} & \vec{\sigma} \\ -\vec{\sigma} & \mathbf{0} \end{pmatrix} \ , \tag{4.15}$$

where all the entries are to be read as 2×2 matrices and $\mathbf{1}$ is the identity in 2×2. This is the standard representation.

As we can see from the form of the Dirac equation (4.12), the object we used to call the wave function, $\psi(\mathbf{x}, t)$, must be a more complex one. The indices corresponding to the gamma matrices appearing in (4.12) must be contracted with indices in $\psi(\mathbf{x}, t)$. So we now should understand that there is an implicit index, i.e. $\psi_a(\mathbf{x}, t)$, where $a = 1, ..., 4$ contracts with the ones in the gamma matrices. This is a *spinorial* index, and $\psi(\mathbf{x}, t)$ is called a spinor. As we will see below, the implication of having this free index is that spinors transform non-trivially under Lorentz transformations. However, we will see that their transformation is different from that of four-vectors. We will see how in the next section.

But before we go to the Lorentz transfomation properties of spinors, we want to check if it is possible to interpret the wave-function $\psi(\mathbf{x}, t)$ satisfying the Dirac equation as a probability amplitude. For this, we complex conjugate (4.9) to obtain

$$\psi^\dagger(i(\gamma^\mu)^\dagger\partial_\mu + m) = 0 \ , \tag{4.16}$$

where the derivative is applied to the *left*. But we have that

$$(\gamma^0)^\dagger = \gamma^0, \qquad \vec{\gamma}^\dagger = \gamma^0\vec{\gamma}\gamma^0 \ , \tag{4.17}$$

which results in

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \ . \tag{4.18}$$

This allows us to rewrite (4.16) as

$$\psi^\dagger(i\gamma^0\gamma^\mu\gamma^0\partial_\mu + m) = 0 \ . \tag{4.19}$$

If we now define

$$\bar{\psi} \equiv \psi^\dagger\gamma^0 \ , \tag{4.20}$$

we arrive at the following form for the conjugate Dirac equation

$$\bar{\psi}(i\gamma^\mu\partial_\mu + m) = 0 \ , \tag{4.21}$$

where, as before, the derivative is acting on the left. Then using both the Dirac equation and its conjugate we have

$$\begin{aligned} i\gamma^\mu\partial_\mu\psi &= m\psi \\ \bar{\psi}i\gamma^\mu\partial_\mu &= -m\bar{\psi} \ , \end{aligned} \tag{4.22}$$

or, by multiplying the first of these equations by $\bar{\psi}$ on the left and the second by ψ on the right, we obtain

$$(\partial_\mu\bar{\psi})i\gamma^\mu\psi + i\bar{\psi}\gamma^\mu\partial_\mu\psi = 0 \ , \tag{4.23}$$

Therefore we can define the conserved current

$$j^\mu \equiv \bar{\psi}\gamma^\mu\psi \ , \tag{4.24}$$

satisfying $\partial_\mu j^\mu = 0$ according to (4.23). We are now in a position to check if the density associated with this conserved current can have a probabilistic interpretation. The wave-function $\psi(\mathbf{x}, t)$ must be a four-component spinor. We will derive its transformation properties later on. But for now it suffices to note that it can be decomposed into two two-component spinors as

$$\psi = \begin{pmatrix} \xi \\ \chi \end{pmatrix} \ . \tag{4.25}$$

The Dirac equation results in two coupled equations for ξ and χ. Using the standard representation of the γ matrices we have

$$\begin{aligned} i\frac{\partial\xi}{\partial t} &= m\xi - i\vec{\sigma}\cdot\vec{\nabla}\chi \\ i\frac{\partial\chi}{\partial t} &= -m\chi - i\vec{\sigma}\cdot\vec{\nabla}\xi \ . \end{aligned} \tag{4.26}$$

It is interesting to notice that the two coupled equations in (4.26) decouple if $m = 0$. In this case it is possible to describe the system using Pauli matrices, as we anticipated at the beginning of this section. In the general case, with $m \neq 0$, we can compute $\rho = j^0$. This is

$$\rho = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = \xi^\dagger\xi + \chi^\dagger\chi \ , \tag{4.27}$$

which is clearly positive definite. So the Dirac equation describes the relativistic evolution of the wave-functions *and* allows for a probabilistic interpretation. We will still have to deal with the problem of negative energies that will inevitably lead us into quantum field theory, i.e. describing the spinor $\psi(\mathbf{x}, t)$ as a quantum field, as opposed to the wave function of a single particle.

