Lecture 3

Canonical Quantization

Since, as we saw before, Quantum Field Theory (QFT) emerges as we attempt to combine quantum mechanics with special relativity it is natural to start with quantum mechanics of a single particle. We will see that when trying to make this conform with relativistic dynamics, we will naturally develop a way of thinking of the solution to this problem that goes by the name of canonical quantization. Besides being conceptually natural, this formalism will be useful when trying to understand the statistics of different states.

3.1 Quantum Mechanics

The Schrödinger equation for the wave-function of a free particle is

$$i\frac{\partial}{\partial t}\,\psi(\mathbf{x},t) = -\frac{1}{2m}\,\nabla^2\psi(\mathbf{x},t)\ , \tag{3.1}$$

where we set $\hbar = 1$. In terms of states and operators, we can define the wave function as $\psi(\mathbf{x},t) = \langle\mathbf{x}|\psi,t\rangle$, i.e. in term of the state $|\psi,t\rangle$ projected onto the position state $|\mathbf{x}\rangle$. More generally, eq. (3.1) can be written as

$$i\frac{\partial}{\partial t}\,|\psi,t\rangle = H\,|\psi,t\rangle\ , \tag{3.2}$$

where H is the hamiltonian which in the non-relativistic free-particle case is just

$$H = \frac{\mathbf{p}^2}{2m}\ , \tag{3.3}$$

resulting in (3.1). We would like to generalize this for the relativistic case, i.e. choosing

$$H = +\sqrt{\mathbf{p}^2 + m^2} \;, \tag{3.4}$$

where again we use $c = 1$. If one uses this Hamiltonian in the Schrödinger equation one gets

$$i\,\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = \sqrt{-\nabla^2 + m^2}\,\psi(\mathbf{x}, t) \;. \tag{3.5}$$

But this is problematic for a relativistic equation since time and space derivatives are of different order. If the equation has to have any chance of being Lorentz invariant, it needs to have the same number of time and space derivatives. One simple way to do this is to apply the time derivative operator twice on both sides. This results in

$$-\frac{\partial^2}{\partial t^2}\,\psi(\mathbf{x}, t) = \left(-\nabla^2 + m^2\right)\,\psi(\mathbf{x}, t) \;. \tag{3.6}$$

This is the Klein-Gordon equation for the wave function $\psi(\mathbf{x}, t)$, and is clearly consistent with the relativistic dispersion relation (3.4), once we make the identifications

$$i\frac{\partial}{\partial t} \leftrightarrow H \qquad\qquad -i\nabla \leftrightarrow \mathbf{p} \;, \tag{3.7}$$

where H and $\mathbf{p}$ are the hamiltonian and momentum operators. In covariant notation, and using

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = (\frac{\partial}{\partial t}, \nabla) \;, \qquad \frac{\partial}{\partial x_\mu} \equiv \partial^\mu = (\frac{\partial}{\partial t}, -\nabla) \;, \tag{3.8}$$

we can write the Klein-Gordon equation as

$$(\partial_\mu\,\partial^\mu + m^2)\,\psi(\mathbf{x}, t) \;= 0 \;. \tag{3.9}$$

This is manifestly Lorentz invariant. However it has several problems. The fact that this equation has two time derivatives implies for instance that $|\psi(\mathbf{x}, t)|^2$ is not generally time independent, so we cannot interpret it as a conserved probability, as it is in the case of the Scrödinger equation. This issue is tackled by Dirac, which derives a relativistic equation for the wave-function that is first order in both time and space derivatives. But this equation will be valid for spinors, not scalar wave-functions. We will study it in the next lecture in detail. But it does not resolve the central issue, as we see below.