4.2 Lorentz transformation of the Dirac Spinor

All we know about the Dirac spinors is that they obey the Dirac equation. But how do they behave under Lorentz transformations ? In order to answer this question we will impose that the Dirac equation be the same in any frame. Suppose we have the following homogenous Lorentz transformation

$$x'^\mu = \Lambda^\mu_\nu\, x^\nu \ , \tag{4.28}$$

relating the coordinates of the two frames. We want that the equation in the unprimed frame

$$i\gamma^\mu\frac{\partial\psi(x)}{\partial x^\mu} - m\psi(x) = 0 \ , \tag{4.29}$$

be obtained again in the new frame. I. e. the Dirac equation in the primed frame must have the form

$$i\gamma^\mu \frac{\partial \psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0 \ , \tag{4.30}$$

where we have defined the spinor in the primed frame by the transformation

$$\psi'(x') \equiv S(\Lambda)\, \psi(x) \ . \tag{4.31}$$

In (4.31) $S(\Lambda)$ is a 4×4 non-singular matrix that depends on the Lorentz transformation Λ on the coordinates. Since

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu_\nu \ , \tag{4.32}$$

then we have that

$$\frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu_\mu \ . \tag{4.33}$$

Then we can do the following replacement in (4.30)

$$\frac{\partial}{\partial x'^\mu} \to \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial x^\nu} \ . \tag{4.34}$$

As a result, (4.30) reads now as

$$i\gamma^\mu (\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial x^\nu} S(\Lambda)\psi(x) = m\, S(\Lambda)\psi(x) \ . \tag{4.35}$$

In order for the above equation to go back to the Dirac equation in the unprimed frame (4.29) we need to impose the following identity on $S(\Lambda)$

$$\boxed{\gamma^\mu \, (\Lambda^{-1})^\nu_\mu = S(\Lambda)\, \gamma^\nu \, S^{-1}(\Lambda) \ .} \tag{4.36}$$

Equation (4.36) defines $S(\Lambda)$, and therefore it also defines the Dirac spinor $\psi(\mathbf{x}, t)$ through its Lorentz transformation properties. In theory group parlance, four-vectors are a "vectorial" representation of the Lorentz group, whereas spinors furnish a spinorial representation.

In order to obtain $S(\Lambda)$ we will expand the Lorentz transformations (4.28) around the identity

$$\Lambda^\mu_\nu \simeq \delta^\mu_\nu + \omega^\mu_\nu \ , \tag{4.37}$$

where we assume ω^μ_ν is infinitesimal, i.e. $\omega^\mu_\nu \ll \delta^\mu_\nu$. The inverse infinitesimal transformation is

$$(\Lambda^{-1})^\mu_\nu = \delta^\mu_\nu - \omega^\mu_\nu \ , \tag{4.38}$$

which is required so that

$$(\Lambda^{-1})^\mu_\nu \Lambda^\nu_\sigma \simeq (\delta^\mu_\nu - \omega^\mu_\nu)(\delta^\nu_\sigma + \omega^\nu_\sigma) = \delta^\mu_\sigma + \omega^\mu_\sigma - \omega^\mu_\sigma + \dots \ , \tag{4.39}$$

where the dots refer to terms of higher order in ω^μ_σ. It is important to notice that $\omega_{\mu\nu}$ must be antisymmetric. For this we must go no further than the invariance of the interval

$$g_{\mu\nu}\, dx'^\mu\, dx'^\nu = g_{\rho\sigma}\, dx^\rho\, dx^\sigma \ , \tag{4.40}$$

resulting in

$$g_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = g_{\mu\nu}\, \Lambda^\mu_\rho\, \Lambda^\nu_\sigma = g_{\rho\sigma} \ . \tag{4.41}$$

Then, for infinitesimal Lorentz transformations we have

$$g_{\rho\sigma} = g_{\mu\nu}\, (\delta^\mu_\rho + \omega^\mu_\rho)\, (\delta^\nu_\sigma + \omega^\nu_\sigma) = g_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \dots \ , \tag{4.42}$$

which clearly implies that

$$\boxed{\omega_{\sigma\rho} = -\omega_{\rho\sigma}} \tag{4.43}$$

We can now expand the spinor transformation matrix $S(\Lambda)$ as

$$S(\Lambda) \simeq \mathbf{1} - \frac{i}{4}\sigma_{\mu\nu}\, \omega^{\mu\nu} + \dots \ , \tag{4.44}$$

where the $\sigma_{\mu\nu}$ are generic 4×4 matrices, which must be antisymmetric under the exchange of μ and ν since $\omega^{\mu\nu}$ is. If we apply (4.36) we have, to first order in ω,

$$\gamma^\mu(\delta^\nu_\mu - \omega^\nu_\mu) = (\mathbf{1} - \frac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta})\, \gamma^\nu\, (\mathbf{1} + \frac{i}{4}\sigma_{\rho\sigma}\omega^{\rho\sigma}) \ . \tag{4.45}$$

Using the antisymmetry of $\omega^{\mu\nu}$, we can write

$$\gamma^\mu \omega^\nu_\mu = \frac{1}{2}\delta^\nu_\alpha \gamma_\beta \omega^{\alpha\beta} + \frac{1}{2}\delta^\nu_\beta \gamma_\alpha \omega^{\beta\alpha} = \frac{1}{2}\left(\delta^\nu_\alpha \gamma_\beta - \delta^\nu_\beta \gamma_\alpha\right)\omega^{\alpha\beta} \ . \tag{4.46}$$

Using the result above in (4.45) we obtain

$$[\gamma^\nu, \sigma_{\alpha\beta}] = 2i\left(\delta^\nu_\alpha \gamma_\beta - \delta^\nu_\beta \gamma_\alpha\right) \ . \tag{4.47}$$

The equality (4.47) indicates that $\sigma_{\alpha\beta}$ must contain the product of two gamma matrices, in addition to being antisymmetric in the Lorentz indices. It is relatively straightforward to see that (4.47) is satisfied by

$$\sigma_{\alpha\beta} = \frac{i}{2}\left[\gamma_\alpha, \gamma_\beta\right] \ . \tag{4.48}$$

Finally, for finite $\omega^{\mu\nu}$'s we have that the Lorentz transformation of spinors is given by

$$S(\Lambda) = e^{-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}} \ . \tag{4.49}$$

4.3 Relation to the Poincaré Group

It is interesting to put the results obtained above in the context of the group theory associated with the most general relativistic transformations. On the spacetime coordinates these are

$$x'^\mu = \Lambda^\mu_\nu \, x^\nu + a^\mu \ , \tag{4.50}$$

where the four-vector a^μ parametrizes translations.

Let us consider the most general situation of the transfomation properties of states Ψ in Hilbert space under such transformations. At the end, we will make contact with the specific case for spinors. States in Hilbert space will then generally transform under (4.50) as

$$\Psi \longrightarrow U(\Lambda, a)\,\Psi \ , \tag{4.51}$$

with $U(\Lambda, a)$ a linear unitary transformation. These transformations form a group. For instance, if we apply two successive transformations, they obey

$$U(\bar{\Lambda},\bar{a})\,U(\Lambda,a) = U(\bar{\Lambda}\,\Lambda,\bar{\Lambda}a+\bar{a})\ . \tag{4.52}$$

To convince ourselves that this is the case, we just consider two successive applications of (4.50), as in

$$\begin{aligned} x''^{\mu} &= \bar{\Lambda}^{\mu}_{\nu}\,x'^{\nu} + \bar{a}^{\mu} = \bar{\Lambda}^{\mu}_{\nu}\left(\Lambda^{\nu}_{\rho}\,x^{\rho} + a^{\nu}\right) + \bar{a}^{\mu} \\ &= \bar{\Lambda}^{\mu}_{\nu}\Lambda^{\nu}_{\rho}\,x^{\rho} + \bar{\Lambda}^{\mu}_{\nu}\,a^{\nu} + \bar{a}^{\mu}\ , \end{aligned} \tag{4.53}$$

which corresponds to the Poincaré transformation shown in the argument of the right hand side in (4.52).

If we now consider infinitesimal transformations, we can expand them according to

$$U(1+\omega,\epsilon) \simeq 1 + \frac{i}{2}\omega_{\rho\sigma}\,J^{\rho\sigma} - i\epsilon_{\mu}\,P^{\mu}\ , \tag{4.54}$$

where ϵ_μ is an infinitesimal translation. Here $J^{\rho\sigma}$ and P^μ are in principle generic coefficients of the expansion that do not depend on $\omega^{\mu\nu}$ and ϵ^μ. Of course, as we will see below, they will have obvious interpretations.