Both the Klein-Gordon and the Dirac equations admit solutions with negative energies. This would imply that the system does not have a ground state, since it would be always energetically favorable to go to the negative energy states. Since the Dirac equation describes fermions, one can use Pauli's exclusion principle and argue, as Dirac did, that all the negative energy states are already occupied. This is the so-called Dirac sea. According to this picture, an electron would not be able to drop to negative energy states since these are already filled. Interestingly, this predicts that in principle it should be possible to kick one of the negative energy states to a positive energy state. Then, one would see an electron appear. But this would leave a hole in the sea, which would appear as a positively charged state. This is Dirac's prediction of the existence of the positron. Is really nice, but now we need an infinite number of particles in the sea, whereas we were supposed to be describing the wave-function of *one* particle. Besides, this only works for wave-functions describing fermions. What about bosons ?

What we are seeing is the inadequacy of the relativistic description of the one-particle wave-function. At best, as in the case of fermions, we were driven from a one-particle description to one with an infinite number of particles. At the heart of the problem is the fact that, although now we have the same number of time and space derivatives, position and time are not treated on the same footing in quantum mechanics. There is in fact a position operator, whereas time is just a parameter labeling the states.

On the other hand, we can consider operators labeled by the *spacetime* position $x^\mu = (t, \mathbf{x})$, such as in

$$\phi(t, \mathbf{x}) = \phi(x) \ . \tag{3.10}$$

These objects are called quantum fields. They are clearly in the Heisenberg picture, whereas if we choose the time-independent Schrödinger picture quantum fields they are only labeled by the spatial component of the position as in $\phi(\mathbf{x})$. These quantum fields will be our dynamical degrees of freedom. All spacetime positions have a value of $\phi(x)$ assigned. As we will see in more detail below, the quantization of these fields will result in infinitely many states. So we will abandon the idea of trying to describe the quantum dynamics of *one* particle. This formulation will allow us to include *antiparticles* and (in the presence of interactions) also other particles associated with other quantum fields. It solves one of the problems mentioned earlier, the fact that relativity and quantum mechanics should allow the presence of these extra particles as long as there is enough energy, and/or the intermediate process that *violates* energy conservation by ΔE lasts a time Δt such that $\Delta E \Delta t \sim \hbar$.

The behavior of quantum fields under Lorentz transformations will define their properties. We can have scalar fields $\phi(x)$, i.e. no Lorentz indices; fields that transform as four-vectors: $\phi^\mu(x)$; as spinors: $\phi_a(x)$, with a a spinorial index; as tensors, as in the rank 2 tensor $\phi^{\mu\nu}(x)$; etc. We will start with the simplest kind, the scalar field.

3.2 Canonical Description of Quantum Fields

First, let us assume a scalar field $\phi(x)$ that obeys the Klein-Gordon equation. The exact meaning of this will become clearer below. But for now it suffices to assume that our dynamical variable obeys a relativistic equation relating space and time derivatives:

$$(\partial^2 + m^2)\phi(x) = 0 \; , \tag{3.11}$$

where we defined the D'Alembertian as $\partial^2 \equiv \partial_\mu \partial^\mu$, and m is the mass of the particle states associated with the field $\phi(x)$. We also assume the scalar field in question is real. That is

$$\phi(x) = \phi^\dagger(x) \; , \tag{3.12}$$

where we already anticipate to elevate the field to an operator, hence the †. It is interesting to solve the Klein-Gordon equation for the classical field in momentum space. The most general solution has the following form

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3}\, N_p\, \left\{ a_p\, e^{-i(\omega_p t - \mathbf{p}\cdot\mathbf{x})} + b_p^\dagger\, e^{i(\omega_p t - \mathbf{p}\cdot\mathbf{x})} \right\} \; , \tag{3.13}$$

where, as defined earlier, $\omega_p = +\sqrt{\mathbf{p}^2 + m^2}$. Here, N_p is a momentum-dependent normalization to be determined later, and the momentum-dependent coefficients a_p and $b_P^\dagger$ will eventually be elevated to operators. In general a_p and $b_p^\dagger$ are independent. However, when we impose (3.12), this results in

$$a_p = b_p \; . \tag{3.14}$$

This is not the case, for instance, if $\phi(x)$ is a complex scalar field.

At this point and before we quantize the system, we remind ourselves of the fact that the Klein-Gordon equation (3.11) is obtained from the lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \frac{1}{2}\, m^2\, \phi^2 \; . \tag{3.15}$$

To convince yourself of this just use the Eüler-Lagrange equations from the previous lecture to derive (3.11) from (3.15). Then, since $\phi(x)$ is our dynamical variable, the canonically conjugated momentum is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = \dot{\phi} \; , \tag{3.16}$$

which, using (3.13), results in

$$\pi(\mathbf{x},t) = \int \frac{d^3x}{(2\pi)^3}\, N_p \left\{-i\omega_p\, a_p\, e^{-i(\omega_p t - \mathbf{p}\cdot\mathbf{x})} + i\omega_p\, a_p^\dagger\, e^{i(\omega_p t - \mathbf{p}\cdot\mathbf{x})}\right\} \ . \tag{3.17}$$

Having the field and its conjugate momentum defined we can then impose quantization conditions. It is useful first to refresh our memory on how this is done in quantum mechanics.

3.2.1 Canonical Quantization in Quantum Mechanics

Let us consider a particle of mass $m = 1$ in some units. Its lagrangian is

$$L = \frac{1}{2}\,\dot{q}^2 - V(q) \quad , \tag{3.18}$$

where $V(q)$ is some still unspecified potential, which we assume it does not depend on the velocities. The associated hamiltonian is

$$H = \frac{1}{2}\,p^2 + V(q) \quad , \tag{3.19}$$

where the conjugate momentum is $p = \partial L/\partial\dot{q} = \dot{q}$. To quantize the system we elevate p and q to operators and impose the commutation relations

$$[q,p] = i, \qquad [q,q] = 0 = [p,p] \ . \tag{3.20}$$

Notice that if we are in the Heisenberg description, the commutators should be evaluated at equal time, i.e. $[q(t),p(t)] = i$, etc. We change to a description in terms of the operators

$$\begin{aligned} a &\equiv \frac{1}{2\omega}\,(\omega\, q + ip) \\ a^\dagger &\equiv \frac{1}{2\omega}\,(\omega\, q - ip) \ , \end{aligned} \tag{3.21}$$

where ω is a constant with units of energy. It is straightforward, using the commutators in (3.20), to prove that these operators satisfy the following commutation relations

$$[a,a^\dagger] = 1, \qquad [a,a] = 0 = [a^\dagger,a^\dagger] \ . \tag{3.22}$$

We define the ground state of the system by the following relation

$$a|0\rangle = 0 \ , \tag{3.23}$$

where the 0 in the state refers to the absence of quanta. Then, assuming the ground state (or vacuum) is a normalized state, we have

$$1 = \langle 0|0\rangle = \langle 0|[a, a^\dagger]|0\rangle = \langle 0|aa^\dagger|0\rangle - \langle 0|a^\dagger a|0\rangle \ . \tag{3.24}$$

Since the last term vanishes when using (3.23), we arrive at

$$\langle 0|0\rangle = \langle 0|aa^\dagger|0\rangle \ . \tag{3.25}$$

This is achieved only if we have

$$\begin{aligned} a^\dagger|0\rangle &= |1\rangle \ , \\ a|1\rangle &= |0\rangle \ , \end{aligned} \tag{3.26}$$

which means that a and $a^\dagger$ are ladder operators. We interpret the state $|1\rangle$ as a state with one particle. In this way a and $a^\dagger$ can also we called annihilation and creation operators. The simplest example is, of course, the simple harmonic oscillator, with $V(q) = \omega^2\, q^2/2$.

3.2.2 Quantizing Fields

We are now ready to generalize the canonical quantization procedure for fields. We will impose commutation relations for the field in (3.13) and its conjugate momentum in (3.17), which means that we elevated them to operators, specifically in the Heisenberg representation. The quantization condition is

$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \ . \tag{3.27}$$

Here we see that the commutator is defined at equal times, as it should for Heisenberg operators. All other possible commutators vanish, i.e.