Now, by making use of the relation (4.52) we can now consider the composition of three transformations, with the second being the infinitesimal one. This is given by

$$U(\Lambda,a)\,U(1+\omega,\epsilon)\,U^{-1}(\Lambda,a) = U(\Lambda(1+\omega)\Lambda^{-1},\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)\ , \tag{4.55}$$

Expanding (4.55) and equating the coefficients of ω and ϵ, it is straightforward to show that

$$U(\Lambda,a)\,J^{\rho\sigma}\,U^{-1}(\Lambda,a) = \Lambda^{\rho}_{\mu}\,\Lambda^{\sigma}_{\nu}(J^{\mu\nu} - a^{\mu}P^{\nu} + a^{\nu}P^{\mu}) \tag{4.56}$$

$$U(\Lambda,a)\,P^{\rho}\,U^{-1}(\Lambda,a) = \Lambda^{\rho}_{\mu}\,P^{\mu}\ . \tag{4.57}$$

If we now consider Λ and a themselves as infinitesimal, i.e. $\Lambda^{\mu}_{\nu} \simeq \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}$, and $a^\mu = \epsilon^\mu$, and we apply these in (4.56) and (4.57), then once again collecting the leading terms in the infinitesimal coefficients we can obtain the following identities:

$$\begin{aligned} i[J^{\mu\nu},J^{\rho\sigma}] &= g^{\nu\rho}\,J^{\mu\sigma} - g^{\mu\rho}\,J^{\nu\sigma} - g^{\sigma\mu}\,J^{\rho\nu} + g^{\sigma\nu}\,J^{\rho\mu} \\ i[P^{\mu},J^{\rho\sigma}] &= g^{\mu\rho}\,P^{\sigma} - g^{\mu\sigma}\,P^{\rho} \\ [P^{\mu},P^{\nu}] &= 0 \end{aligned} \tag{4.58}$$

This collection of commutators is referred to as the Poincaré algebra. As we know, the quantities

$$\begin{aligned} P^0 &= H \ , \\ \vec{P} &= \{P_1, P_2, P_3\} \ , \\ \vec{J} &= \{J^{23}, J^{31}, J^{12}\} \ , \end{aligned} \tag{4.59}$$

are conserved quantities. On the other hand,

$$\vec{K} = J^{10}, J^{20}, J^{30} \ , \tag{4.60}$$

are the so-called boost vectors and are not conserved. We recover, for instance, the well known commutation relations for angular momentum

$$[J_i, J_k] = \epsilon_{ijk}\, J_k \tag{4.61}$$

for $i, j = 1, 2, 3$ the spatial indices. In the case of interest here, i.e. the spinorial representation of the homogenous Lorentz group ($a^\mu = 0$), we have that we identify the generators of the group $J_{\mu\nu}$ with the matrices $\sigma_{\mu\nu}$ (up to factors of 1/2 in the definitions). This means that σ_{ij} are the generators of spatial rotations in the spinorial representation, whereas σ_{0i} generates boosts along the axis $\hat{i}$. In explicit form we have (in the standard representation)

$$\sigma_{0i} = \frac{i}{2}\,[\gamma^0, \gamma^i] = i \left(\begin{array}{cc} \mathbf{0} & \sigma^{\mathbf{i}} \\ \sigma^{\mathbf{i}} & \mathbf{0} \end{array} \ , \right) \tag{4.62}$$

for the boosts, and

$$\sigma_{ij} = \frac{i}{2}\,[\gamma^i, \gamma^j] = \epsilon_{ijk} \left(\begin{array}{cc} \sigma^{\mathbf{k}} & \mathbf{0} \\ \mathbf{0} & \sigma^{\mathbf{k}} \end{array} \ , \right) \tag{4.63}$$

for spatial rotations. As we will see in a later lecture, The σ_{ij} are, up to a factor of two, the components of the spin. Then, just as the components of the angular momentum are the generators of rotations in the vetor representation of the Poincaré group, we will see that the spin components are the generators of spatial rotations in the spinorial representation.