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = 0 = [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] \tag{3.28}$$

Now, when we turned $\phi(x)$ into an operator, so did a_p and $a_p^\dagger$. In order to see what the imposition of (3.27) and (3.28) implies for the commutators of the operators a_p and $a_p^\dagger$, we write out (3.27) using the explicit expressions (3.13) and (3.17) for the field and its momentum in terms of them. We obtain

$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] = \int \frac{d^3p}{(2\pi)^3}\,N_p \int \frac{d^3p'}{(2\pi)^3}\,N_{p'}\,\Big\{i\omega_{p'}e^{-i(\omega_p-\omega_{p'})t}e^{i\mathbf{p}\cdot\mathbf{x}-i\mathbf{p}'\cdot\mathbf{x}'}\,[a_p,a^\dagger_{p'}] - i\omega_{p'}e^{i(\omega_p-\omega_{p'})t}e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{p}'\cdot\mathbf{x}'}\,[a^\dagger_p,a_{p'}]\Big\}\ , \tag{3.29}$$

where we have already assumed that

$$[a_p,a_{p'}] = 0 = [a^\dagger_p,a^\dagger_{p'}]\ . \tag{3.30}$$

The question is what are the commutation rules for $[a_p,a^\dagger_{p'}]$. Now we will show that in order for (3.27) to be satisfied, we need to impose

$$[a_p,a^\dagger_{p'}] = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{p}')\ . \tag{3.31}$$

If we do this in (3.27) we see that $\omega_p = \omega_{p'}$, $N_p = N_{p'}$, and we obtain

$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] = i\int \frac{d^3p}{(2\pi)^3}\,N_p^2\,\omega_p\,\Big\{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}+e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\Big\}\ . \tag{3.32}$$

But we notice that since

$$\delta^{(3)}(\mathbf{x}-\mathbf{x}') = \int \frac{d^3p}{(2\pi)^3}\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} = \int \frac{d^3p}{(2\pi)^3}\,e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\ , \tag{3.33}$$

then if

$$N_p^2\,\omega_p = \frac{1}{2}\ , \tag{3.34}$$

we recover the result of (3.27). In other words

$$\boxed{[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}')} \longleftrightarrow \boxed{[a_p,a^\dagger_{p'}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')}\ , \tag{3.35}$$

as long as

$$N_p = \frac{1}{\sqrt{2\omega_p}} \ . \tag{3.36}$$

We can now go back to the expression (3.13) for the real scalar field, and rewrite it in covariant form as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \left\{ a_p e^{-ip_\mu x^\mu} + a_p^\dagger e^{ip_\mu x^\mu} \right\} \ , \tag{3.37}$$

where we used that

$$p_\mu x^\mu = p_0 x_0 - \mathbf{p}\cdot\mathbf{x} = \omega_p t - \mathbf{p}\cdot\mathbf{x} \tag{3.38}$$

Once again, since we define the vacuum state by

$$a_p|0\rangle = 0 \ , \tag{3.39}$$

we conclude that a_p and $a_p^\dagger$ are ladder operators, just as in the quantum mechanical case seen above. In other words we have

$$a_p^\dagger|0\rangle = |1_p\rangle \ , \tag{3.40}$$

where $|1_p\rangle$ corresponds to the state containing one particle of momentum $\mathbf{p}$. Conversely, and analogously to the quantum mechanical case, we have

$$a_p|1_p\rangle = |0\rangle \ . \tag{3.41}$$

This allows us to interpret the operators $\phi(x)$ and $\phi^\dagger(x)$ in the following form:

<u>The operator $\phi(x)$</u>:

- Annihilates a *particle* of momentum $\mathbf{p}$
- Creates an *anti-particle* of momentum $\mathbf{p}$

On the other hand,

<u>The operator $\phi^\dagger(x)$</u>:

- Annihilates an *anti-particle* of momentum $\mathbf{p}$

- Creates a *particle* of momentum $\mathbf{p}$

Of course in our case, a real scalar field, particles and anti-particles are the same due to (3.14). On the other hand, if ϕ was for instance complex, particles and anti-particles would be created and annihilated by different operators, and they would carry different "charges" under the global $U(1)$ symmetry of the lagrangian. We will discuss this later in the course.

3.3 Statistics

One of the advantages of this formulation is that we will be able to determine the statistics of our Klein-Gordon quanta. In general a state of N particles can be defined by

$$|p_1 p_2 \dots p_j p_k \dots p_N\rangle = a^\dagger_{p_1} a^\dagger_{p_2} \dots a^\dagger_{p_j} a^\dagger_{p_k} \dots a^\dagger_{p_N} |0\rangle \ , \tag{3.42}$$

But since the $a^\dagger_p$'s commute, then the state on the left is even under the exchange of any two particles. Also, it is not a problem to have any two of the particles to have the same momentum. For instance, if we have two particles

$$|pq\rangle = a^\dagger_p a^\dagger_q |0\rangle \ , \tag{3.43}$$

the fact that $[a^\dagger_p, a^\dagger_q] = 0$ implies that it is allowed to have $\mathbf{p} = \mathbf{q}$. All this implies that the quanta of the real scalar field $\phi(x)$ obeying the Klein-Gordon equation obey Bose-Einstein statistics.

We will see that for other fields, we will need to impose *anti-commuting* rules . These anti-commuting operators satisfy

$$b^\dagger_p b^\dagger_q + b^\dagger_q b^\dagger_p \equiv \{b^\dagger_p, b^\dagger_q\} = 0 \ . \tag{3.44}$$

For these, a state of two particles is given by

$$|pq\rangle = b^\dagger_p b^\dagger_q |0\rangle \ . \tag{3.45}$$

But then the exchange of the two particles results in

$$|qp\rangle = b^\dagger_q b^\dagger_p |0\rangle = -b^\dagger_p b^\dagger_q |0\rangle = -|pq\rangle \ , \tag{3.46}$$

meaning that the state is odd under the particle exchange. These particles (fermions) obey Fermi statistics. This clearly means that these two particles cannot have the same momentum. This is called the *Pauli exclusion principle*. In the next lecture we will see that fermions must have ladder operators obeying anti-commutations rules such as (3.44), and therefore are subject to the Pauli exclusion principle.

3.4 The Feynman Propagator

Once again, just as we did in the first lecture, we consider here the amplitude of creating a particle at some spacetime point y and then annihilating it at some other point x.

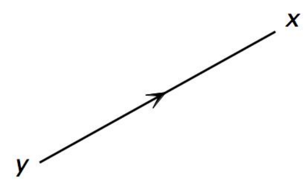

Figure 3.1: Creation, propagation and annihilation of a particle.

Since now we know we can use the field operator $\phi(x)$ to create and annihilate, it looks reasonable to compute the following amplitude

$$\langle 0|\phi(x)\,\phi(y)|0\rangle\ . \tag{3.47}$$

From the momentum expansion of $\phi(x)$ in terms of creation and annihilation operators a_p and $a_p^\dagger$ it is easy to realize that only terms like $a_p\,a_{p'}^\dagger$ will contribute to the vacuum expectation value (VEV) in (3.47). This is because

$$\langle 0|a_p\,a_{p'}|0\rangle = 0 = \langle 0|a_p^\dagger\,a_{p'}^\dagger|0\rangle\ , \tag{3.48}$$

whereas

$$\begin{aligned}\langle 0|a_p\,a_{p'}^\dagger|0\rangle &= \langle 0|a_p\,a_{p'}^\dagger - a_{p'}^\dagger\,a_p|0\rangle \\ &= \langle 0|[a_p, a_{p'}^\dagger]|0\rangle = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{p}')\ ,\end{aligned} \tag{3.49}$$

Then (3.47) is

$$\begin{aligned}\langle 0|\phi(x)\,\phi(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\, e^{-ip\cdot x}\, e^{+ik\cdot y}\, \langle 0|a_p\, a_k^\dagger|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, e^{-ip\cdot(x-y)} \equiv D(x-y)\ ,\end{aligned} \tag{3.50}$$

where in the last line we used (3.49) and we define the two-point function $D(x-y)$, the propagator. But, as we discussed at length in the first lecture, this just gives us the amplitude for one particular temporal order: $x_0 > y_0$. We actually need an amplitude that is causal, i.e. that is valid for whatever temporal order.