4.4 Transformation Properties of Bilinear Operators

We would like to know the behavior under Lorentz transformations of fermion bilinear operators. This way we will know how to construct a Lorentz invariant lagrangian. We will start by the simplest fermion bilinear operator: $\psi^\dagger\psi$. The spinor transformation (4.31) results in

$$\psi^\dagger\psi \longrightarrow \psi^\dagger S^\dagger(\Lambda)S(\Lambda)\,\psi\ . \tag{4.64}$$

But it turns out $S(\Lambda)$ is not a unitary transformation, so $S^\dagger(\Lambda) \neq S^{-1}(\Lambda)$. This stems from the fact that $\sigma^\dagger_{\mu\nu} \neq \sigma_{\mu\nu}$. On the one hand, using (4.62) we have

$$(\sigma_{0i})^\dagger = -\sigma_{0i}\ , \tag{4.65}$$

whereas from (4.63) we can easily infer that

$$(\sigma_{ij})^\dagger = +\sigma_{ij}\ . \tag{4.66}$$

Then, from (4.36) is clear that $S^\dagger(\Lambda) \neq S^{-1}(\Lambda)$, and thus $\psi^\dagger\psi$ is not invariant. On the other hand, if we define

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0\ , \tag{4.67}$$

this object now transforms into

$$\bar{\psi}'(x') = \psi^\dagger\, S^\dagger(\Lambda)\gamma^0 = \psi^\dagger\gamma^0\gamma^0 S^\dagger(\Lambda)\gamma^0\ , \tag{4.68}$$

where the in last equality we used $(\gamma^0)^2 = 1$. But making use of (4.65) and (4.66) it is straightforward to prove

$$\gamma^0\, S^\dagger(\Lambda)\gamma^0 = \gamma^0 e^{+(i/4)\sigma^\dagger_{\mu\nu}\omega^{\mu\nu}}\gamma^0 = e^{+(i/4)\sigma_{\mu\nu}\omega^{\mu\nu}}\ , \tag{4.69}$$

where to prove the last equality we need to use that[1] $\sigma_{0i}\gamma^0 = -\gamma^0\sigma_{0i}$, but $\sigma_{ij}\gamma^0 = +\gamma^0\sigma_{ij}$. With this we have then that

$$\gamma^0\, S^\dagger(\Lambda)\,\gamma^0 = S^{-1}(\Lambda)\ , \tag{4.70}$$

[1] These are satisfied independently of the representation used for the Dirac matrices.

which implies that

$$\bar{\psi}'(x') = \bar{\psi}(x)S^{-1}(\Lambda) \ . \tag{4.71}$$

This then allows us to show that

$$\bar{\psi}(x)\psi(x) \longrightarrow \bar{\psi}(x)\, S^{-1}(\Lambda)\, S(\Lambda)\, \psi(x) = \bar{\psi}(x)\psi(x) \ , \tag{4.72}$$

so $\bar{\psi}\psi$ is Lorentz invariant.

In general, we would like to know the transformation properties of various fermion bilinears, schematically represented by

$$\bar{\psi}(x)\, \mathbf{A}\, \psi(x) \ , \tag{4.73}$$

where the general Dirac structure is $\mathbf{A} = \{\mathbf{1}, \gamma^\mu, \sigma^{\mu\nu}, \dots\}$.

For instance, for $\bar{\psi}(x)\, \gamma^\mu\, \psi(x)$ we have

$$\bar{\psi}'(x')\, \gamma^\mu\, \psi'(x') = \bar{\psi}(x)\, S^{-1}(\Lambda)\, \gamma^\mu\, S(\Lambda)\, \psi(x) \ . \tag{4.74}$$

Earlier we derived that

$$S^{-1}(\Lambda)\, \gamma^\mu\, S(\Lambda) = \Lambda^\mu_\nu\, \gamma^\nu \ , \tag{4.75}$$

which means that

$$\bar{\psi}(x)\, \gamma^\mu\, \psi(x) \longrightarrow \Lambda^\mu_\nu\, \bar{\psi}(x)\, \gamma^\nu\, \psi(x) \ . \tag{4.76}$$

The result is that $\bar{\psi}(x)\, \gamma^\mu\, \psi(x)$ transforms like a regular four-vector. This should be the case since this is the fermion conserved four-current. The transformation properties of other fermion bilinear operators can be derived in similar fashion.

Additional suggested readings

- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Chapter 2.
- *The Quantum Theory of Fields*, by S. Weinberg. See Sections 2.3 and 2.4.