We will see that the following object is the propagator we are looking for. We define the Feynman propagator as the two-point function given by

$$D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot(x-y)}\ , \tag{3.51}$$

where m is the mass of the particle and ϵ is an infinitesimal, positive, real constant. Its meaning will soon be clear. Let us just say for now that the choice of sign with wich ϵ enters in (3.51) is what matters, not its precise value.

To expose the issue of temporal order we rewrite the Feynman propagator as

$$D_F(x-y) = \int \frac{d^3p}{(2\pi)^4}\, dp_0\, \frac{i}{p_0^2 - \omega_p^2 + i\epsilon}\, e^{-ip_0(x_0-y_0)}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\ , \tag{3.52}$$

where we rewrote the denominator as

$$p_0^2 - |\mathbf{p}|^2 - m^2 + i\epsilon = p_0^2 - \omega_p^2 + i\epsilon\ . \tag{3.53}$$

We will perform the integral in p_0. For this purpose we analytically continue it to the complex plane. This is the reason for the introduction of the small imaginary part $i\epsilon$. First, we identify the poles of the integrand. They are at

$$p_0 = \left\{ \begin{array}{c} \omega_p - i\epsilon \\ -\omega_p + i\epsilon \end{array} \right\}\ , \tag{3.54}$$

in other words infinitesimally below and above the real p_0 axis. To perform the integral in the complex p_0 plane we need to choose how to close the contour to apply Cauchy's residue theorem. Of course, we want the contour to only contribute for real values of p_0 so we get the integral we actually want to perform. But there are two possible choices to close the contour: at very large and positive $Im[p_0]$ or very large and negative $Im[p_0]$. As we can see from (3.52), the choice that leads to a vanishing contribution from the infinte radius semicircle depends on the sign of $x_0 - y_0$, that is on the temporal order. We will consider each case separately.

<u>$x_0 > y_0$:</u>
For this temporal order, the factor

$$e^{-ip_0(x_0-y_0)} \tag{3.55}$$

decays exponentially (what we want to have a non-contributing semicircle in the contour) if

$$Im[p_0] < 0 \ . \tag{3.56}$$

This contour is shown in Figure 3.2.

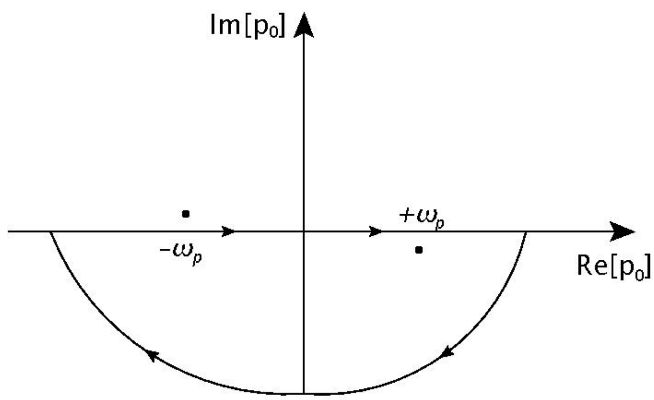

Figure 3.2: Closing the contour below for $x_0 > y_0$ to perform the p_0 integral.

Since only the $p_0 = \omega_p - i\epsilon$ pole is enclosed in the chosen contour, we have

$$\begin{aligned} D_F(x-y) &= \int \frac{d^3p}{(2\pi)^4} \int dp_0 \, \frac{i}{(p_0-\omega_p)(p_0+\omega_p)} \, e^{-ip\cdot(x-y)} \\ &= \int \frac{d^3p}{(2\pi)^4} \, (-2\pi i) \, \frac{i}{2\omega_p} \, e^{-ip\cdot(x-y)} \ . \end{aligned} \tag{3.57}$$

We then conclude that for the temporal order $x_0 > y_0$ the Feynman propagator is

$$D_F(x-y) = \int \frac{d^3p}{(2\pi)^3 2\omega_p}\, e^{-ip\cdot(x-y)} = D(x-y)\ , \tag{3.58}$$

which is the value of the amplitude computed in (3.50). We will now evaluate the Feynman propagator for the other possible temporal order.

$\underline{y_0 > x_0}$:
For this case, of course, we need to close the contour above in order to have an exponentially decaying factor in (3.55). That is, we need

$$Im[p_0] > 0\ . \tag{3.59}$$

The contour is shown is Figure 3.3. We see that now the pole that contributes is $p_0 = -\omega_p + i\epsilon$.

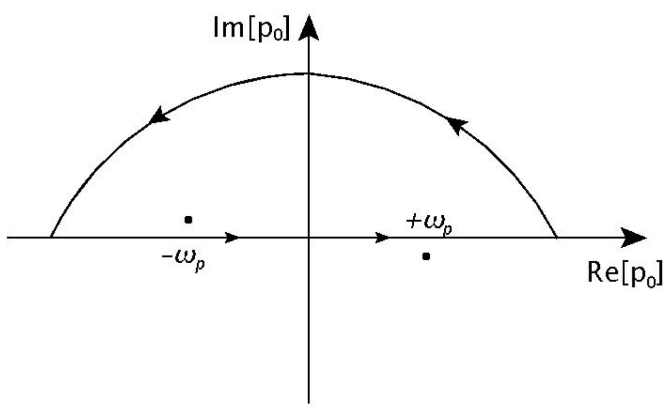

Figure 3.3: Closing the contour below for $y_0 > x_0$ to perform the p_0 integral.

Now the result of the integration is

$$D_F(x-y) = \int \frac{d^3p}{(2\pi)^4}\,(+2\pi i)\,\frac{i}{-2\omega_p}\, e^{-ip\cdot(x-y)}\ . \tag{3.60}$$

Given that, neglecting ϵ, the pole is here given by $p_0 = -\omega_p$, we can write

$$e^{-ip_0(x_0-y_0)} = e^{-i\omega_p(y_0-x_0)}\ . \tag{3.61}$$

This, together with the fact that we can change $\mathbf{p} \to -\mathbf{p}$ by switching the integration limits in the d^3p integrals, allows us to make the replacement

$$e^{-ip\cdot(x-y)} \to e^{-ip\cdot(y-x)} \;, \tag{3.62}$$

for this temporal order. Then, for $y_0 > x_0$ the Feynman propagator is

$$D_F(x-y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_p}\,e^{-ip\cdot(y-x)} = D(y-x)\;. \tag{3.63}$$

We can summarize the results of (3.58) and (3.63) for both temporal orders as

$$D_F(x-y) = \theta(x_0-y_0)\,D(x-y) + \theta(y_0-x_0)\,D(y-x)\;. \tag{3.64}$$

Or, using our result from (3.50), we can write

$$D_F(x-y) = \theta(x_0-y_0)\,\langle 0|\phi(x)\,\phi(y)|0\rangle + \theta(y_0-x_0)\,\langle 0|\phi(y)\,\phi(x)|0\rangle\;. \tag{3.65}$$

Defining the time order operator T, we can write (3.65) in a compact form as

$$\boxed{D_F(x-y) = \langle 0|T\phi(x)\,\phi(y)|0\rangle}\;. \tag{3.66}$$

This means that the Feynman or causal propagator is given by the VEV of the time-ordered product of the fields. Finally, we can see that we obtained a causal propagator with the choice of positive ϵ. Had we chosen ϵ to be negative we would have obtained a non-causal object. Thus, the choice of ϵ is crucial in order to get the correct propagator.

Additional suggested readings

- *Quantum Field Theory*, by M. Srednicki, Chapter 3.
- *Quantum Field Theory in a Nutshell*, by A. Zee. See the discussion in Chapter 1.8 about canonical quantization.